2.2 Functions and Their Properties; Domain and Range

Recall: **relation** – any set of ordered pairs
domain of a relation – the set of all **first coordinates** of the relation
range of a relation – the set of all **second coordinates** of the relation
function - a relation that have assigned a **unique value** from the range for every input from the domain.

relation F relation H

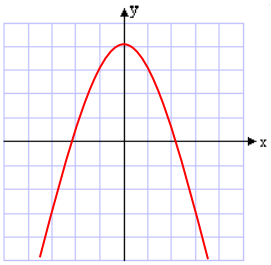
F is a function H is not a function

VERTICAL LINE TEST: A graph represents a function iff no vertical line intersects the graph at more than one point.

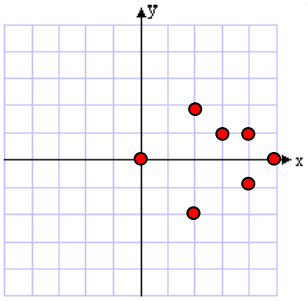
Example 1: Decide whether each relation defines a function and give its domain and range.

a) $\{(-3,1),(1,-3),(2,0),(3,1)\}$

b)



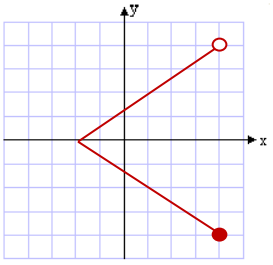
c)



d) $y = \sqrt{x-1}$

e) $x^2 + y^2 = 4$

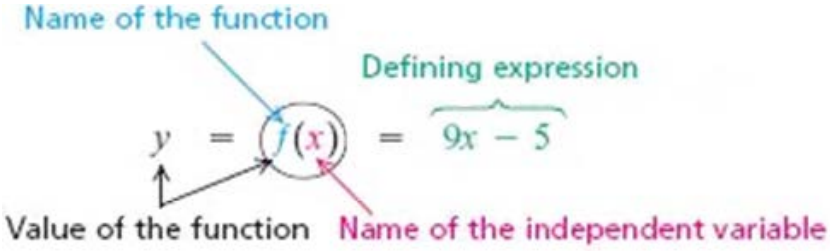
f)



g) $x = -y^2$

h) $x + y < 3$

Function Notation: To denote the *y*-value assigned to the particular *x*, we write $f(x)$ (*read: "**f** of **x**".*)



In the above example we have $f(1) = 4$, which means that the value $y = 4$ is assigned to the input $x = 1$ by the function *f*; or that the point $(1,4)$ belongs to the graph of function *f*.

Example 2: Given the function $f(x) = x^2 - 2x$, evaluate

a) $f(a + 1)$

b) $f(-2c)$

c) $f(2 + h) - f(2)$

graph of a function *f* – the set of points $\{(x, f(x)) | x \in D_f\}$

piecewise function – a function defined by different expressions in different parts of the domain

Example 3: Let $f(x) = \begin{cases} |x - 2|, & if \quad x \leq -2 \\ x^2 - 4, & if \quad -2 < x < 3 \\ 2x + 1, & if \quad x \geq 3 \end{cases}$.

A) Find

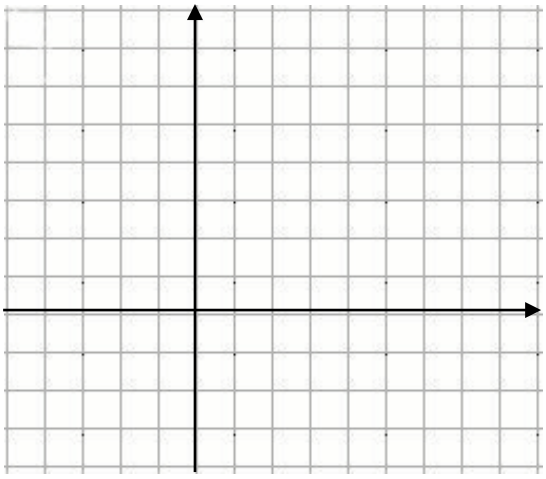
i) $f(-3)$

ii) $f(-2)$

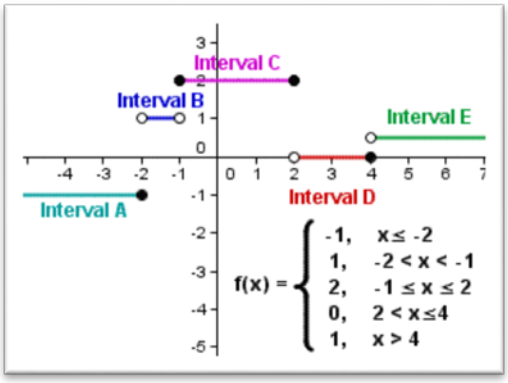
iii) $f(-1)$

iv) $f(5)$

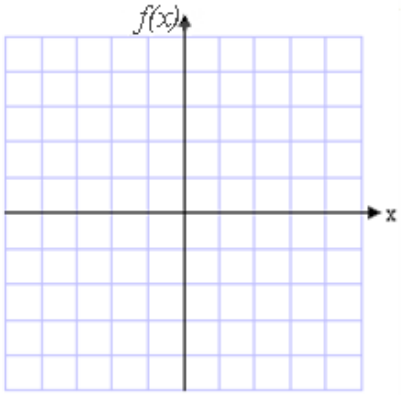
B) Graph function *f*.



step function – a piecewise-constant function



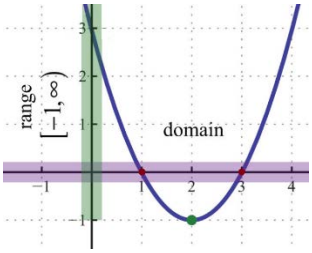
The Greatest Integer Function
(Floor Function) denoted $[\![x]\!]$, $\lfloor x \rfloor$, or $int(x)$ is defined as the greatest integer not exceeding x. It means that $[\![x]\!] = n$ for all $x \in [n, n+1)$.



Example 4: Graph $f(x) = [\![x]\!]$

Important characteristics of functions:

domain – the set of all input values that can be used to evaluate the function
range – the set of all output values of the function
zero of a function – any value $a \in D_f$ satisfying the equation $f(a) = 0$



Example 5: Determine the domain of each function.

a) $f(x) = \frac{2x}{3x-5}$ b) $g(x) = \frac{\sqrt{x}}{\sqrt{3-x}}$

c) $h(x) = \sqrt{9 - x^2}$

d) $p(x) = \begin{cases} x^2 - 1, & for \quad 1 \le x \le 3 \\ -\frac{1}{2}x + 8, & for \quad 3 < x \le 10 \end{cases}$

Example 6: Find all *a*-values satisfying $f(a) = 5$, if $f(x) = |x - 3|$.

Example 7: Find all zero's of $f(x) = x^2 - x - 30$.

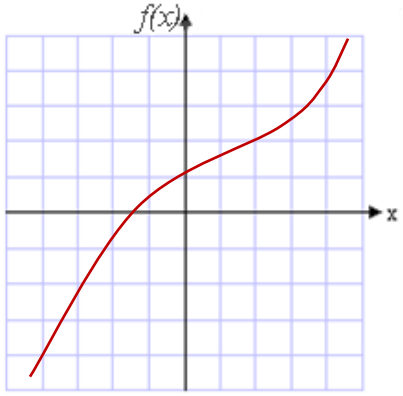
Let I be an interval in the domain of a function *f*. Then

- f is **increasing** on I iff $f(a) < f(b)$ for every choice of $a, b \in I$, such that $a < b$,
- f is **decreasing** on I iff $f(a) > f(b)$ for every choice of $a, b \in I$, such that $a < b$,
- f is **constant** on I iff $f(a) = f(b)$ for every choice of $a, b \in I$.

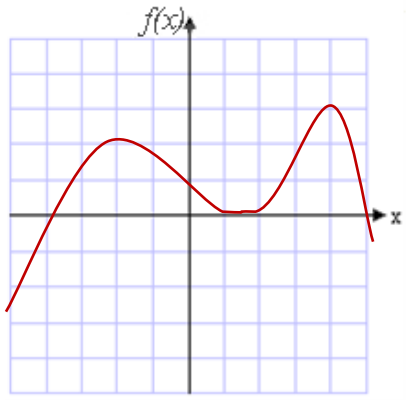
monotonic function – a function that is either **always increasing or always decreasing** in its domain;
one-to-one function – a function that pairs any given **output *y* with exactly one input *x***; a function that satisfies the **horizontal line test** (If any horizontal line intersects the graph of a function at most once, then the function is 1-1.)

Example 8: For the given graphs of functions, decide whether or not the function is one-to-one, or monotonic. State on which intervals the function increases, decreases, or remains constant.

a)

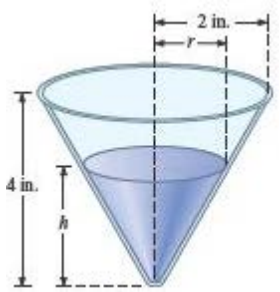


b)



Some Applications:

Example 9: Water is flowing into a conical drinking cup with an altitude of 4 inches and radius of 2 inches.

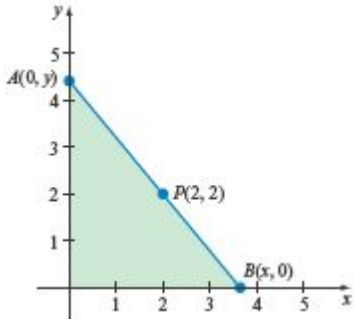


A) Write the radius r of the surface of the water as a function of its depth h.

B) Write the volume V of the water as a function of its depth h.

Example 10: A triangle is bounded by the x- and y-axes and must pass through $P(2,2)$.

A) Find the area of the triangle as a function of x.
(*Hint:* use similar triangles)



B) What is the domain of the above function?

Example 11: A man in a rowboat that is 2 miles from the nearest point A on a shoreline wishes to reach a house located at a point B that is 6 miles farther down the shoreline. He plans to row to a point P that is between A and B and is x miles from the house, and then he will walk the remainder of the distance. Suppose he can row at a rate of 3 mi/hr and can walk at a rate of 5 mi/hr. If T is the total time required to reach the house, express T as a function of x.