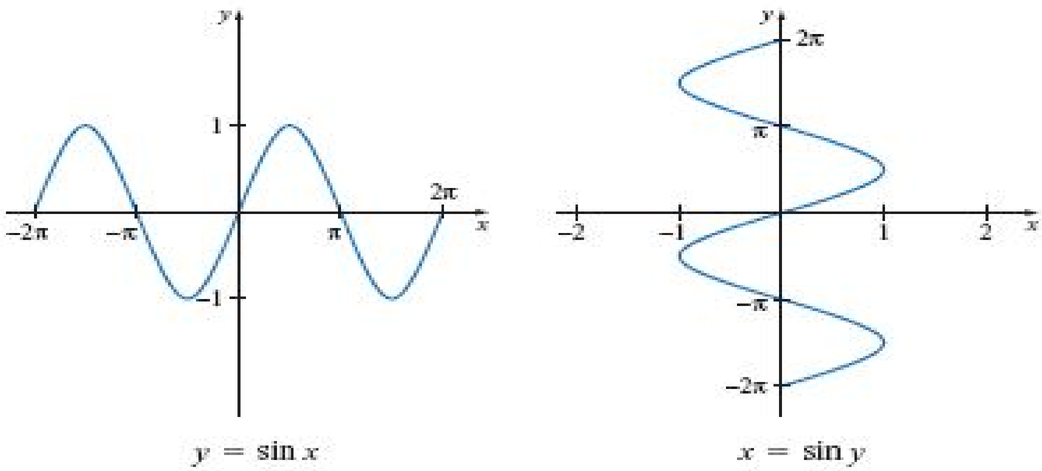
6.5 Inverse Trigonometric Functions

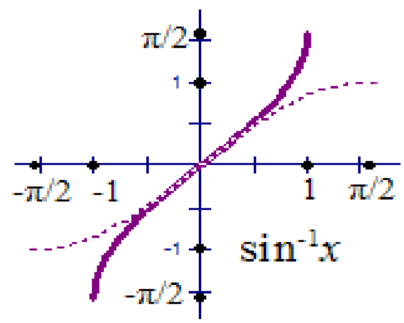
Inverse relation for sine:



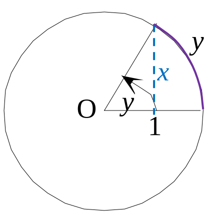
Recall that a function f has an inverse f^{-1} iff f is a 1-1 function.

To find the **inverse** of $f(x) = \sin x$, we restrict the domain of sine to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Such inverse will be denoted $\mathbf{sin^{-1}}$ or **arcsin**.



Definition:
$\boldsymbol{y = \sin^{-1} x} \iff \boldsymbol{\sin y = x}$ for $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



Why the name "**arcsine**"?
Notice that arcsin x is the **angle *y*** (equivalently **the length of an arc** on a unit circle) whose sine is equal to *x*.

Properties of **arcsine** ($\sin^{-1}$) function:

- Domain: Range:
- increasing; odd;
- $\sin(\sin^{-1} x) = x$, if $-1 \le x \le 1$;
- $\sin^{-1}(\sin y) = \begin{cases} y, & if \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2} \\ y_{domain}, & if \quad y \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \end{cases}$, where y_{domain} represents the angle corresponding to *y* but moved to the restricted domain.

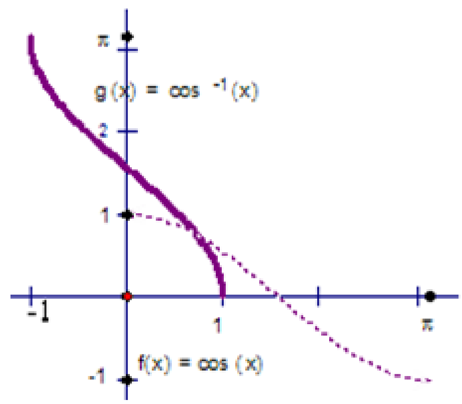
Example 1: Find the exact value of:

a) $\sin^{-1}\left(-\frac{1}{2}\right)$ b) $\sin(\sin^{-1} 0.7)$

c) $\sin^{-1}\left(\sin\frac{\pi}{3}\right)$ d) $\sin^{-1}\left(\sin\frac{4\pi}{3}\right)$

e) $\tan\left(\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right)$ f) $\cos\left(\sin^{-1}\left(\frac{1}{5}\right)\right)$

To find the **inverse** of $f(x) = \cos x$, we restrict the domain of cosine to $[0, \pi]$. Such inverse will be denoted $\mathbf{cos^{-1}}$ or **arccos**.



> *Definition:*
> $\boldsymbol{y = \cos^{-1} x} \iff \boldsymbol{\cos y = x}$ for $-1 \le x \le 1$ and $0 \le y \le \pi$

Properties of **arccosine** ($\cos^{-1}$) function:

- Domain: Range:
- decreasing;
- $\cos(\cos^{-1} x) = x$, if $-1 \le x \le 1$;
- $\cos^{-1}(\cos y) = \begin{cases} y, & if \quad 0 \le y \le \pi \\ y_{domain}, & if \quad y \notin [0, \pi] \end{cases}$.

Example 2: Find the exact value of:

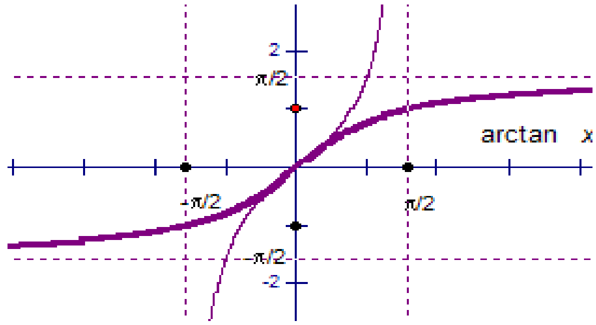
a) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ b) $\cos(\cos^{-1}(-1))$

c) $\cos^{-1}\left(\cos\frac{\pi}{2}\right)$ d) $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$

e) $\sin\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$ f) $\tan\left(\cos^{-1}\left(\frac{1}{5}\right)\right)$

To find the **inverse** of $f(x) = \tan x$, we restrict the domain of tangent to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
Such inverse will be denoted $\mathbf{tan^{-1}}$ or **arctan**.

Definition:
$\boldsymbol{y = \tan^{-1} x \iff \tan y = x}$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$



Properties of **arctangent** ($\tan^{-1}$) function:

- Domain: Range:
- increasing; odd;
- asymptotes:
- $\tan(\tan^{-1} x) = x$, for every real x;
- $\tan^{-1}(\tan y) = \begin{cases} y, & if \quad -\frac{\pi}{2} < y < \frac{\pi}{2} \\ y_{domain}, & if \quad y \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{cases}$.

Example 3: Find the exact value of:

a) $\tan^{-1}(1)$ b) $\tan\left(\tan^{-1}(100)\right)$

c) $\tan^{-1}\left(\tan\left(-\frac{\pi}{4}\right)\right)$ d) $\tan^{-1}\left(\tan\left(-\frac{5\pi}{4}\right)\right)$

e) $\sin\left(\tan^{-1}\sqrt{3}\right)$ f) $\cos\left(\tan^{-1}\left(\frac{2}{3}\right)\right)$

Here are the inverses of cotangent, secant, and cosecant:

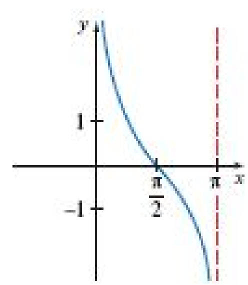
arccotangent:

$\boldsymbol{y = \cot^{-1} x \iff \cot y = x}$, for $x \in \mathbb{R}$ and $0 < y < \pi$

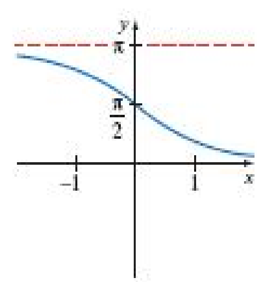
Notice that since $\boldsymbol{\cot y = x \iff \tan y = \frac{1}{x}}$, then

$$\boldsymbol{\cot^{-1} x = \begin{cases} \tan^{-1}\frac{1}{x} & , for \quad x > 0 \\ \tan^{-1}\frac{1}{x} + \pi & , for \quad x < 0 \\ \frac{\pi}{2} & , for \quad x = 0 \end{cases}}$$

Domain: Range:
Asymptotes:



$y = \cot x$



$y = \cot^{-1} x$

arcsecant:

$\boldsymbol{y = sec^{-1}x} \iff \boldsymbol{sec\, y = x}$, for $x \in (-\infty, -1] \cup [1, \infty)$ and $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

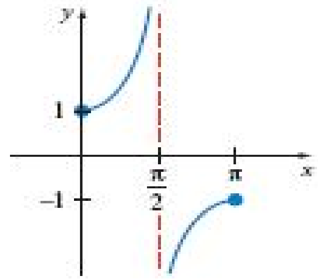
Notice that since $\boldsymbol{sec\, y = x} \iff \boldsymbol{cos\, y = \frac{1}{x}}$, then

$$\boldsymbol{\sec^{-1}x = \cos^{-1}\frac{1}{x}}$$

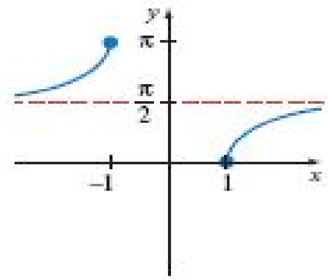
Domain:

Range:

Asymptote:



$y = \sec x$



$y = \sec^{-1}x$

arccosecant:

$\boldsymbol{y = csc^{-1}x} \iff \boldsymbol{csc\, y = x}$, for $x \in (-\infty, -1] \cup [1, \infty)$ and $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

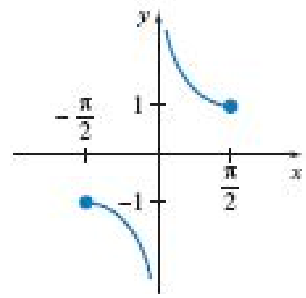
Notice that since $\boldsymbol{csc\, y = x} \iff \boldsymbol{sin\, y = \frac{1}{x}}$, then

$$\boldsymbol{\csc^{-1}x = \sin^{-1}\frac{1}{x}}$$

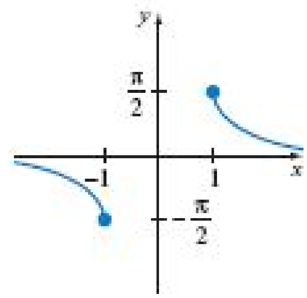
Domain:

Range:

Asymptote:



$y = \csc x$



$y = \csc^{-1}x$

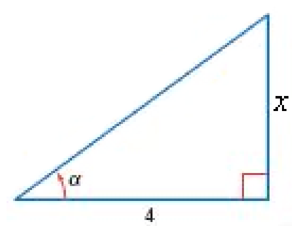
Example 4: Evaluate. Give the exact value if possible.

a) $\csc^{-1}2$

b) $\cot^{-1}\sqrt{3}$

c) $\sec^{-1}(-1.5)$

Example 5: Given the triangle, express α as a function of x.



Example 6: Rewrite the expression in terms of x.

a) $\cos(\sec^{-1} x)$

b) $\sin(\sec^{-1} x)$ (*attention: the range of* $\sec^{-1} x$ *is* $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, *so* $\sin(\sec^{-1} x) \geq 0$)

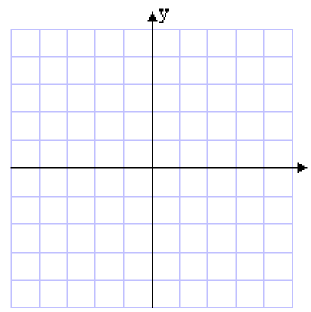
Example 7: Evaluate $\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)$.

Example 8: Solve the equation $\tan\left(x + \frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$ for *x*.

Example 9: Solve for *y* in terms of *x*: $x + 1 = \frac{1}{3}\sin^{-1} 5y$

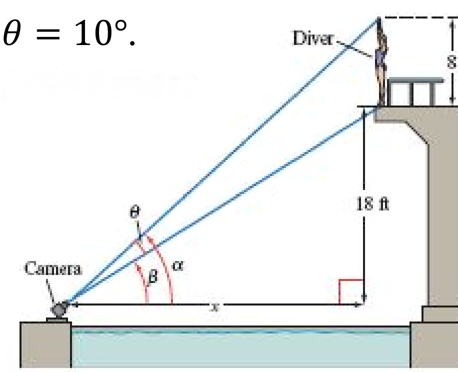
Example 10: Verify the identity: $\tan(\csc^{-1} x) = \frac{\sqrt{x^2-1}}{x^2-1}$, for $x > 1$.

Example 11: Graph $f(x) = \tan^{-1}(x+1) - 2$ without a calculator.



Example 12:

a) Use the diagram to determine the values of *x* for which $\theta = 10°$.



b) What value of *x* maximizes θ?