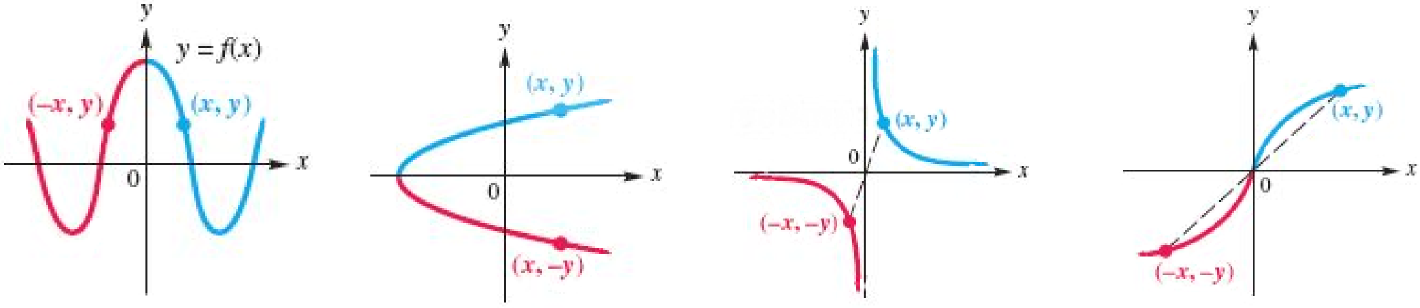
2.5 Symmetries and Transformations of Graphs

Symmetries:



y-axis symmetry x-axis symmetry Origin symmetry

Definition:

⇨ A graph of $y = f(x)$ is **symmetric** with respect to ***y*-axis** $(\boldsymbol{S_y})$ iff $\boldsymbol{f(-x) = f(x)}$
Such function is called **EVEN**.

⇨ A graph of $x = f(y)$ is **symmetric** with respect to ***x*-axis** $(\boldsymbol{S_x})$ iff $\boldsymbol{f(-y) = f(y)}$.

⇨ A graph of $y = f(x)$ is **symmetric** with respect to the **origin** $(\boldsymbol{S_O})$ iff $\boldsymbol{f(-x) = -f(x)}$
Such function is called **ODD**.

Example 1: Without graphing, determine symmetries of graphs of the given equations.

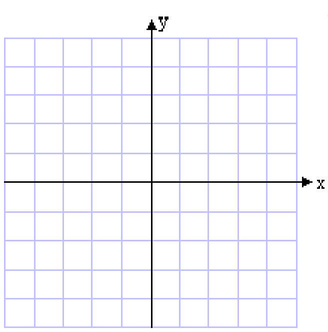
a) $y = x^4 + 6x^2$ b) $y = x^3 - x$

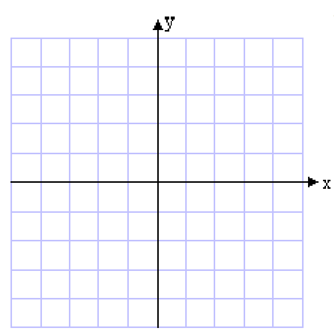
c) $xy = 8$ d) $x^2 + y^2 = 9$

e) $x = |y| + 1$ f) $y = |x + 1|$

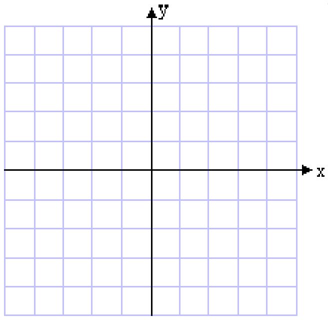
Example 2: Graph each relation. State its domain, *x*- and *y*-intercepts, and symmetries.

a) $y = \sqrt{|x|}$ b) $|x| + |y| = 4$

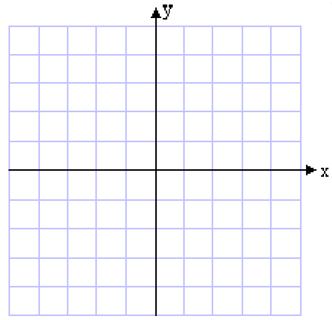




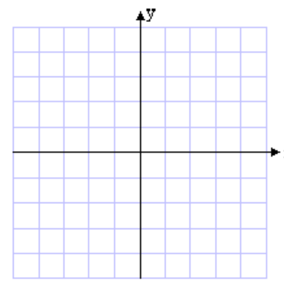
c) $y = x^3 - x$



d) $y = \sqrt{9 - x^2}$



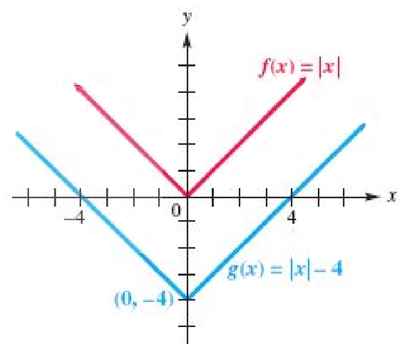
Example 3: Find a function that is both even and odd.

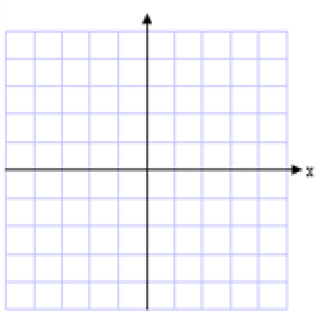


Transformations:

∗ **vertical translations:**

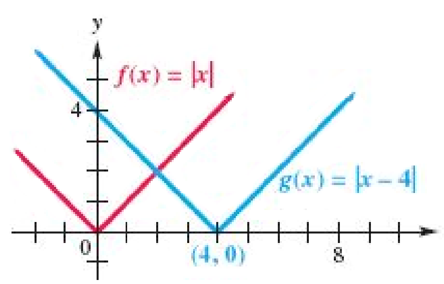
- to obtain $g(x) = \boldsymbol{f(x)} + \boldsymbol{c}$ from $f(x)$, **translate** the graph of f by a vector $\langle \mathbf{0}, \boldsymbol{c} \rangle$; this translation will be denoted $T_{\langle 0,c \rangle}$
- so $f \ni (\boldsymbol{x}, \boldsymbol{y}) \xrightarrow{T_{\langle 0,c\rangle}} (\boldsymbol{x}, \boldsymbol{y} + \boldsymbol{c}) \in g$
- *example:* If $f(x) = x^2$, graph $g(x) = x^2 + 2$.

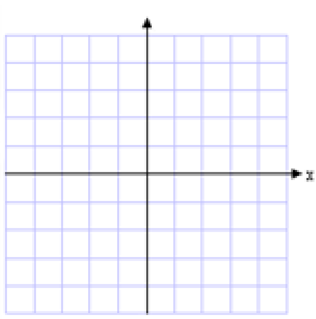




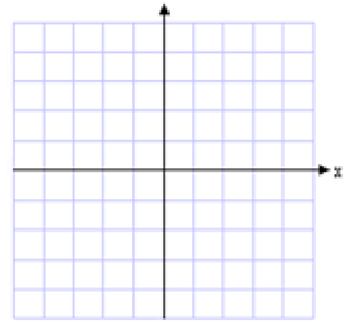
∗ **horizontal translations:**

- to obtain $g(x) = \boldsymbol{f}(\boldsymbol{x} + \boldsymbol{c})$ from $f(x)$, **translate** the graph of f by a vector $\langle -\boldsymbol{c}, \mathbf{0} \rangle$; this translation will be denoted $T_{\langle -c,0 \rangle}$
- so $f \ni (\boldsymbol{x}, \boldsymbol{y}) \xrightarrow{T_{\langle -c,0\rangle}} (\boldsymbol{x} - \boldsymbol{c}, \boldsymbol{y}) \in g$
- *example:* If $f(x) = x^2$, graph $g(x) = (x + 2)^2$.



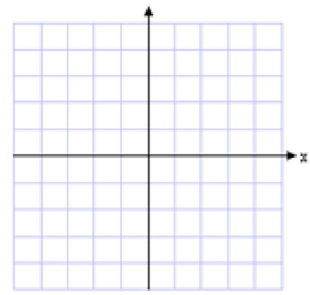


Example 4: List the translations needed to obtain the graph of $g(x) = \sqrt{x+3} - 2$ from the graph of $(x) = \sqrt{x}$. Then, graph the function g.

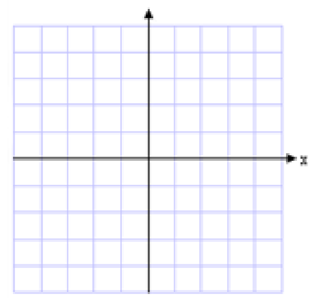


$*$ **reflections:**

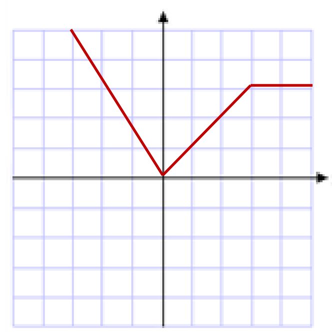
- to obtain $g(x) = -\boldsymbol{f}(\boldsymbol{x})$ from $f(x)$, **reflect** the graph of f with respect to $\boldsymbol{x}$**-axis**; this transformation will be denoted $\boldsymbol{S_x}$
- so $f \ni (\boldsymbol{x}, \boldsymbol{y}) \xrightarrow{S_x} (\boldsymbol{x}, -\boldsymbol{y}) \in g$
- *example:* If $f(x) = x^2$, graph $g(x) = -x^2$.



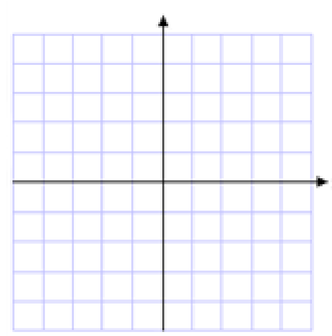
- to obtain $g(x) = \boldsymbol{f}(-\boldsymbol{x})$ from $f(x)$, **reflect** the graph of f with respect to $\boldsymbol{y}$**-axis**; this transformation will be denoted $\boldsymbol{S_y}$
- so $f \ni (\boldsymbol{x}, \boldsymbol{y}) \xrightarrow{S_y} (-\boldsymbol{x}, \boldsymbol{y}) \in g$
- *example:* If $f(x) = (x+2)^2$, graph $g(x) = (-x+2)^2$.



Example 5: Using the given graph of $f(x)$, graph the function $g(x) = -f(-x)$. Observe the action of symmetries.

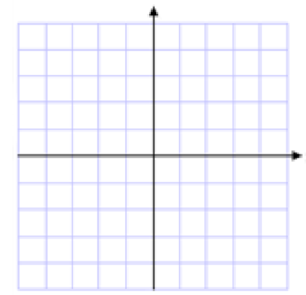


Example 6: List the transformations needed to obtain the graph of $g(x) = 2 - |1 - x|$ from the graph of $(x) = |x|$. Then, graph the function g.



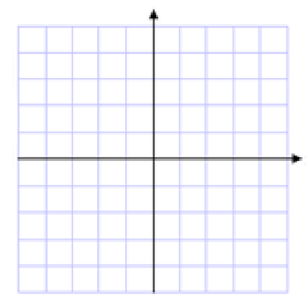
∗ **dilations:**

- to obtain $g(x) = \boldsymbol{k f(x)}$ from $f(x)$, **dilate** (stretch or shrink) the graph of f **vertically** by the factor $\boldsymbol{k}$; this transformation will be denoted $\boldsymbol{D_y^k}$



- so $f \ni (\boldsymbol{x}, \boldsymbol{y}) \xrightarrow{D_y^k} (\boldsymbol{x}, \boldsymbol{ky}) \in g$
- *example:* If $f(x) = x^2$, graph $g(x) = 2x^2$.

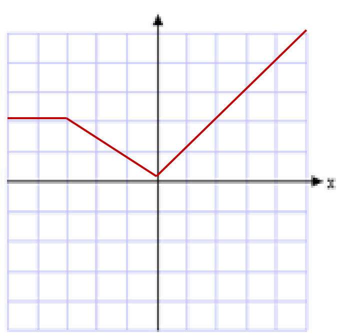
- to obtain $g(x) = \boldsymbol{f(kx)}$ from $f(x)$, **dilate** (stretch or shrink) the graph of f **horizontally** by the factor $\boldsymbol{\frac{1}{k}}$; this transformation will be denoted $\boldsymbol{D_x^{\frac{1}{k}}}$



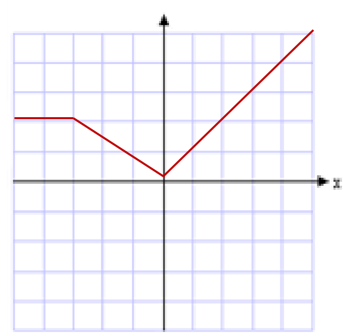
- so $f \ni (\boldsymbol{x}, \boldsymbol{y}) \xrightarrow{D_x^{\frac{1}{k}}} \left(\boldsymbol{\frac{1}{k} x}, \boldsymbol{y}\right) \in g$
- *example:* If $f(x) = x^2$, graph $g(x) = \left(\frac{1}{2}x\right)^2$.

Example 7: Using the given graph of $f(x)$, graph

a) $g(x) = -\frac{1}{2} f(3x)$



b) $h(x) = 2f(x + 1) - 3$

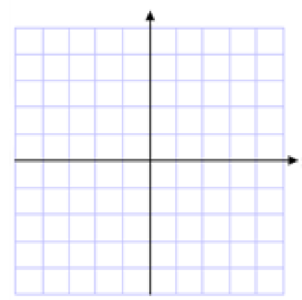


Example 8: Let f be a function such that $f(-1) = 3$ and $f(2) = -3$. Give the coordinates of two points on the graph of $g(x) = 2f(x - 1) + 3$

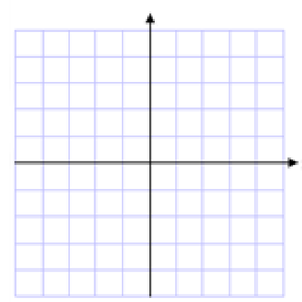
Example 9: Let f be a function with domain $D = [-4,1]$ and range $R = [-2,6]$. Find a sequence of transformations of the graph of f to obtain the graph of $g(x) = -\frac{1}{2}f(2x+6)+1$ and state the domain and range of the function g.

* **absolute value:**

- to obtain $g(x) = |\boldsymbol{f(x)}|$ from $f(x)$, **reflect the negative part** of the graph of f with respect to x-axis and leave the positive part unchanged
- *example:* If $f(x) = x^2 - 4$, graph $g(x) = |x^2 - 4|$.



- to obtain $g(x) = \boldsymbol{f(|x|)}$ from $f(x)$, **erase the left** part of the graph of f and **redraw the right part** of the graph **in reflection** with respect **to *y*-axis**
- *example:* If $f(x) = (x-1)^2$, graph $g(x) = (|x|-1)^2$.



Example 10: Graph $f(x) = \big||x-1|-1\big|$.