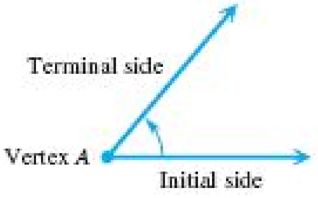
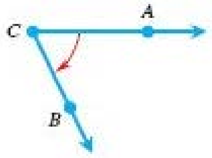
14.1-14.4 Trigonometric Ratios for Angles from 0° to 360°, Special Angles, and Basic Identities

angle – rotational space between two rays, called the **initial** and **terminal side**, coming from the same point, called the **vertex**; if the rotation is counterclockwise (ccw), the angle measure is positive, otherwise - negative

degree – a unit of measure of an angle;
360° corresponds to a complete rotation;
$1^\circ = 60'$, $1' = 60''$



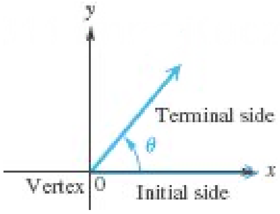
positive angle



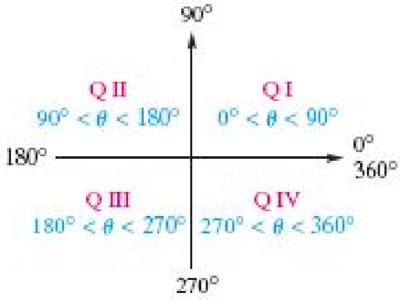
negative angle

complementary angles – angles that add up to 90°
supplementary angles – angles that add up to 180°

standard position – vertex at the origin and initial side on the positive *x*-axis

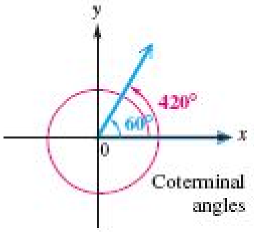


quadrants – four infinite regions of the Cartesian plane, bounded by two half-axes



quadrantal angles – angles in standard position with terminal side on one of the axes, such as 0°, 90°, 180°, 270°, and so on

coterminal angles – angles in standard position with the same terminal side, for example 60° and 420°;
An angle coterminal to $\boldsymbol{\alpha}^\circ$ has a form $\boldsymbol{\alpha^\circ + n \cdot 360^\circ}$, for some $n \in \mathbb{Z}$.



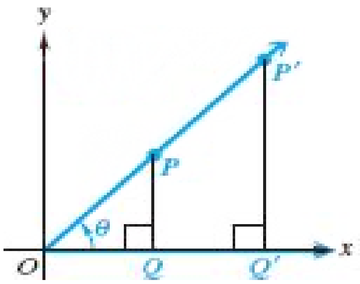
Example 1:
Convert angles between decimal degrees form and DMS form.

decimal degrees	DMS
15.25°	
	65°30′45″
80.125°	
	32°10′12″

Example 2: Find a positive and a negative coterminal angle to the given one.

a) 74° b) −81° c) 115°

$$\underbrace{\textbf{TRI} – \textbf{GONO}}_{triangle} – \underset{measurement}{\textbf{METRY}}$$

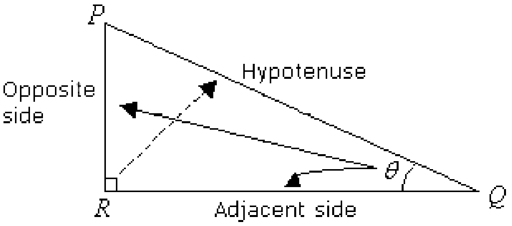


In all **right angle similar triangles** with the same acute angle $\boldsymbol{\theta}$, the ratios between their corresponding sides are equal.
These ratios depend only on the angle θ, and are called as follows:

sine: $\sin\theta = \dfrac{Opposite}{Hypotenuse}$

cosine: $\cos\theta = \dfrac{Adjacent}{Hypotenuse}$

tangent: $\tan\theta = \dfrac{Opposite}{Adjacent}$



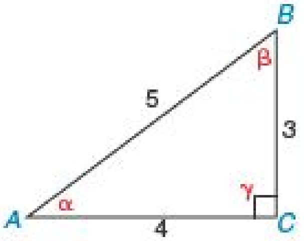
Here is an easy way to remember these ratios: **SOH – CAH – TOA** “so-ka-toe-ah”

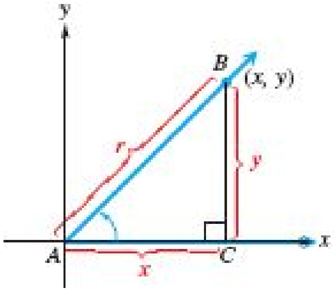
Example 1: Refer to the given triangle to find the following ratios:

a) $\sin\alpha =$ $\sin\beta =$

b) $\cos\alpha =$ $\cos\beta =$

c) $\tan\alpha =$ $\tan\beta =$



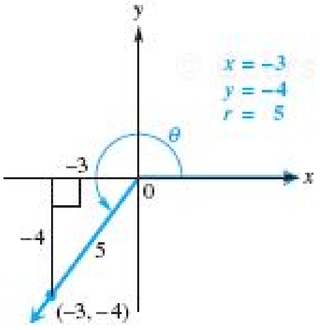


If we place our right angle triangle in a system of coordinates in a way so the acute angle θ will be in standard position, the definitions of the three trigonometric ratios (in terms of *x*, *y*, and *r*) could be stated as follows:

$$\boldsymbol{\sin\theta = \frac{y}{r}} \qquad \boldsymbol{\cos\theta = \frac{x}{r}} \qquad \boldsymbol{\tan\theta = \frac{y}{x}}$$

This setting allows us to extend the definitions of trigonometric functions $\sin\theta$, $\cos\theta$, and $\tan\theta$ to angles $\theta \in [0°, 360°)$.

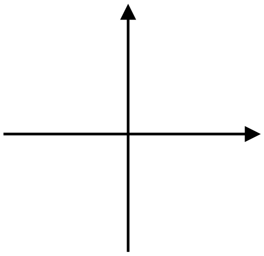
Example 2: The terminal side of an angle θ in standard position passes through the point $(-3, -4)$. Find the values of the three trigonometric functions.

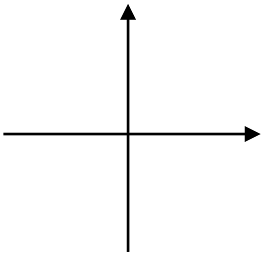


Practice:

1. Sketch an angle θ in standard position having the given point on its terminal side. Then, find the values of the three trigonometric functions for the angle θ.

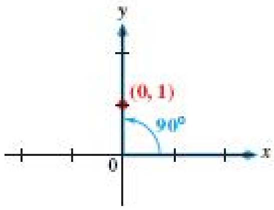
a) $\left(-2\sqrt{3}, 2\right)$ b) $(5, -12)$



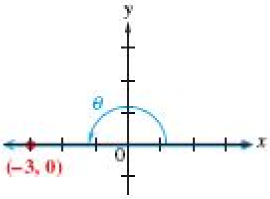


Example 3:

Find the values of the three trigonometric functions for the quadrantal angles.

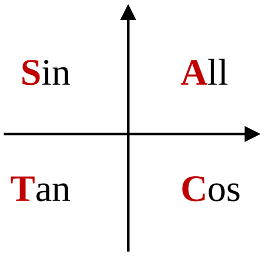


$\boldsymbol{\theta}$	$\mathbf{sin}\,\boldsymbol{\theta} = \frac{y}{r}$	$\mathbf{cos}\,\boldsymbol{\theta} = \frac{x}{r}$	$\mathbf{tan}\,\boldsymbol{\theta} = \frac{y}{x}$
$0°, 360°$			
$90°$			
$180°$			
$270°$			



Example 4:

Observing signs of $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{r}$ in different quadrants, find the signs of trigonometric functions in each quadrant to complete the table.



$\boldsymbol{\theta}$ **in quadrant**	$\mathbf{sin}\,\boldsymbol{\theta} = \frac{y}{r}$	$\mathbf{cos}\,\boldsymbol{\theta} = \frac{x}{r}$	$\mathbf{tan}\,\boldsymbol{\theta} = \frac{y}{x}$
I			
II			
III			
IV			

Example 5:

Given the information, identify the quadrant(s) of the angle θ.

a) $\sin\theta > 0$ and $\cos\theta < 0$ b) $\sin\theta < 0$ and $\tan\theta < 0$

Example 6:

Using definitions in terms of x, y, r, find the values of the other trigonometric functions of the angle θ, knowing that

a) $\cos\theta = -\frac{1}{4}$, and θ is in II quadrant

b) $\sin\theta = \frac{\sqrt{3}}{2}$, and $\tan\theta > 0$

Notice! The Pythagorean equation $x^2 + y^2 = r^2$ and the definitions of the three trigonometric functions lead us to the following identities:

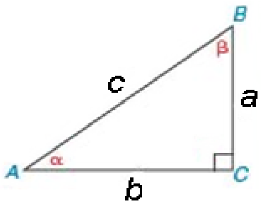
$$\boldsymbol{\sin^2\theta + \cos^2\theta = 1} \qquad \text{and} \qquad \boldsymbol{\tan\theta = \frac{\sin\theta}{\cos\theta}}$$

Proof:

Also, notice that the **cofunctions** of complementary angles are equal:

$$\boldsymbol{\sin\alpha} = \frac{a}{c} = \cos\beta = \boldsymbol{\cos(90° - \alpha)}$$

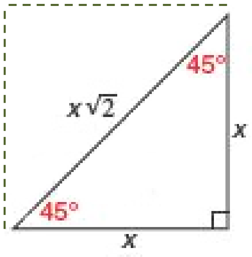
For example: $\sin 18° = \cos 72°$

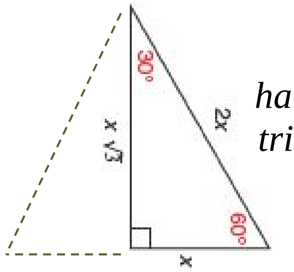


Angles such as $30°, 45°, 60°$ occurs quite often in applications and the trig functions of such angles can be calculated exactly, using Pythagorean relations in following triangles:

Special triangles: $(45° - 45° - 90°$, and $30° - 60° - 90°)$

half of a square



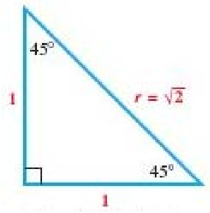


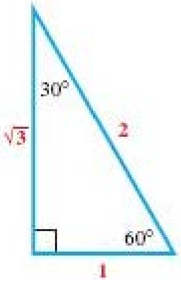
half of an equilateral triangle (Golden Triangle)

Practice:

2. Using The Pythagorean Theorem, prove the relationship between sides of special triangles as shown above.

Example 7: Use special triangles to complete the following table of values:

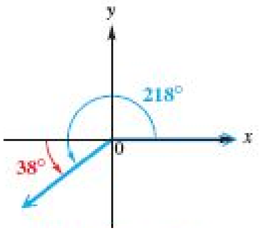


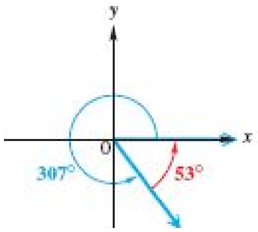


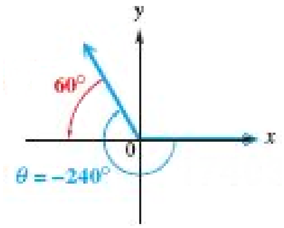
$\boldsymbol{\theta}$	$\mathbf{sin}\,\boldsymbol{\theta}$	$\mathbf{cos}\,\boldsymbol{\theta}$	$\mathbf{tan}\,\boldsymbol{\theta}$
30°			
45°			
60°			

reference angle – the **positive acute angle** formed by the **terminal** side and ***x*-axis**

examples:







Example 8: Find the reference angle for each given angle.

a) 375° b) 281° c) 169°

$$\mathbf{trig\ function}(\boldsymbol{angle}) = \pm\ \mathbf{trig\ function}(\boldsymbol{reference\ angle})$$

The sign is determined according to the sign rule: **CAST**

Example 9:

Use reference angles to find exact values of the indicated trig functions.

a) $\tan 225°$

b) $\sin 150°$

c) $\cos 300°$

Example 10:

Find all values of θ, if $\theta \in [0°, 360°)$ and

a) $\sin\theta = \frac{\sqrt{2}}{2}$

b) $\tan\theta = -\sqrt{3}$