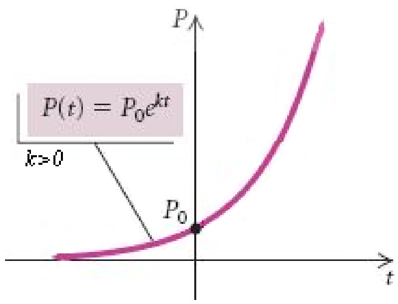
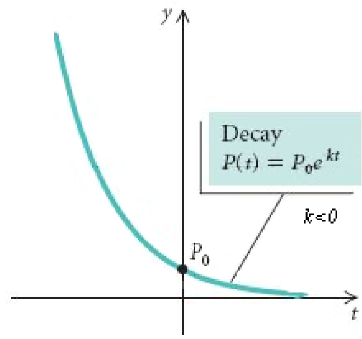
4.6 Exponential Growth and Decay

If a quantity $P(t)$ changes at a **rate proportional to the quantity** itself, then such quantity can be modelled by a function

$$\boldsymbol{P(t) = P_0 e^{kt}}$$

where $P_0 = P(0)$ is the initial quantity and $|\boldsymbol{k}|$ is a **proportionality constant**, called **exponential growth** (if $\boldsymbol{k} > 0$) or **decay** (if $\boldsymbol{k} < 0$) **rate**.





Example 1:
In 2011, the population of Mexico was about 113.7 million, and the exponential growth rate was 1.1% per year.

a) Find the exponential growth function.

b) Estimate the population in 2015.

c) When would we expect the population to double the one from 2011?

Example 2: The number of cable TV networks has grown exponentially from 28 in 1980 to 565 in 2010.

a) Find an exponential growth function $N(t) = N_0 e^{kt}$ that describes the number of cable TV networks after ***t*** years, where ***t*** in the number of years since 1980.

b) According to this model, find the predicted number of cable TV networks in 2020.

c) At this growth rate, in which year will the number of cable TV networks be 1000?

Example 3: Verify that the **growth rate** $\boldsymbol{k}$ and the **doubling** time $\boldsymbol{T}$ are in the relation:

$$\boxed{\boldsymbol{kT = \ln 2}}$$

Example 4: The population of Kenya is doubling every 28.2 years. What is the exponential growth rate?

Example 5: Verify that the **decay rate** $\boldsymbol{k}$ and the **half-life** time $\boldsymbol{T}$ are in the relation:

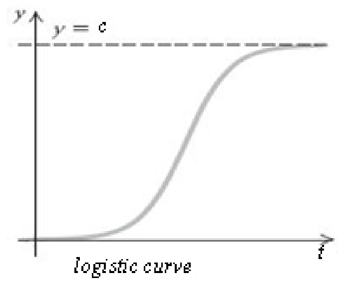
$$\boxed{\boldsymbol{kT = \ln 2}}$$

Example 6: In 1970, scientists discovered teeth and jawbones that had lost 77.2% of their carbon-14. If half-life of carbon-14 is 5730 years, how old were the bones at the time they were discovered.

Example 7: Isotope Strontium has a half-life of 28.9 years. If after Chernobyl explosion in 1986, there was about 7000 units of this substance per kilogram of fish,

a) find a formula for the number of units $N(t)$ of strontium per kilogram of fish after $\boldsymbol{t}$ years.

b) When this number of units will decay below 1000 units per kilogram that is safe for consumption?



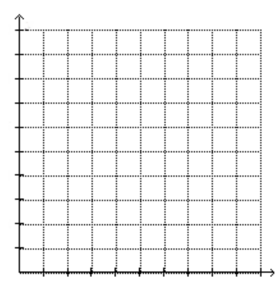
If there are some limitations for the growth of a population, for example limitations on food, living space, or other natural resources, a logistic model of growth can be used.

Logistic model: $\boldsymbol{P(t) = \frac{c}{1+ae^{-kt}}}$

where $\boldsymbol{c}$ is the **carrying capacity**, $\boldsymbol{k} > 0$ is the **growth rate**, and $a = \frac{c-P_0}{P_0}$.

Example 7: A lake is stocked with fish of a new variety. The size of the lake, the availability of food, and the number of other fish restrict the growth of that type of fish in the lake to a limiting value of 2500. The population of fish in the lake after time t, in months, is given by the logistic function $P(t) = \frac{2500}{1+5.25e^{-0.32t}}$.

a) What is the initial number of fish P_0?



b) Graph the function.

c) Find the population after 2, 6, 12, and 20 months.

d) When the population will reach 2000 fish?

Compound Interest Formulas:

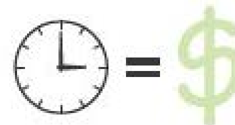
If the initial amount of money $\boldsymbol{P}$ is invested at an annual interest rate $\boldsymbol{r}$ (as a decimal) and compounded $\boldsymbol{n}$ times per year, then the amount of money $\boldsymbol{A(t)}$ after $\boldsymbol{t}$ years equals

a) $A(t) = P(1+r)^t$, if $n = 1$ (interest compounded **annually**)

b) $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$, if $n > 1$ (interest compounded ***n*** times per year)

c) $A(t) = Pe^{rt}$, if $n \to \infty$ (interest compounded **continuously**)

Example 8: If a \$1000 is invested at 5% interest, find the value of the investment after 3 years, if the interest is compounded



a) yearly,

b) monthly,

c) continuously.

d) How long will it take the investment to double if the interest is compounded semiannually.