

6.3 Double-Angle, Half-Angle and Power-Reducing Identities

Recall: $\cos 2\theta = \cos^2\theta - \sin^2\theta$

Exercise 1: Show that $\cos 2\theta = 2\cos^2\theta - 1$ and also $\cos 2\theta = 1 - 2\sin^2\theta$.

Double-Angle Identities:

$$\begin{aligned}\sin 2\theta &= 2 \sin \theta \cos \theta \\ \cos 2\theta &= \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta \\ \tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2\theta}\end{aligned}$$

From the above identities we can derive the following

Power-Reducing Identities:

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2} \quad \cos^2 \theta = \frac{1 + \cos 2\theta}{2} \quad \tan^2 \theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$$

Now, if we replacing θ by $\frac{\theta}{2}$ and take square root of both sides in the power-reducing identities, then we have

Half-Angle Identities:

$$\begin{aligned}\sin \frac{\theta}{2} &= \pm \sqrt{\frac{1 - \cos \theta}{2}} & \cos \frac{\theta}{2} &= \pm \sqrt{\frac{1 + \cos \theta}{2}} \\ \tan \frac{\theta}{2} &= \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} = \frac{\sin \theta}{1 + \cos \theta} = \frac{1 - \cos \theta}{\sin \theta}, \text{ where " } \pm \text{ " depends on the quadrant for } \frac{\theta}{2}.\end{aligned}$$

Exercise 2: Show that $\pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} = \frac{\sin \theta}{1 + \cos \theta}$ and also $\pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} = \frac{1 - \cos \theta}{\sin \theta}$.

Example 3: Rewrite $\cos^2 5\theta - \sin^2 5\theta$ in terms of a single trig function.

Example 4: Knowing that $\cos \alpha = \frac{5}{6}$ and $270^\circ < \alpha < 360^\circ$, find the exact value of

a) $\sin 2\alpha$

b) $\cos 2\alpha$

c) $\tan 2\alpha$

d) $\sin \frac{\alpha}{2}$

e) $\cos \frac{\alpha}{2}$

f) $\tan \frac{\alpha}{2}$

Example 5: Using power-reducing identities, rewrite $\cos^4 x$ in terms of first power of cosine function.

Example 6: Using half-angle identities, find the exact values of

a) $\cos 105^\circ$

b) $\tan 67.5^\circ$

c) $\sin \frac{9\pi}{8}$

d) $\cos \frac{5\pi}{12}$

Example 7: Verify the identity.

a) $\frac{1}{1-\cos 2x} = \frac{1}{2} \csc^2 x$

b) $\tan \frac{x}{2} = \csc x - \cot x$

c) $\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \frac{1}{2} \csc x \sin 2x$