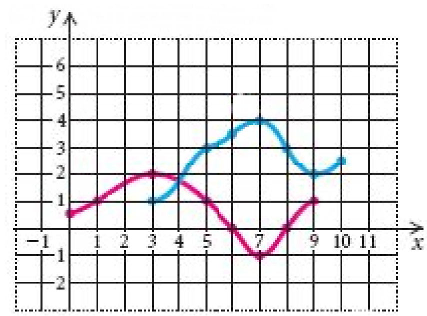
2.6 Algebra of Functions

Definition 1:



For any given function $\boldsymbol{f}$ (with the domain D_f), and $\boldsymbol{g}$ (with the domain D_g), we can define

the **sum** $(\boldsymbol{f}+\boldsymbol{g})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x}),$

the **difference** $(\boldsymbol{f}-\boldsymbol{g})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x}),$

the **product** $(\boldsymbol{fg})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x})\boldsymbol{g}(\boldsymbol{x}),$ and

the **quotient** function: $\left(\frac{\boldsymbol{f}}{\boldsymbol{g}}\right)(\boldsymbol{x}) = \frac{\boldsymbol{f}(\boldsymbol{x})}{\boldsymbol{g}(\boldsymbol{x})}$.

The **domain** of the **sum**, **difference**, and **product** function is the intersection $\boldsymbol{D_f} \cap \boldsymbol{D_g}$of the domains of functions $\boldsymbol{f}$ and $\boldsymbol{g}$.

The **domain** of the **quotient** function is $\boldsymbol{D_f} \cap \boldsymbol{D_g} \setminus \{\boldsymbol{x} \in \mathbb{R} |\ \boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{0}\}$, the intersection of the domains of functions $\boldsymbol{f}$ and $\boldsymbol{g}$, excluding all $\boldsymbol{x}$-values such that $g(x) = 0$.

Example 1: Given $f(x) = x + 2$, and $g(x) = x^2 - 4$, find the following functions and their domains:

a) $(f + g)(x)$

b) $(g - f)(x)$

c) $2 \cdot (fg)(x)$

d) $(ff - g)(x)$

e) $\left(\frac{g}{f}\right)(x)$

f) $\left(\frac{f}{g}\right)(x)$

Example 2: Let $f(x) = \sqrt{x + 4}$, and $g(x) = \frac{1}{\sqrt{x}}$. If possible, evaluate the following:

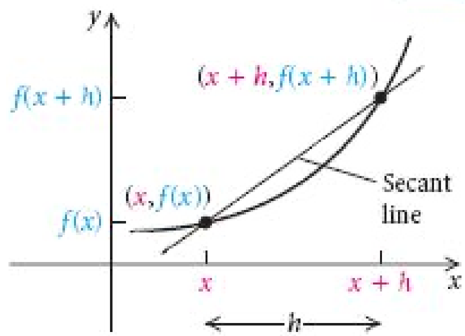
a) $(fg)(4)$

b) $f(0) - 2g(9)$

c) $(f + g)(0)$

d) $\left(\frac{f}{g}\right)(2)$

Difference Quotient – slope of the **secant line** connecting two points on a curve; represents the **average rate of change** (velocity) between the two points;



Difference quotient = $\dfrac{\boldsymbol{f(x+h)-f(x)}}{\boldsymbol{h}}$

Example 3: Find the difference quotient for the given function.

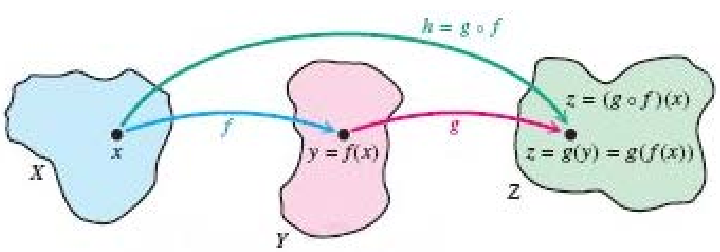
a) $f(x) = 2x + 1$

b) $f(x) = \frac{1}{x}$

c) $f(x) = x^2 + x$

Composition of Functions

Definition 2: For any given function $\boldsymbol{f}$ and $\boldsymbol{g}$, we can define the **composition**: $(\boldsymbol{g} \circ \boldsymbol{f})(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}))$ as long as $f(x)$ is in the domain of $\boldsymbol{g}$.



The **domain of the composition** $\boldsymbol{D_{g\circ f} = D_f \cap D_{res}}$, where D_f is the domain of the inner function f and D_{res} is the domain of the simplified resulting expression for $g \circ f$.

Example 4: Find $(f \circ g)(x)$ and $(g \circ f)(x)$ and the domains of each composition, if

a) $f(x) = 2x + 1,\ g(x) = x^2 - 1$

b) $f(x) = \frac{1}{x+1},\ g(x) = \frac{x}{x-1}$

c) $f(x) = \sqrt{x+5},\ g(x) = x^2 - 5$

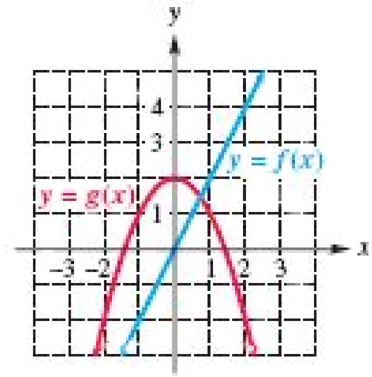
Observation: Generally, composition of functions is not commutative.

Example 4: Let $f(x) = x^2 - x$ and $(x) = |x - 1|$. Evaluate

a) $(g \circ f)(-2)$

b) $(f \circ g)(-2)$

Example 5: Use the graph to evaluate each expression.



a) $(g \circ f)(1)$

b) $(g \circ g)(-1)$

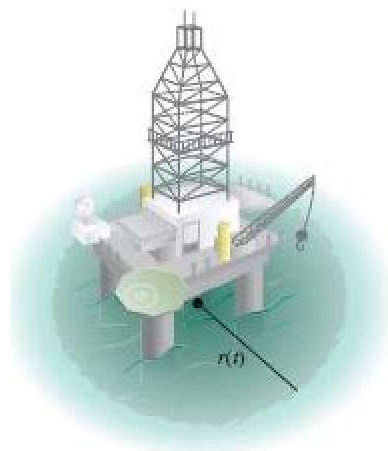
c) $(f + g)(2)$

d) $(fg)(0)$

e) $\left(\frac{g}{f}\right)(-1)$

f) $\frac{g(1)-g(-2)}{1-(-2)}$

Example 6: An oil well off the Gulf Coast is leaking, with the leak spreading oil over the water's surface. At any time ***t***, in minutes, after the beginning of the leak, the radius of the circular oil slick on the surface is $r(t) = 4t$ feet. Let $A(r) = \pi r^2$ represent the area of a circle of radius ***r***.



a) Find and interpret $(A \circ r)(t)$.

b) What is the area of the oil slick after 3 minutes?