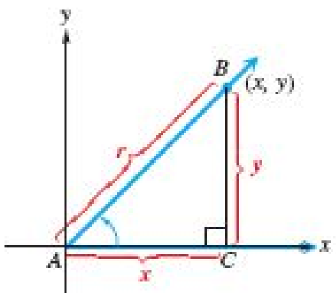
5.3 Trigonometric Functions of Any Angle



If we place the right angle triangle in a system of coordinates in such a way that the acute angle θ will be in standard position, the definitions of the six trigonometric ratios (in terms of *x*, *y*, and *r*) could be stated as follows:

$\sin\theta = \frac{y}{r}$	$\cos\theta = \frac{x}{r}$	$\tan\theta = \frac{y}{x}$
$\csc\theta = \frac{r}{y}$	$\sec\theta = \frac{r}{x}$	$\cot\theta = \frac{x}{y}$

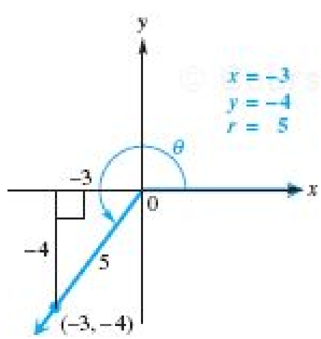
Notice that the above definitions work even if the angle is not acute.
Also, we have the following identities:

- **reciprocal identities**: $\csc\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, $\cot\theta = \frac{1}{\tan\theta}$
- **ratio identities**: $\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$

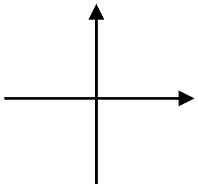
Exercise 1: Prove ratio identities.

To find values of the six trigonometric functions of any angle θ in standard position, it is enough to know a point (x, y) on the terminal ray of this angle and the Pythagorean relation $r^2 = x^2 + y^2$.

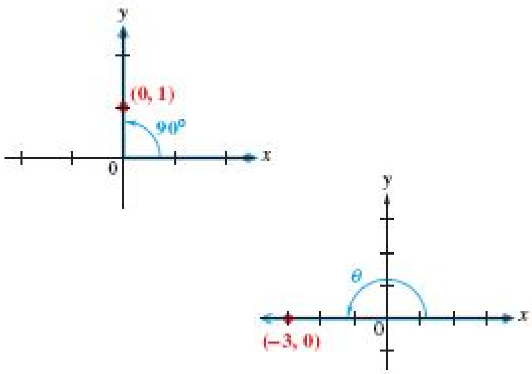
Example 2: The terminal side of an angle θ in standard position passes through the point $(-3, -4)$. Find values of the six trigonometric functions of θ.



Example 3: Sketch an angle θ in standard position having point $\left(-2\sqrt{3}, 2\right)$ on its terminal side. Then, find values of the first three trigonometric functions of the angle θ.

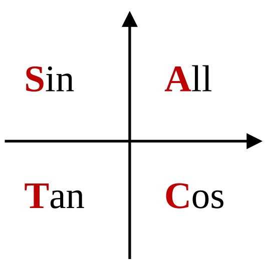


Exercise 4: Find values of the three trigonometric functions for the quadrantal angles.



$\boldsymbol{\theta}$	$\mathbf{sin}\,\boldsymbol{\theta}$	$\mathbf{cos}\,\boldsymbol{\theta}$	$\mathbf{tan}\,\boldsymbol{\theta}$	$\mathbf{csc}\,\boldsymbol{\theta}$	$\mathbf{sec}\,\boldsymbol{\theta}$	$\mathbf{cot}\,\boldsymbol{\theta}$
$0°, 360°$						
$90°$						
$180°$						
$270°$						

Exercise 5:
Observing signs of $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{r}$ in different quadrants, find signs of trigonometric functions in each quadrant to complete the table.



$\boldsymbol{\theta}$ **in quadrant**	$\mathbf{sin}\,\boldsymbol{\theta}$ & $\mathbf{csc}\,\boldsymbol{\theta}$	$\mathbf{cos}\,\boldsymbol{\theta}$ & $\mathbf{sec}\,\boldsymbol{\theta}$	$\mathbf{tan}\,\boldsymbol{\theta}$ & $\mathbf{cot}\,\boldsymbol{\theta}$
I			
II			
III			
IV			

Example 6: Let $\boldsymbol{\theta}$ be an angle in standard position. Which quadrant $\boldsymbol{\theta}$ could be in if

a) $\sin\theta > 0$ and $\tan\theta < 0$

b) $\sin\theta \cdot \cos\theta > 0$

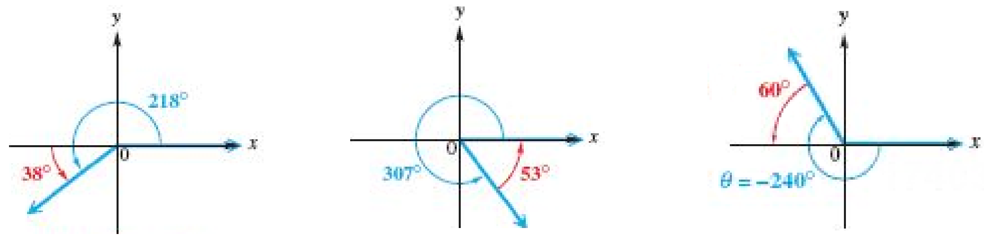
Example 7:

a) Knowing that $\tan\theta = \frac{\sqrt{3}}{3}$ and $\sin\theta < 0$, find the exact value of $\sec\theta$.

b) Knowing that $\csc\theta = -2$ and $\tan\theta < 0$, find the exact value of $\cos\theta$.

reference angle $\boldsymbol{\theta_{ref}}$ – the **positive acute angle** formed by the **terminal** side and ***x*-axis**

examples:



Example 8: Find the reference angle for each given angle.

a) $315°$ b) $\frac{7\pi}{6}$ c) $\frac{2\pi}{3}$ d) $-15°$

$$\textbf{trig function}(\textit{angle}) = \pm\ \textbf{trig function}(\textit{reference angle})$$

The sign is determined according to the sign rule: **CAST**

Example 9: Use reference angles to find exact values of the indicated trig functions.

a) $\tan 225°$

b) $\cos\frac{5\pi}{6}$

c) $\csc 300°$

Example 10: Find the exact value of each expression.

a) $\sin 210° + \cos 135° \cdot \tan 315°$

b) $\sin\frac{3\pi}{2} \cdot \tan\frac{\pi}{4} - \cos\frac{\pi}{3}$

Example 11: Find all values of $\theta \in [0°, 360°)$ satisfying the equation

a) $\cot\theta = 1$

b) $\sin\theta = -\frac{\sqrt{3}}{2}$

Example 12: Find all values of $\theta \in [0, 2\pi)$ satisfying the equation

a) $\cos\theta = \frac{1}{2}$

b) $\csc\theta = -\sqrt{2}$

Example 13: Use definitions of the six trigonometric functions and the Pythagorean equation $x^2 + y^2 = r^2$ to prove the following **Pythagorean identities**:

b) $\boldsymbol{\sin^2\theta + \cos^2\theta = 1}$

c) $\boldsymbol{1 + \tan^2\theta = \sec^2\theta}$

d) $\boldsymbol{\cot^2\theta + 1 = \csc^2\theta}$

Exercise 14: Justify geometrically that **cofunctions** of complementary angles are equal:

$\boldsymbol{\sin\theta = \cos(90° - \theta)}$ and $\boldsymbol{\cos\theta = \sin(90° - \theta)}$

$\boldsymbol{\csc\theta = \sec(90° - \theta)}$ and $\boldsymbol{\sec\theta = \csc(90° - \theta)}$

$\boldsymbol{\tan\theta = \cot(90° - \theta)}$ and $\boldsymbol{\cot\theta = \tan(90° - \theta)}$

For example: $\sin 18° = \cos 72°$; $\tan 80° = \cot 10°$; $\sec\frac{\pi}{3} = \csc\frac{\pi}{6}$