6.3 Double-Angle, Half-Angle and Power-Reducing Identities

Recall: $\cos 2\theta = \cos^2\theta - \sin^2\theta$

Exercise 1: Show that $\cos 2\theta = 2\cos^2\theta - 1$ and also $\cos 2\theta = 1 - 2\sin^2\theta$.

Double-Angle Identities:

$$\boldsymbol{\sin 2\theta = 2\sin\theta\cos\theta}$$
$$\boldsymbol{\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta}$$
$$\boldsymbol{\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}}$$

From the above identities we can derive the following

Power-Reducing Identities:

$$\boldsymbol{\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}}$$

Now, if we replacing θ by $\frac{\theta}{2}$ and take square root of both sides in the power-reducing identities, then we have

Half-Angle Identities:

$$\boldsymbol{\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \qquad \cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}}$$

$$\boldsymbol{\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}}$$, where " $\pm$ " depends on the quadrant for $\frac{\theta}{2}$.

Exercise 2: Show that $\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta}$ and also $\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta}$.

Example 3: Rewrite $\cos^2 5\theta - \sin^2 5\theta$ in terms of a single trig function.

Example 4: Knowing that $\cos\alpha = \frac{5}{6}$ and $270° < \alpha < 360°$, find the exact value of

a) $\sin 2\alpha$

b) $\cos 2\alpha$

c) $\tan 2\alpha$

d) $\sin\frac{\alpha}{2}$

e) $\cos\frac{\alpha}{2}$

f) $\tan\frac{\alpha}{2}$

Example 5: Using power-reducing identities, rewrite $\sin^4 x \cos^2 x$ in terms of first power of cosine function.

Example 6: Using half-angle identities, find the exact values of

a) $\cos 105°$

b) $\tan 67.5°$

c) $\sin\frac{9\pi}{8}$

d) $\cos\frac{5\pi}{12}$

Example 7: Verify the identity.

a) $\frac{1}{1-\cos 2x} = \frac{1}{2}\csc^2 x$

b) $\tan\frac{x}{2} = \csc x - \cot x$

c) $\cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1}{2}\csc x \sin 2x$