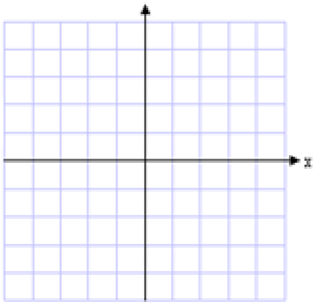
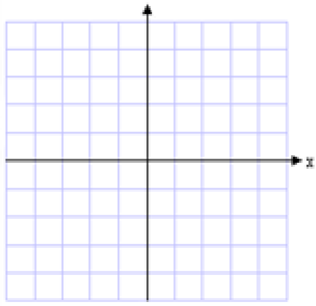
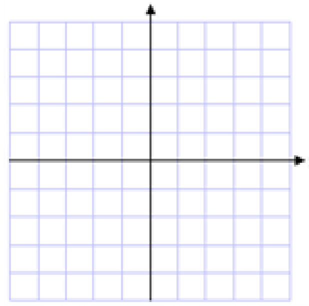
3.2 Analysis of Graphs of Polynomial Functions; Maximization Problems

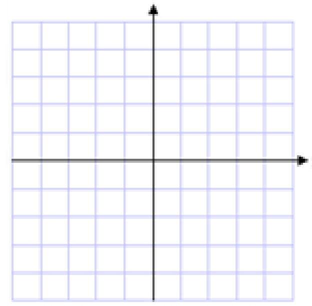
Graph the following functions and observe their behaviour at infinities.

$f(x) = x^4$ $f(x) = x^4 - 2x^2 + x - 3$ $f(x) = -x^4$ $f(x) = -x^4 + x^2 + 2$

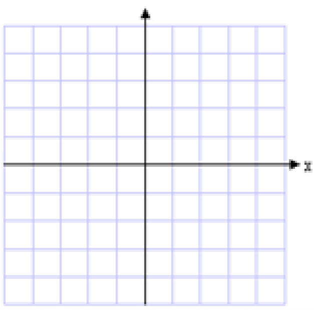


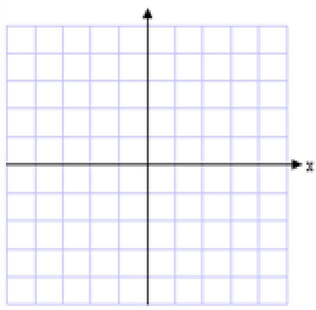


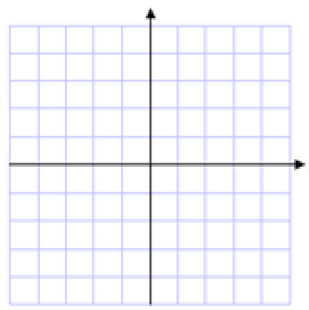


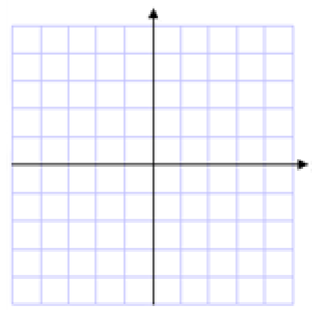


$g(x) = x^3$ $g(x) = x^3 - x$ $f(x) = -x^3$ $f(x) = -x^3 + x^2 + x$





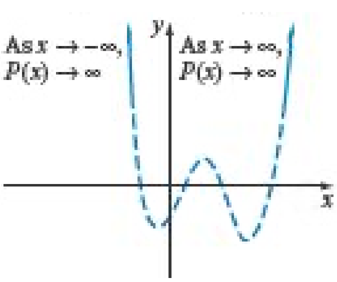




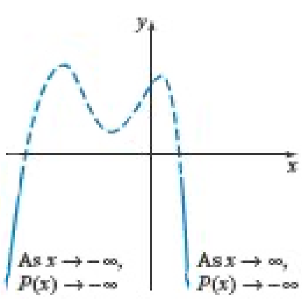
Conclusion:

The **end behaviour** of a polynomial function $\boldsymbol{P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}$ is determined by the **dominating term** (the leading term).

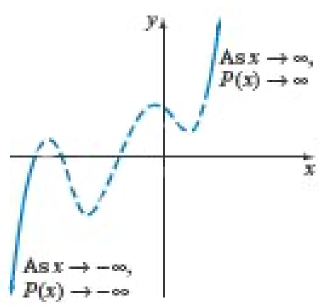
***n* even:** ***n* odd:**



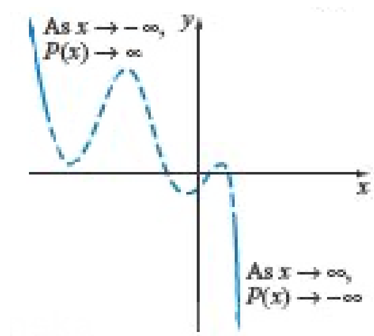
$a_n > 0$



$a_n < 0$



$a_n > 0$

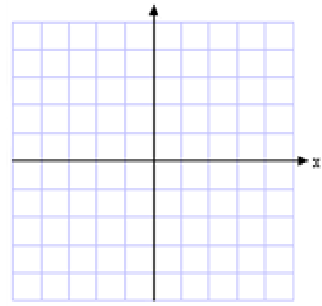


$a_n < 0$

To graph a polynomial function $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$

- find the **roots** (*x*-intercepts) and ***y*-intercepts**;
- use the **end behaviour** and test selected points if necessary
- check for **symmetry** (even or odd function)

Example 1: Graph $f(x) = x^4 - 4x^2$

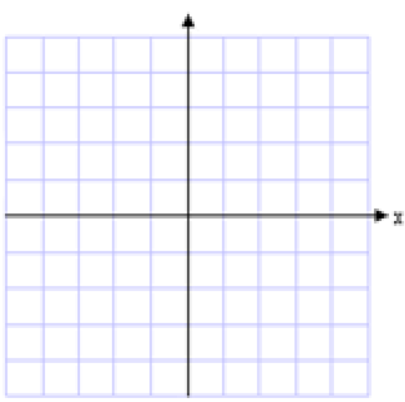


How many real roots a polynomial can have?

If r is a root of a polynomial $f(x)$, then it is a solution of $f(x) = 0$, and by the **Factor Theorem**, $f(x)$ is divisible by $(x - r)$.
Therefore, the number of real roots of $f(x)$ is the same as the number of different linear factors of $f(x)$.

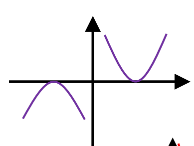
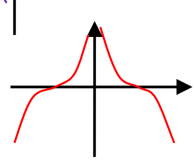
Conclusion: The maximum number of roots (zeros) of an *n*-th degree polynomial is ....

Example 2: Find the zeros of $f(x) = -\frac{1}{8}(x-1)^3(x-3)(x+2)^2$ and then graph $f(x)$.

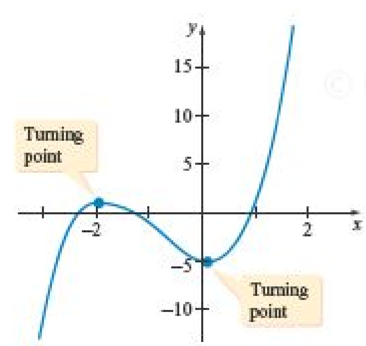


Notice:

- factor $(x + 2)$ appears twice; we say that -2 is the root of **multiplicity** 2
- factor $(x - 1)$ appears three times; we say that 1 is the root of **multiplicity** 3

- even multiplicity – the graph looks like 
- odd multiplicity – the graph looks like 

turning points – points on the graph when the function changes its direction
The turning points are the **maximum** or **minimum** points in the locality of the turning point.



Definition:

- A function *f* has a **local minimum** at a point c iff there is an open interval $I \subset D_f$ containing *c*, such that $\boldsymbol{f(c) \leq f(x)}$ for all $x \in I$. $\boldsymbol{f(c)}$ is the **local minimum value.**
- A function *f* has a **local maximum** at a point c iff there is an open interval $I \subset D_f$ containing *c*, such that $\boldsymbol{f(c) \geq f(x)}$ for all $x \in I$. $\boldsymbol{f(c)}$ is the **local maximum value**.
- If $I = D_f$, we talk about the absolute (global) maximum or minimum (extremum).

Example 3: Let $f(x) = x^3 - 2x^2 - x + 1$. Using a graphing calculator,

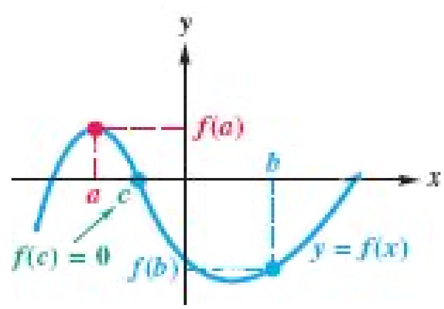
a) find all local extreme points of *f*,

b) find all zeros of *f*.

If we can't find the exact root by factoring, it is useful to know that a root exists in a specified interval.

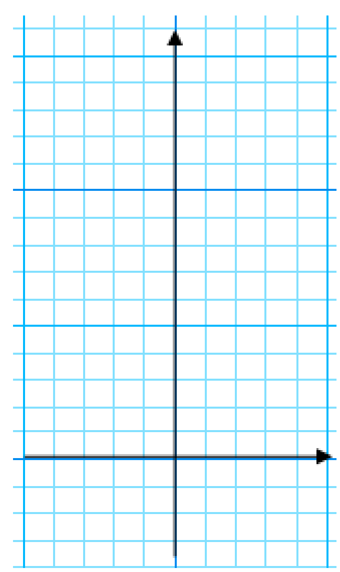
Intermediate Value Theorem:

Let $f(x)$ be a polynomial with real coefficients. If $a < b$ are two real numbers s.t. $f(a)$ and $f(b)$ are opposite in sign, then there exists at least one real zero in the interval (a, b).

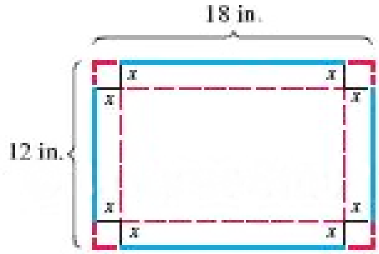


Example 4: Verify (using IVT) that the polynomial $f(x) = -x^4 + x^3 + 5x - 1$ has a zero between 0.1 and 0.2.

Example 5: A) Give the transformations needed to obtain a graph of $Q(x) = (x+2)^4 - 3$ from the graph of $P(x) = x^4$. Then, graph both functions on the same grid.



B) The point $(2,16)$ belongs to the graph of $P(x)$. What are the coordinates of the corresponding point on the graph of $Q(x)$?



Example 6: A rectangular piece of cardboard measuring 12 in. by 18 in. is to be made into a box with an open top by cutting equal-size squares from each corner and folding up the sides. Let *x* represent the length of a side of each such square, in inches.

a) Give the restrictions on *x*.

b) Determine a function *V* that gives the volume of the box as a function of *x*.

c) For what value of *x* will the volume be a maximum? What is this maximum volume?

d) For what values of *x* will the volume be greater than 80 in^3?

Example 7: Find the value of *x* in the diagram that will maximize the area of rectangle *ABCD*.