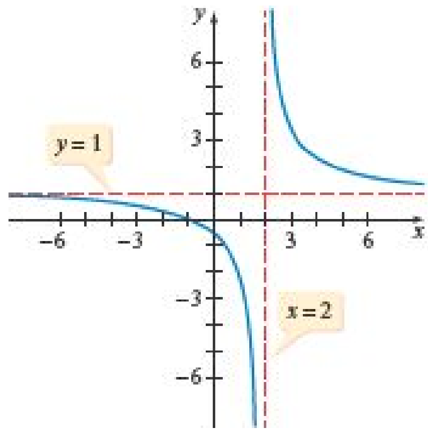
3.5 Graphs of Rational Functions and Their Applications

rational function – any function of the form $\boldsymbol{f(x) = \frac{P(x)}{Q(x)}}$, where $P(x)$ and $Q(x)$ are any polynomials, and $Q(x) \neq 0$;

The **domain** of a rational function consists of all real numbers except for those that will make the denominator $Q(x)$ equal to 0. So $\boldsymbol{D_f = \mathbb{R} \setminus \{x | Q(x) = 0\}}$



Example 1: Analyse the graph of the function $G(x) = \frac{x+1}{x-2}$

a) state the **domain** of *G*

b) find the ***x*- and *y*-intercepts**

c) observe the behaviour of *G* for *x*-values ***approaching 2 from the right***;

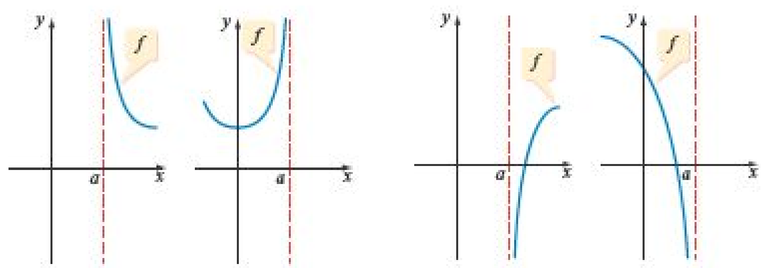
d) observe the behaviour of *G* for *x*-values ***approaching 2 from the left***;

e) observe the behaviour of *G* for *x*-values ***approaching infinity***;

f) observe the behaviour of *G* for *x*-values ***approaching negative infinity***;

Definition 1:

If $\boldsymbol{f(x)} \to \pm\infty$ (*read: f* approaches positive or negative infinity) as $\boldsymbol{x \to a^{\pm}}$ (*read*: *x* approaches ***a*** from the right or left), then the graph of *f* has the **vertical asymptote** given by the equation $\boldsymbol{x = a}$.



a. $f(x) \to \infty$ as $x \to a^+$ b. $f(x) \to \infty$ as $x \to a^-$ c. $f(x) \to -\infty$ as $x \to a^+$ d. $f(x) \to -\infty$ as $x \to a^-$

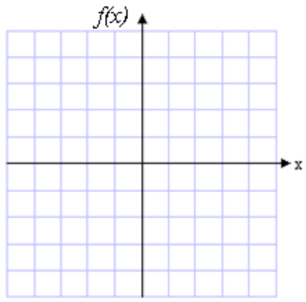
Theorem 1 ***(Vertical Asymptotes):***

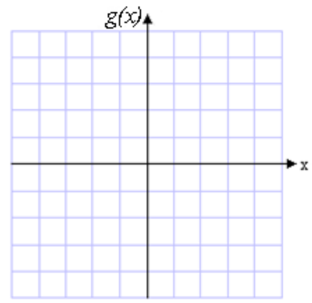
If $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ have no common factors, and $a \in \mathbb{R}$ is a **zero of the denominator** $Q(x)$, then *f* has the **vertical asymptote** $\boldsymbol{x = a}$.

Example 2: Find the equations of all vertical asymptotes of the given function. Then using a graphing calculator, show the approximate graph of the function.

a) $f(x) = \frac{x-3}{x^2-5x+6}$

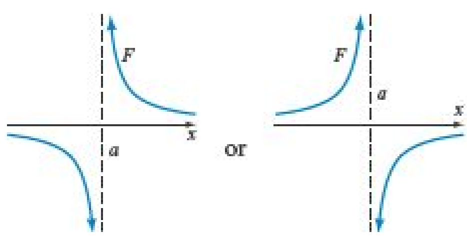
b) $g(x) = \frac{x^2+1}{x^3-4x^2+4x}$



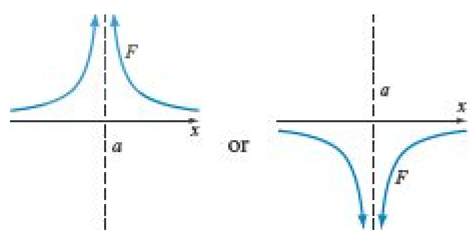


Observation: Let $\boldsymbol{a}$ be the zero of multiplicity $\boldsymbol{n}$ of the denominator $Q(x)$. Then the graph of a rational function $f(x) = \frac{P(x)}{Q(x)}$ around $\boldsymbol{a}$ looks as follows:

for **ODD** $\boldsymbol{n}$:

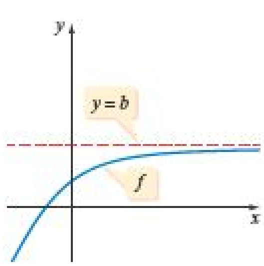


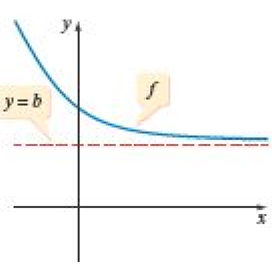
for **EVEN** $\boldsymbol{n}$:

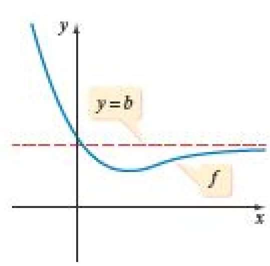


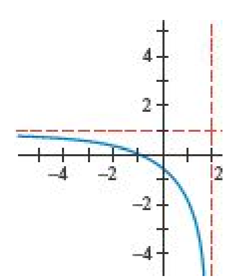
Definition 2:

If $\boldsymbol{f(x) \to b}$ (*read: f* approaches $\boldsymbol{b}$) as $\boldsymbol{x \to \pm\infty}$ (read: *x* approaches positive or negative infinity), then the graph of *f* has the **horizontal asymptote** given by the equation $\boldsymbol{y = b}$.



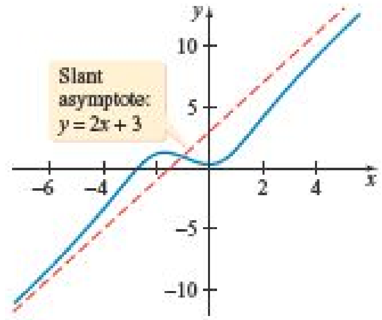






Definition 3:

If $\boldsymbol{f(x) \to mx + b}$ $(m \neq 0)$ as $\boldsymbol{x \to \pm\infty}$, then the graph of *f* has the **slant asymptote** given by the equation $\boldsymbol{y = mx + b}$.



$f(x) = \frac{2x^3 + 5x^2 + 1}{x^2 + x + 3}$

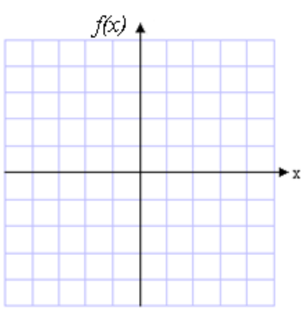
Theorem 1 ***(Horizontal or Slant(curved) Asymptotes):***

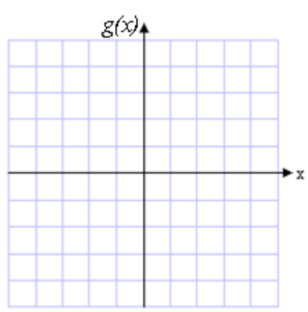
Let $f(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$, where $a_n, b_m \neq 0$. Then the graph of *f* has the following asymptotes:

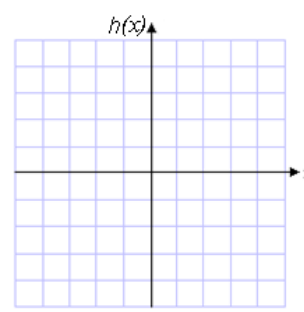
- horizontal asymptote: $y = 0$, if $m > n$;
- horizontal asymptote: $y = \frac{a_n}{b_m}$, if $m = n$;
- slant (or curved) asymptote: $y = quotient\ from\ division\ of\ P(x)\ by\ Q(x)$, if $m < n$

Example 3: Find the equations of a horizontal or slant asymptote of the given function. Then using a graphing calculator, show the approximate graph of the function.

a) $f(x) = \frac{2x^4}{x^4+1}$ b) $g(x) = \frac{x^2-9}{2x-4}$ c) $h(x) = \frac{x-1}{x^2-x-6}$





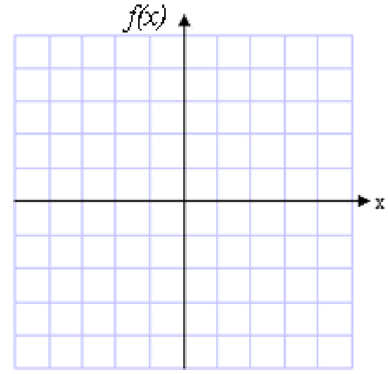


How to graph a rational function?

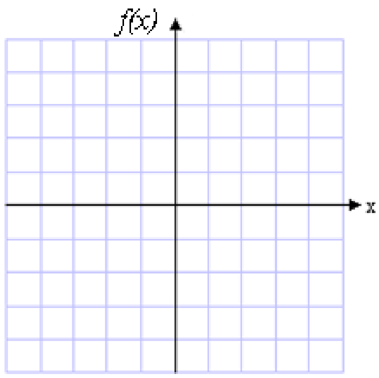
- factor the numerator and denominator
- find the **domain**
- cancel common factors – roots of common factors correspond to **holes** in the graph
- find all **asymptotes** and graph them as dashed lines
- find all **intercepts** – watch multiplicity of *x*-intercepts to know if the graph crosses *x*-axis or not

Example: $f(x) = \frac{x+2}{x^2-4}$

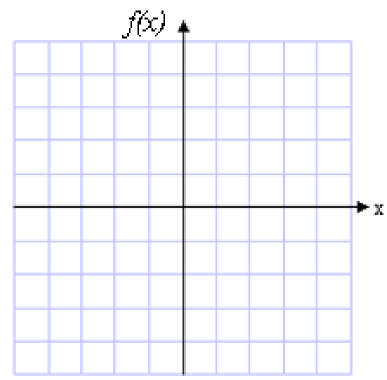
- check the **symmetry** (function even or odd)
- check the behaviour near asymptotes
- check some additional points if needed



Example 3: Graph $f(x) = \frac{x^2}{x^2-x-2}$.

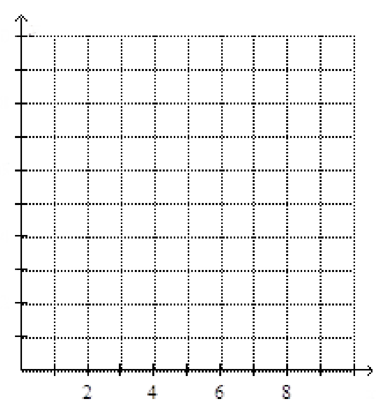


Example 4: Graph $f(x) = \frac{x^3}{x^2+2}$.



Example 5: A cylindrical soft drink can is to be made so that it will have a volume of 354 millilitres. Let ***r*** be the radius of the can in centimetres. Then, the total surface area of the can ***A***, in square centimetres, is given by the rational function $A(r) = \frac{2\pi r^3+708}{r}$.

a) Using a graphing calculator, graph ***A*** and find the ***r***-value that produces the minimum surface area of the can.



b) What is the minimum surface area?

c) Explain the meaning of: if $r \to \infty$, then $A \to 2\pi r^2$