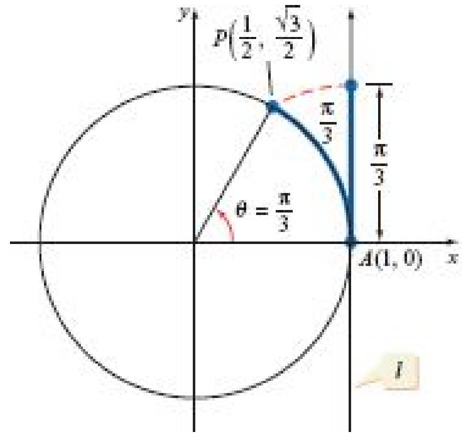
5.4 Properties of Trigonometric Functions of Real Numbers and Trigonometric Identities

unit circle – a circle given by the equation $x^2 + y^2 = 1$

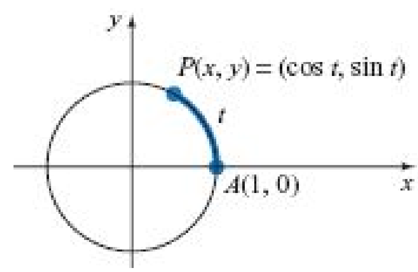


circular functions – trigonometric functions defined by coordinates of a point on a **unit circle** with domains of real numbers corresponding to **radian measurements**.

$\mathbf{\sin t = y}$ $\quad\mathbf{\cos t = x}$ $\quad\mathbf{\tan t = \frac{y}{x}}$ $(x \neq 0)$

$\mathbf{\csc t = \frac{1}{y}}$ $(y \neq 0)$ $\quad\mathbf{\sec t = \frac{1}{x}}$ $(x \neq 0)$ $\quad\mathbf{\cot t = \frac{x}{y}}$ $(y \neq 0)$

Notice: The coordinates of a point $\boldsymbol{P(x,y)}$ on a unit circle can be seen as $(\mathbf{\cos t, \sin t})$, where $\boldsymbol{t}$ is the arc length from $A(1,0)$ to $P(x,y)$ in counterclockwise direction.



Exercise 1: Using properties of a unit circle, find domains and ranges of circular functions.

Function	***Domain***	***Range***
$f(t) = \sin t$	$\mathbb{R}$	$[-1,1]$
$f(t) = \cos t$		
$f(t) = \tan t$	$\left\{t \in \mathbb{R} \middle\vert\, \boldsymbol{t \neq \frac{\pi}{2} + n\pi},\ n \in \mathbb{Z}\right\}$	$\mathbb{R}$
$f(t) = \csc t$		
$f(t) = \sec t$		
$f(t) = \cot t$		

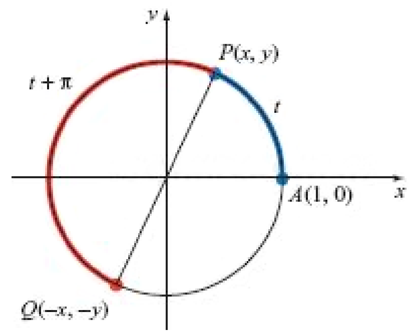
Definition:

A function $\boldsymbol{f}$ is **periodic** iff there exists a positive number $\boldsymbol{p}$ such that

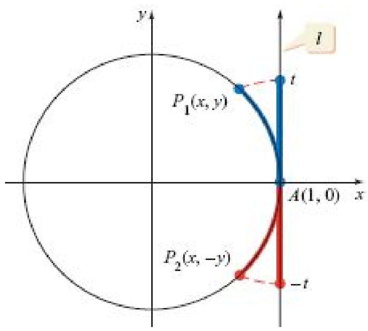
$$\boxed{\boldsymbol{f(t+p) = f(t)}} \quad \text{for all } \boldsymbol{t \in D_f}.$$

The smallest such number $\boldsymbol{p}$ is called the **period** of the function $\boldsymbol{f}$.

Are the circular functions periodic?
What are their periods?



function	***period***	***equation*** $(k \in \mathbb{Z})$
$\sin t$		$\sin(t + 2k\pi) = \sin t$
$\cos t$		
$\sec t$		
$\csc t$		
$\tan t$		
$\cot t$		



What kind of symmetries can we observe?

relation	***symmetry***	***even or odd?***
$\sin(-t) = -\sin t$	S_O	
$\csc(-t) =$		
$\cos(-t) =$		
$\sec(-t) =$		
$\tan(-t) =$		
$\cot(-t) =$		

Example 2: Determine whether the function is even, odd, or neither.

a) $f(x) = x^2 - \cos x$

b) $g(x) = \frac{x}{\sec x} - \tan x$

c) $h(x) = 1 + \cot x$

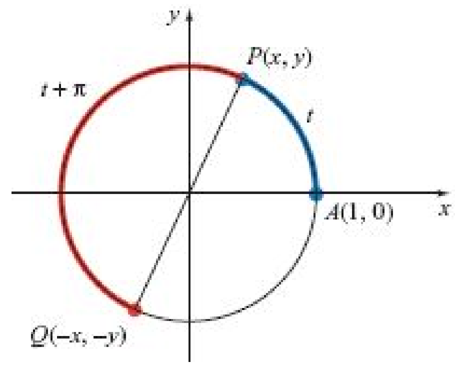
Example 3: Find the exact value of each expression.

a) $\sec\left(-\frac{7\pi}{6}\right)$ b) $\cot\left(\frac{13\pi}{3}\right)$

Example 4: Use a calculator to find an approximate value of each expression. Round your answer to 4 decimal places.

a) $\cot\left(-\frac{\pi}{5}\right)$ b) $\csc 2.51$

Example 5:
Use the unit circle and the definitions of the trigonometric functions to verify the identity $\sin(t+\pi) = -\sin t$.



Example 6: Rewrite the expression in terms of the given function.

a) $\frac{\csc t}{\cot t}$ in terms of $\sec t$

b) $\cot t$ in terms of $\cos t$ for $\frac{3\pi}{2} < t < 2\pi$

c) $\frac{1}{1-\cos t} + \frac{1}{1+\cos t}$ in terms of $\csc t$

Example 7: Perform the operations and simplify.

a) $\cos t - \frac{1}{\cos t}$

b) $(\sec t - 1)(\sec t + 1)$

c) $\frac{1 - \sin t}{\cos t} - \frac{1}{\tan t + \sec t}$

Example 8: Factor the expression.

a) $\tan^2 t + 4\tan t + 4$

b) $2\sin^2 t + 9\sin t - 5$

Example 9: Given that $\sin t = \frac{1}{2}$ and $\frac{\pi}{2} < t < \pi$, find $\cot t$.