Area and Volume Handout

Math 063/075

8.3 Area and Volume

8.3 OBJECTIVES

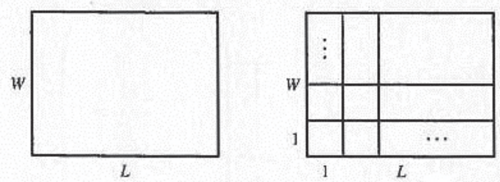
1. Determine areas of rectangles, squares, triangles, and parallelograms
2. Convert between units of area
3. Determine the area of a circle
4. Determine volumes of rectangular solids, cubes, spheres, and cylinders

In Section 1.5, we examined the area of a rectangle. In this section, we build on that idea. We begin with a definition.

Definition: Area

The **area** of an object is the number of unit squares that cover its surface. We usually designate area with the letter *A*.

For a rectangle with length L and width W, we can cover the rectangle with $L \cdot W$ unit squares, as illustrated below.



This leads to the following formulas for the areas of squares and rectangles.

Property: Formula for Area of a Square

$A = s^2$

in which s is the length of a side.

NOTE A square is a rectangle in which a side $s = L = W$.

Property: Formula for Area of a Rectangle

$A = L \cdot W$

in which L is the length and W is the width.

OBJECTIVE 1

Example 1 Finding the Area of a Square

Find the area of the square.

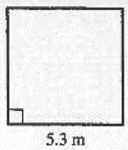
The area of a square is the square of the length of one side. Here we have

$$A = s^2 = (3.2\text{ cm})^2 = 10.24\text{ cm}^2$$

It is important to note that the units for area are *always* square units.

CHECK YOURSELF 1

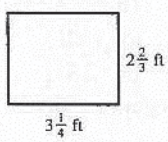
Find the area of the square.



In Example 2, we find the area of a rectangle.

Example 2 Finding the Area of a Rectangle

Find the area of the rectangle.



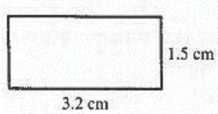
Using the formula $A = L \cdot W$, we get

$$A = 3\frac{1}{4}\text{ ft} \cdot 2\frac{2}{3}\text{ ft} = \frac{13}{4} \cdot \frac{8}{3}\text{ ft}^2 = \frac{104}{12}\text{ ft}^2 = \frac{26}{3}\text{ ft}^2 = 8\frac{2}{3}\text{ ft}^2$$

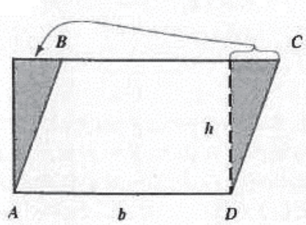
Note that the area is just under 9 ft^2.

CHECK YOURSELF 2

Find the area of the rectangle.



Two other figures that are frequently encountered are parallelograms and triangles.



In the figure, *ABCD* is a *parallelogram* (its opposite sides are parallel and of equal length). We can draw a line from *D* that forms a right angle with side *BC*. This cuts off one corner of the parallelogram. Now imagine that we move that corner over to the left side of the figure, as shown. This gives us a rectangle instead of a parallelogram. Because the area of the figure is not changed by moving the corner, the parallelogram has the same area as the rectangle, the product of the base and the height.

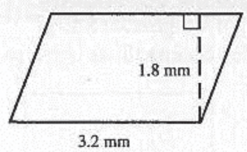
Property: Formula for the Area of a Parallelogram

$A = b \cdot h$

in which *b* is the base and *h* is the height.

Example 3 **Finding the Area of a Parallelogram**

A parallelogram has the dimensions shown in the figure. What is its area?



Use the formula for the area of a parallelogram, with $b = 3.2$ mm and $h = 1.8$ mm.

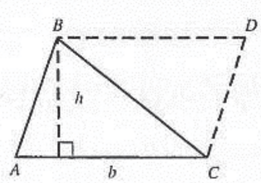
$$A = b \cdot h$$

$$= 3.2 \text{ mm} \cdot 1.8 \text{ mm} = 5.76 \text{ mm}^2$$

CHECK YOURSELF 3

If the base of a parallelogram is $3\frac{1}{2}$ *in. and its height is* $1\frac{1}{2}$ *in., what is its area?*

As we saw in Section 8.1, another common geometric figure is the *triangle.* It has three sides. Triangle *ABC* is shown in the figure.



b is the base of the triangle.
h is the height, or the *altitude,* of the triangle.

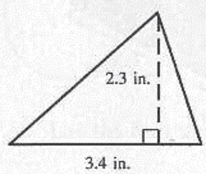
Once we have a formula for the area of a parallelogram, it is not hard to find the area of a triangle. If we draw the dotted lines from *B* to *D* and from *C* to *D* parallel to two sides of the triangle, we form a parallelogram. The area of the triangle is then one-half the area of the parallelogram (which is $b \cdot h$).

Property: Formula for the Area of a Triangle

$$A = \frac{1}{2} \cdot b \cdot h$$

Example 4 Finding the Area of a Triangle

A triangle has an altitude of 2.3 in., and its base is 3.4 in. What is its area?



Using the formula for the area of a triangle, with $b = 3.4$ in. and $h = 2.3$ in.

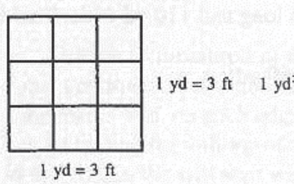
$$A = \frac{1}{2} \cdot b \cdot h$$

$$= \frac{1}{2} \cdot 3.4 \text{ in.} \cdot 2.3 \text{ in.} = 3.91 \text{ in.}^2$$

CHECK YOURSELF 4

A triangle has a base of 10 kilometers (km) and an altitude of 6 km. Find its area.

Sometimes we want to convert from one square unit to another. For instance, look at 1 yd^2 in the figure.



The table gives some useful relationships.

Square Units and Equivalents

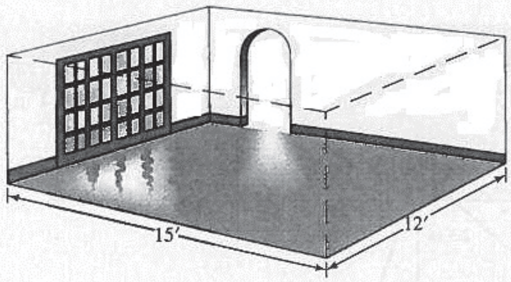
1 square foot (ft^2)	= 144 square inches ($in.^2$)
1 square yard (yd^2)	= 9 ft^2
1 acre	= 4,840 yd^2 = 43,560 ft^2
1 square mile (mi^2)	= 640 acres

NOTE Originally, an acre was the area that could be plowed by a team of oxen in a day!

OBJECTIVE 2

Example 5 Converting between Square Feet and Square Yards in Finding Area

A room has the dimensions 12 ft by 15 ft. How many square yards of linoleum will be needed to cover the floor?



NOTE We first find the area in square feet and then convert to square yards.

$$A = 12 \text{ ft} \cdot 15 \text{ ft} = 180 \text{ ft}^2$$

$$= \overset{20}{\cancel{180}} \text{ ft}^2 \cdot \frac{1 \text{ yd}^2}{\underset{1}{\cancel{9}} \text{ ft}^2}$$

Recall that $\frac{1 \text{ yd}^2}{9 \text{ ft}^2}$ is a units fraction.

$$= 20 \text{ yd}^2$$

CHECK YOURSELF 5

A hallway is 27 ft long and 4 ft wide. How many square yards of carpeting will be needed to carpet the hallway?

Example 6 illustrates the use of a common unit of area, the acre.

Example 6 Converting between Square Yards and Acres in Finding Area

A rectangular field is 220 yd long and 110 yd wide. Find its area in acres.

$$A = 220 \text{ yd} \cdot 110 \text{ yd} = 24{,}200 \text{ yd}^2$$

$$= \overset{5}{\cancel{24{,}200}} \text{ yd}^2 \cdot \frac{1}{\underset{1}{\cancel{4840}}} \frac{\text{acre}}{\text{yd}^2}$$

$$= 5 \text{ acres}$$

CHECK YOURSELF 6

A proposed site for an elementary school is 220 yd long and 198 yd wide. Find its area in acres.

The number pi (π), which we used to find circumference in Section 8.2, is also used in finding the area of a circle. If r is the radius of a circle, we have this formula.

NOTE This is read as, "Area equals pi r squared." You can multiply the radius by itself and then by pi.

Property: Formula for the Area of a Circle

$A = \pi r^2$

OBJECTIVE 3

Example 7 Find the Area of a Circle

A circle has a radius of 7 in. What is its area?

Use the formula for the area of a circle, 3.14 for π, and $r = 7$ in.

$A \approx 3.14 \cdot (7 \text{ in.})^2$

$\approx 3.14 \cdot (49 \text{ in.}^2)$

$\approx 153.9 \text{ in.}^2$

Again the area is an approximation because we use 3.14, an approximation for π.

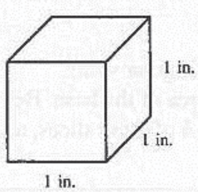
CHECK YOURSELF 7

Find the area of a circle whose diameter is 4.8 cm. Remember that the formula refers to the radius. Use 3.14 for π and round your result to the nearest tenth of a square centimeter.

Our next measurement deals with finding the **volume** of a solid, which is the measure of the space contained in the solid.

Definition: Solids

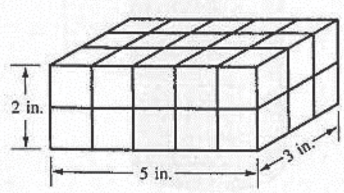
A solid is a three-dimensional figure. It has length, width, and height.



1 cubic inch

Volume is measured in **cubic units.** Examples include cubic inches, cubic feet, and cubic centimeters. A cubic inch, for instance, is the measure of the space contained in a cube that is 1 in. on each edge.

In finding the volume of a figure, we want to know how many cubic units are contained in that figure. We will start with a simple example, a **rectangular solid.** A rectangular solid is a very familiar figure. A box, a crate, and most rooms are rectangular solids. Say that the dimensions of the solid are 5 in. by 3 in. by 2 in. as pictured. If we divide the solid into units of 1 in.^{3}, we have two layers, each containing 3 units by 5 units, or 15 in.^{3} Because there are two layers, the volume is 30 in.^{3}



In general, we can see that the volume of a rectangular solid is the product of its length, width, and height.

Property: Formula for the Volume of a Rectangular Solid

$V = L \cdot W \cdot H$

OBJECTIVE 4

Example 8 Finding a Volume

NOTE We are not particularly worried about which is the length, which is the width, and which is the height, because the order in which we multiply does not change the result.

A crate has dimensions 4 ft by 2 ft by 3 ft. Find its volume.

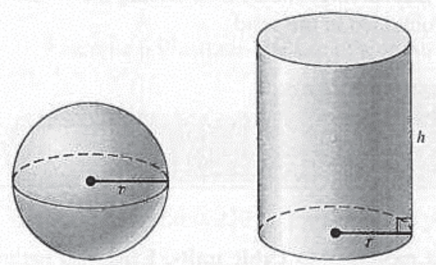
Use the formula for the volume of a rectangular solid, with $L = 4$ ft, $W = 2$ ft, and $H = 3$ ft.

$$
\begin{aligned}
V &= L \cdot W \cdot H \\
&= 4 \text{ ft} \cdot 2 \text{ ft} \cdot 3 \text{ ft} \\
&= 24 \text{ ft}^3
\end{aligned}
$$

CHECK YOURSELF 8

A room is 15 ft long, 10 ft wide, and 8 ft high. What is its volume?

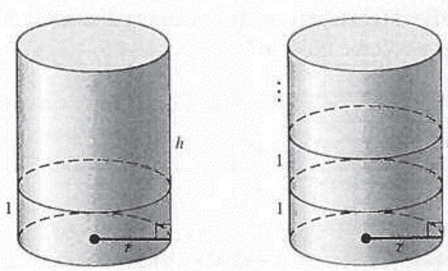
The formulas for the volume of a sphere (the shape of a ball) and a right circular cylinder (the shape of a can) both involve the number π.



NOTE Archimedes of Syracuse studied both cylinders and spheres in the third century B.C.

The process of finding the volume of a cylinder is similar to a rectangular solid.

A one-unit high "slice" gives a piece with a volume equal to the area of the base. Because the base is a circle, we get an area of πr^2 for this slice. There are h of these slices, as illustrated below.



The formula for the area of a sphere is less intuitive. You may learn its derivation in a more advanced math class.

Property: Volume of a Sphere

$$V = \frac{4}{3}\pi r^3$$

in which r is the radius.

Property: Volume of a Right Circular Cylinder

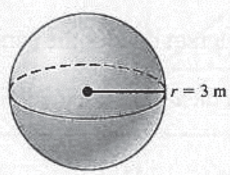
$$V = \pi r^2 h$$

in which r is the radius and h is the height.

Example 9 Finding a Volume

Find the volume of each solid shape.

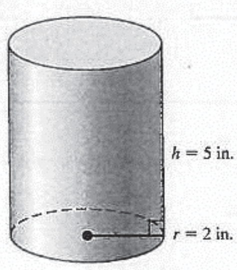
(a)



This is a sphere with radius 3 m. Using the formula for the volume of a sphere,

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi(3\text{ m})^3 \\ &= \frac{4}{3}\pi \cdot 27\text{ m}^3 \\ &= \frac{4}{3} \cdot \pi \cdot \frac{27}{1}\text{ m}^3 \\ &= 36\pi\text{ m}^3 \\ &\approx 36(3.14)\text{ m}^3 \\ &\approx 113.0\text{ m}^3 \end{aligned}$$

(b)



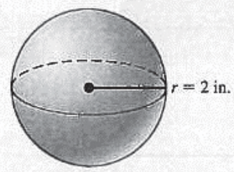
This is a right circular cylinder with radius 2 in. and height 5 in. Using the formula for the volume of a right circular cylinder,

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi(2\text{ in.})^2(5\text{ in.}) \\ &= \pi 4\text{ in.}^2(5\text{ in.}) \\ &= 20\pi\text{ in.}^3 \\ &\approx 20(3.14)\text{ in.}^3 \\ &\approx 62.8\text{ in.}^3 \end{aligned}$$

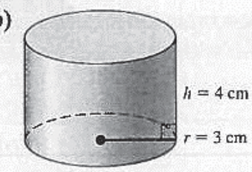
CHECK YOURSELF 9

Find the volume of each solid shape.

(a)



(b)



READING YOUR TEXT

The following fill-in-the-blank exercises are designed to assure that you understand the key vocabulary used in this section. Each sentence comes directly from the section. You will find the correct answers in Appendix C.

Section 8.3

(a) Area is always measured in __________ units.

(b) A __________ has the same area as a rectangle with the same base and height.

(c) A diagonal in a parallelogram divides the parallelogram into two __________.

(d) __________ is always measured in cubic units.

CHECK YOURSELF ANSWERS

1. 28.09 m^2 **2.** 4.8 cm^2 **3.** $A = 5\frac{1}{4}$ in.^{2}

4. $A = 30$ km^2 **5.** 12 yd^2 **6.** 9 acres **7.** ≈ 18.1 cm^2

8. 1,200 ft^3 **9.** **(a)** ≈ 33.5 in.^{3}; **(b)** ≈ 113.0 cm^3