4.3 Logarithmic Functions and Their Applications

Recall: Any exponential function $y = b^x$ is one-to-one, so it should have an inverse.

Definition: The inverse of the exponential function $g(x) = \boldsymbol{b^x}$ is the **logarithmic function** $g^{-1}(x) = \boldsymbol{\log_b x}$, defined as follows:
For $b > 0$, $b \neq 1$, and $x > 0$, we have

$$\log_b x = y \quad \text{iff} \quad b^y = x.$$

So **log** base ***b*** of ***x*** is such an exponent ***y*** that ***b*** raised to the ***y*** gives the original ***x***.

Notice: If $g(x) = b^x$ and $f(x) = \log_b x$ then
$f(g(x)) = \boxed{\log_b b^x = x}$ and $g(f(x)) = \boxed{b^{\log_b x} = x}$

Also notice that $\boxed{\log_b 1 = \quad}$ and $\boxed{\log_b b = \quad}$

The most commonly used bases are:
$b = \mathbf{10}$, then we write just $\boldsymbol{\log x}$ instead of $\log_{10} x$ (**Common Logarithm**), or
$b = \boldsymbol{e}$, then we write $\boldsymbol{\ln x}$ instead of $\log_e x$ (**Natural Logarithm**)

Example 1: Change the form of the given equation from **logarithmic** to **exponential** or otherwise.

a) $\log_2 8 = x$ b) $\log_5 x = 3$

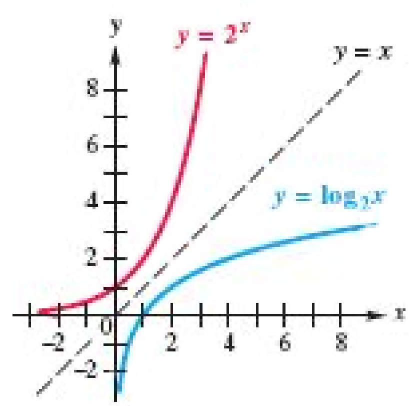
c) $x^5 = 32$ d) $2^x = 16$

e) $\ln x = -1$ f) $10^{-2} = x$

Example 2: Evaluate each expression (without calculator).

a) $\log_7 1$ b) $\log_x x^2$ c) $2^{\log_2 17}$

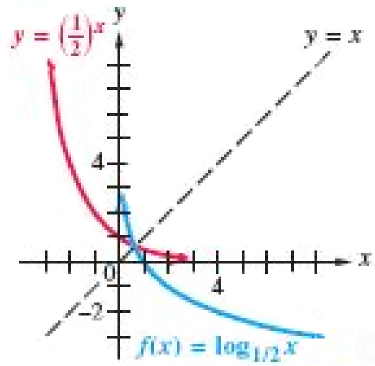
d) $\log_3 \sqrt[2]{3}$ e) $3\ln e^{-10}$



Properties of logarithmic function $\boldsymbol{f(x) = \log_b x}$:

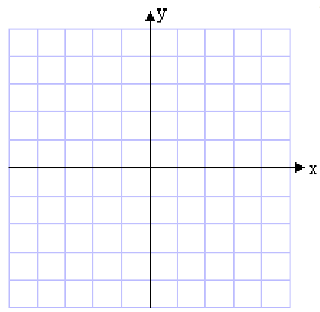
- Domain = Range =
- ***x*-intercept** at $(\mathbf{1}, \mathbf{0})$, no *y*-intercept, and the graph passes through $(\boldsymbol{b}, \mathbf{1})$
- It is **one-to-one**, so $\log_b n = \log_b m$ iff $n = m$
- **vertical asymptote:** $\boldsymbol{x} = 0$
- If $\boldsymbol{b} > 1$, then it is **increasing**; so if $n < m$ then $\log_b n < \log_b m$

- If $\boldsymbol{b} > 1$, we have: as $x \to 0$ from the right, $f(x) \to -\infty$; and as $x \to \infty$, $f(x) \to \infty$
- If $0 < \boldsymbol{b} < 1$, then the function is **decreasing**; so if $n < m$ then $\log_b n > \log_b m$
- If $0 < \boldsymbol{b} < 1$, we have: as $x \to 0$ from the right, $f(x) \to \infty$; and as $x \to \infty$, $f(x) \to -\infty$

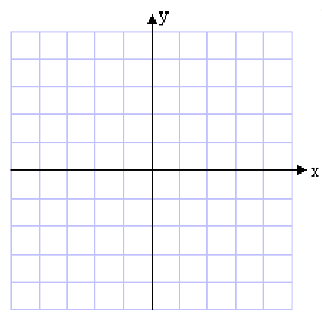


Example 3: Use transformation of the corresponding basic graph of $f(x) = \log_b x$ to produce a graph of

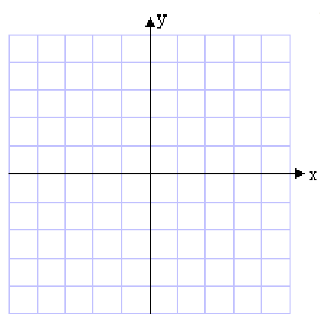
a) $g(x) = \log_2(x-3) + 1$



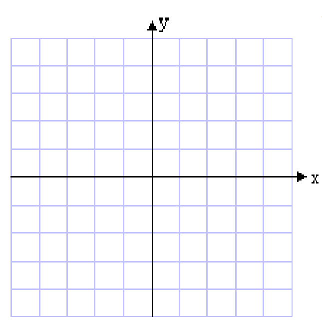
b) $h(x) = -\ln(x) + 3$



c) $p(x) = -\log_2(-x)$



d) $s(x) = \log_{\frac{1}{2}}(x+4)$



Example 4: Find the domain of the following functions:

a) $f(x) = \ln(x^2 - 9)$

b) $f(x) = \ln\sqrt{2x - 5}$

c) $f(x) = \ln\left(\frac{x+5}{x}\right)$

d) $f(x) = 2\log_5(x-1)^2$

Example 5:

The number of digits N in the expansion of b^x, where $b, x \in \mathbb{Z}_+$, is $N = [\![x \cdot \log b]\!] + 1$.

a) We know that $2^{10} = 1024$, so 2^{10} has 4 digits.
Use the equation $N = [\![x \cdot \log b]\!] + 1$ to verify this result.

b) Find the number of digits in 3^{200}.

c) Find the number of digits in 2012^{2012}.

d) Find the number of digits in $9^{(9^9)}$.