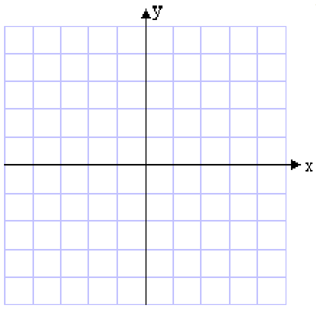
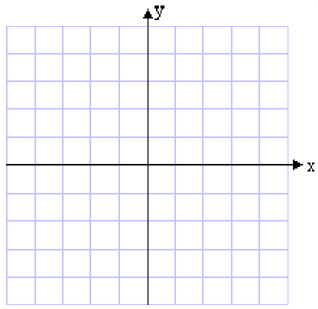
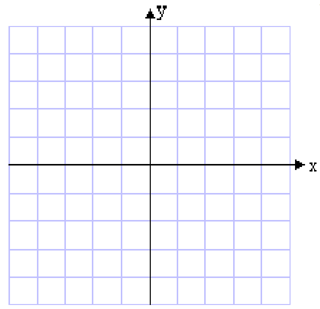
4.1 Inverse Functions

How can we reverse the correspondence given by a function $\boldsymbol{f}$?
For example:
$g(x) = \frac{1}{2}x$ reverses the action of $f(x) = 2x$, as $g\big(f(x)\big) = \frac{1}{2}(2x) = x$;

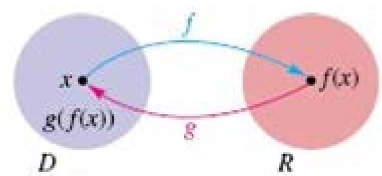
$g(x) = \sqrt[3]{x}$ reverses the action of $f(x) = x^3$, as $g\big(f(x)\big) = \sqrt[3]{x^3} = x$; however,
$g(x) = \sqrt{x}$ reverses the action of $f(x) = x^2$ only for nonnegative *x*-values, as
$g\big(f(x)\big) = \sqrt{x^2} = |x| = x$ for $x \geq 0$.
Observe properties of graphs of the above functions:







Definition: A function $\boldsymbol{g}$ that reverses the correspondence given by a function $\boldsymbol{f}$ is called the **inverse function** and is denoted by f^{-1}.



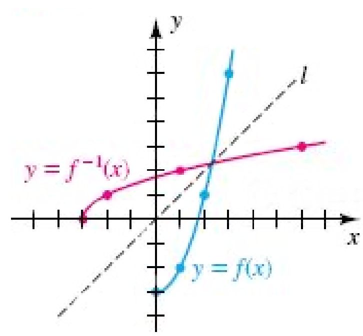
Can every function be reversed?

Which functions can be reversed?
Observe that for a function $y = f(x)$ to have an inverse function f^{-1}, for every *y*-value we should be able to determine a unique *x*-value, such that $f^{-1}(y) = x$.

This means that a function $\boldsymbol{f}$ is **invertible** only if one of the following equivalent conditions apply:

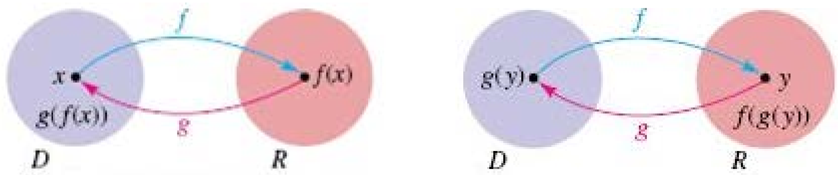
- the function $\boldsymbol{f}$ satisfies the **horizontal line test**:
  Any horizontal line intersects the graph of $\boldsymbol{f}$ *in at most one point.*
- the function $\boldsymbol{f}$ is a **one-to one** function:
  - ➢ if $a \neq b$, then $f(a) \neq f(b)$ for any $a, b \in D_f$; or equivalently
  - ➢ if $f(a) = f(b)$, then $a = b$.

Properties of inverse functions:



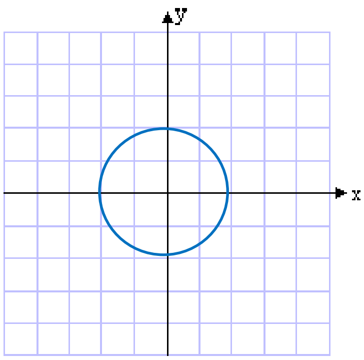
- The graph of the inverse function f^{-1} is symmetric to the graph of f in respect to the diagonal $y = x$.
- If (a, b) belongs to the graph of $\boldsymbol{f}$, then (b, a) belongs to the graph of f^{-1}.

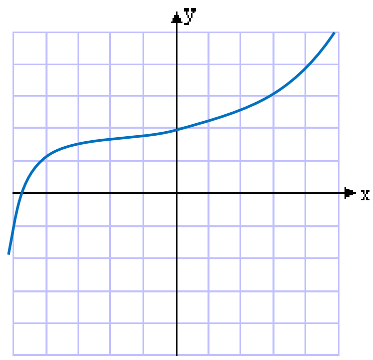
- Any **monotonic** (always increasing or always decreasing) function is **one-to-one**, so it has an **inverse function**
- Let $f: D \to R$. A function $g: R \to D$ is the **inverse function** of $\boldsymbol{f}$ iff
  $\boldsymbol{g(f(x)) = x}$ for every $x \in D$, and $\boldsymbol{f(g(y)) = y}$ for every $y \in R$.
  The inverse function is unique.

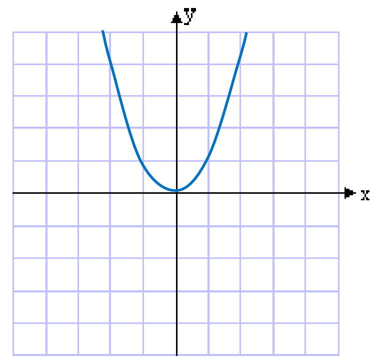


- $D_{f^{-1}} = R_f$ and $R_{f^{-1}} = D_f$

Example 1: Graph the inverse relation for the given graph. Is the inverse a function?







Example 2: Find $f^{-1}(2)$ knowing that $f(5) = 2$.

Example 3: Determine whether the given functions are inverses to each other.

a) $f = \{(-3,6), (2,1), (5,7)\}$, $g = \{(1,2), (6,-3)\}$

b) $f(x) = \frac{1}{x+1}$, $g(x) = \frac{1-x}{x}$

Example 4: Determine the domain and range of the two functions in *Example 3b*. Then, give equations of horizontal and vertical asymptotes for each of the above functions.

How can we find a formula for an inverse function?

Example 5: Find $f^{-1}(x)$ for the given function f.

a) $f(x) = 2x - 3$

use y instead of $f(x)$ and **solve** the equation **for x**

$y = 2x - 3$

$y + 3 = 2x$

$x = \frac{y+3}{2}$

$f^{-1}(y) = \frac{y+3}{2}$

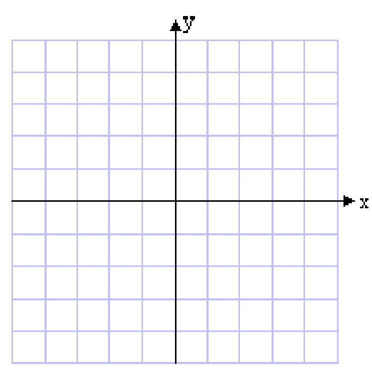
write the formula for the inverse function by using $f^{-1}(y)$ instead of x

$f^{-1}(x) = \frac{x+3}{2}$

rewrite the formula for the inverse in terms of x (simply **replace the variable y by x**)

b) $f(x) = \frac{3x+1}{x-2}$

c) $f(x) = x^2 - 2x$ for $x \leq -1$



Example 6: Suppose $f(x)$ is the number of cars that can be built for *x* dollars. What does $f^{-1}(1000)$ represent?

Example 7: A landscaping company uses the function $c(x) = \frac{600+140x}{x}$ to determine the amount, in dollars, it charges per tree to deliver and plant *x* palm trees.

a) Find $c(5)$ and interpret the meaning of this value.

b) Find $c^{-1}(160)$ and interpret the meaning of this value.