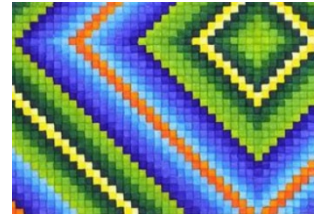


Graphs and Linear Functions

A 2-dimensional graph is a visual representation of a relationship between two variables given by an equation or an inequality. Graphs help us solve algebraic problems by analysing the geometric aspects of a problem. While equations are more suitable for precise calculations, graphs are more suitable for showing patterns and trends in a relationship. To fully utilize what graphs can offer, we must first understand the concepts and skills involved in graphing that are discussed in this chapter.



G.1

System of Coordinates, Graphs of Linear Equations and the Midpoint Formula

In this section, we will review the rectangular coordinate system, graph various linear equations and inequalities, and introduce a formula for finding coordinates of the midpoint of a given segment.

The Cartesian Coordinate System

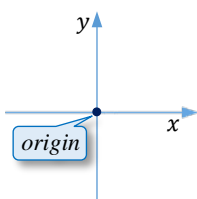


Figure 1a

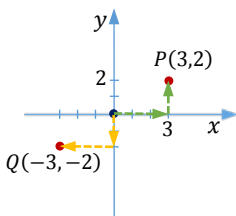


Figure 1b

A rectangular coordinate system, also called a **Cartesian coordinate system** (in honor of French mathematician, *René Descartes*), consists of two perpendicular number lines that cross each other at point zero, called the **origin**. Traditionally, one of these number lines, usually called the **x-axis**, is positioned horizontally and directed to the right (see *Figure 1a*). The other number line, usually called **y-axis**, is positioned vertically and directed up. Using this setting, we identify each point P of the plane with an **ordered pair** of numbers (x, y) , which indicates the location of this point with respect to the origin. The first number in the ordered pair, the **x-coordinate**, represents the horizontal distance of the point P from the origin. The second number, the **y-coordinate**, represents the vertical distance of the point P from the origin. For example, to locate point $P(3,2)$, we start from the origin, go 3 steps to the right, and then two steps up. To locate point $Q(-3,-2)$, we start from the origin, go 3 steps to the left, and then two steps down (see *Figure 1b*).

Observe that the coordinates of the origin are $(0,0)$. Also, the second coordinate of any point on the x -axis as well as the first coordinate of any point on the y -axis is equal to zero. So, points on the x -axis have the form $(x, 0)$, while points on the y -axis have the form of $(0, y)$.

To **plot** (or **graph**) an ordered pair (x, y) means to place a dot at the location given by the ordered pair.

Example 1

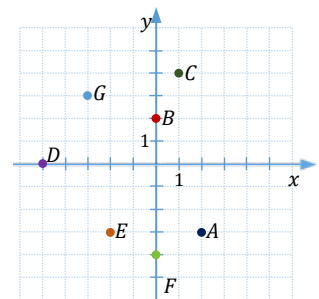
Plotting Points in a Cartesian Coordinate System

Plot the following points:

$$A(2, -3), \quad B(0, 2), \quad C(1, 4), \quad D(-5, 0), \\ E(-2, -3), \quad F(0, -4), \quad G(-3, 3)$$

Solution

Remember! The order of numbers in an ordered pair is important! The **first** number represents the **horizontal** displacement and the **second** number represents the **vertical** displacement from the origin.



Graphs of Linear Equations

A **graph of an equation** in two variables, x and y , is the set of points corresponding to **all ordered pairs** (x, y) that **satisfy** the equation (make the equation true). This means that a graph of an equation is the visual representation of the **solution set** of this equation.

To determine if a point (a, b) belongs to the graph of a given equation, we check if the equation is satisfied by $x = a$ and $y = b$.

Example 1 ▶ Determining if a Point is a Solution of a Given Equation

Determine if the points $(5, 3)$ and $(-3, -2)$ are solutions of $2x - 3y = 0$.

Solution ▶ After substituting $x = 5$ and $y = 3$ into the equation $2x - 3y = 0$, we obtain

$$\begin{aligned} 2 \cdot 5 - 3 \cdot 3 &= 0 \\ 10 - 9 &= 0 \\ 1 &= 0, \end{aligned}$$

which is not true. Since the coordinates of the point $(5, 3)$ do not satisfy the given equation, the point **$(5, 3)$ is not a solution** of this equation.

Note: The fact that the point $(5, 3)$ does not satisfy the given equation indicates that it does not belong to the graph of this equation.

However, after substituting $x = -3$ and $y = -2$ into the equation $2x - 3y = 0$, we obtain

$$\begin{aligned} 2 \cdot (-3) - 3 \cdot (-2) &= 0 \\ -6 + 6 &= 0 \\ 0 &= 0, \end{aligned}$$

which is true. Since the coordinates of the point $(-3, -2)$ satisfy the given equation, the point **$(-3, -2)$ is a solution** of this equation.

Note: The fact that the point $(-3, -2)$ satisfies the given equation indicates that it belongs to the graph of this equation.

To find a solution to a given equation in two variables, we choose a particular value for one of the variables, substitute it into the equation, and then solve the resulting equation for the other variable.

For example, to find a solution to $3x + 2y = 6$, we can choose for example $x = 0$, which leads us to

$$\begin{aligned} 3 \cdot 0 + 2y &= 6 \\ 2y &= 6 \\ y &= 3. \end{aligned}$$

This means that the point $(0, 3)$ satisfies the equation and therefore belongs to the graph of this equation. If we choose a different x -value, for example $x = 1$, the corresponding y -value becomes

$$\begin{aligned} 3 \cdot 1 + 2y &= 6 \\ 2y &= 3 \\ y &= \frac{3}{2}. \end{aligned}$$

So, the point $\left(1, \frac{3}{2}\right)$ also belongs to the graph.

Since any real number could be selected for the x -value, there are infinitely many solutions to this equation. Obviously, we will not be finding all of these infinitely many ordered pairs of numbers in order to graph the solution set to an equation. Rather, based on the location of several solutions that are easy to find, we will look for a pattern and predict the location of the rest of the solutions to complete the graph.

To find more points that belong to the graph of the equation in our example, we might want to solve $3x + 2y = 6$ for y . The equation is equivalent to

$$\begin{aligned} 2y &= -3x + 6 \\ y &= -\frac{3}{2}x + 3 \end{aligned}$$

Observe that if we choose x -values to be multiples of 2, the calculations of y -values will be easier in this case. Here is a table of a few more (x, y) points that belong to the graph:

x	$y = -\frac{3}{2}x + 3$	(x, y)
-2	$-\frac{3}{2}(-2) + 3 = 6$	$(-2, 6)$
2	$-\frac{3}{2}(2) + 3 = 0$	$(2, 0)$
4	$-\frac{3}{2}(4) + 3 = -3$	$(4, -3)$

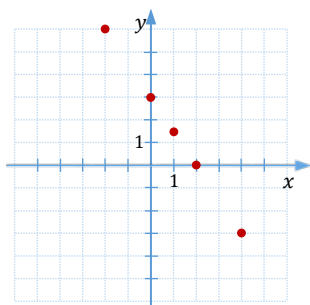


Figure 2a

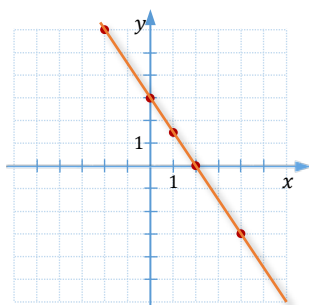


Figure 2b

After plotting the obtained solutions, $(-2, 6)$, $(0, 3)$, $\left(1, \frac{3}{2}\right)$, $(2, 0)$, $(4, -3)$, we observe that the points appear to lie on the same line (see *Figure 2a*). If all the ordered pairs that satisfy the equation $3x + 2y = 6$ were graphed, they would form the line shown in *Figure 2b*. Therefore, if we knew that the graph would turn out to be a line, it would be enough to find just two points (solutions) and draw a line passing through them.

How do we know whether or not the graph of a given equation is a line? It turns out that:

For any equation in two variables, the graph of the equation is a **line** if and only if (iff) the equation is **linear**.

So, the question is how to recognize a linear equation?

Definition 1.1 ▶ Any equation that can be written in the form

$$Ax + By = C, \text{ where } A, B, C \in \mathbb{R}, \text{ and } A \text{ and } B \text{ are not both } 0,$$

is called a **linear equation** in two variables.

The form $Ax + By = C$ is called **standard form** of a linear equation.

Example 2 ▶ **Graphing Linear Equations Using a Table of Values**

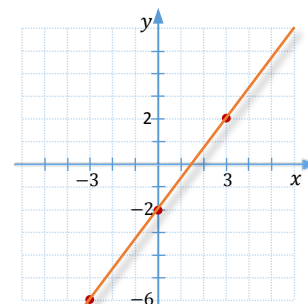
Graph $4x - 3y = 6$ using a table of values.

Solution ▶ Since this is a linear equation, we expect the graph to be a line. While finding two points satisfying the equation is sufficient to graph a line, it is a good idea to use a third point to guard against errors. To find several solutions, first, let us solve $4x - 3y = 6$ for y :

$$\begin{aligned} -3y &= -4x + 6 \\ y &= \frac{4}{3}x - 2 \end{aligned}$$

We like to choose x -values that will make the calculations of the corresponding y -values relatively easy. For example, if x is a multiple of 3, such as -3 , 0 or 3 , the denominator of $\frac{4}{3}$ will be reduced. Here is the table of points satisfying the given equation and the graph of the line.

x	$y = \frac{4}{3}x - 2$	(x, y)
-3	$\frac{4}{3}(-3) - 2 = -6$	$(-3, -6)$
0	$\frac{4}{3}(0) - 2 = -2$	$(0, -2)$
3	$\frac{4}{3}(3) - 2 = 2$	$(3, 2)$



To graph a linear equation in standard form, we can develop a table of values as in *Example 2*, or we can use the x - and y -intercepts.

Definition 1.2 ▶ The **x -intercept** is the point (if any) where the line intersects the x -axis. So, the x -intercept has the form $(x, 0)$.

The **y -intercept** is the point (if any) where the line intersects the y -axis. So, the y -intercept has the form $(0, y)$.

Example 3 ▶ **Graphing Linear Equations Using x - and y -intercepts**

Graph $5x - 3y = 15$ by finding and plotting the x - and y -intercepts.

Solution ▶ To find the x -intercept, we substitute $y = 0$ into $5x - 3y = 15$, and then solve the resulting equation for x . So, we have

$$\begin{aligned} 5x &= 15 \\ x &= 3. \end{aligned}$$

To find y -intercept, we substitute $x = 0$ into $5x - 3y = 15$, and then solve the resulting equation for x . So,

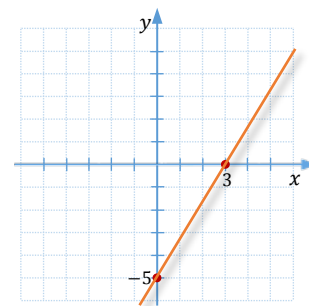
$$\begin{aligned} -3y &= 15 \\ y &= -5. \end{aligned}$$

Hence, we have

x -intercept

y -intercept

x	y
3	0
0	-5



To find several points that belong to the graph of a linear equation in two variables, it was easier to solve the standard form $Ax + By = C$ for y , as follows

$$\begin{aligned} By &= -Ax + C \\ y &= -\frac{A}{B}x + \frac{C}{B}. \end{aligned}$$

This form of a linear equation is also very friendly for graphing, as the graph can be obtained without any calculations. See *Example 4*.

Any equation $Ax + By = C$, where $B \neq 0$ can be written in the form

$$y = mx + b,$$

which is referred to as the **slope-intercept form** of a linear equation.

The value $m = -\frac{A}{B}$ represents the **slope** of the line. Recall that **slope** = $\frac{\text{rise}}{\text{run}}$.

The value b represents the y -intercept, so the point $(0, b)$ belongs to the graph of this line.

Example 4 ▶ Graphing Linear Equations Using Slope and y -intercept

Determine the slope and y -intercept of each line and then graph it.

a. $y = \frac{2}{3}x + 1$

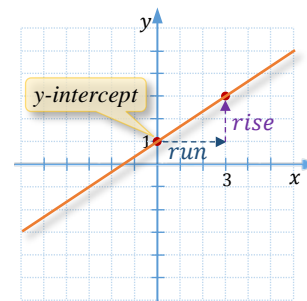
b. $5x + 2y = 8$

Solution

▶ a. The slope is the coefficient by x , so it is $\frac{2}{3}$.

The y -intercept equals 1.

So we plot point $(0, 1)$ and then, since $\frac{2}{3} = \frac{\text{rise}}{\text{run}}$, we rise 2 units and run 3 units to find the next point that belongs to the graph.



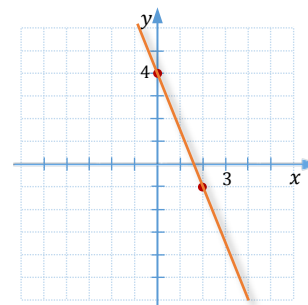
b. To see the slope and y -intercept, we solve

$$5x + 2y = 8 \text{ for } y.$$

$$2y = -5x + 8$$

$$y = \frac{-5}{2}x + 4$$

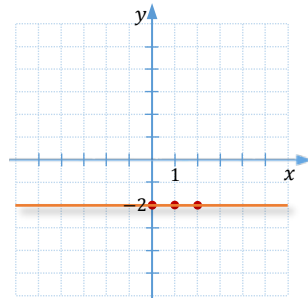
So, the slope is $\frac{-5}{2}$ and the y -intercept is 4. We start from $(0,4)$ and then run 2 units and fall 5 units (because of -5 in the numerator).



Note: Although we can *run* to the right or to the left, depending on the sign in the denominator, we usually **keep the denominator positive and always run forward** (to the right). If the slope is negative, we **keep the negative sign in the numerator** and either *rise* or *fall*, depending on this sign. However, when finding additional points of the line, sometimes we can repeat the *run/rise* movement in either way, to the right, or to the left from one of the already known points. For example, in *Example 4a*, we could find the additional point at $(-3, -2)$ by *running* 3 units to the left and 2 units down from $(0,1)$, as the slope $\frac{2}{3}$ can also be seen as $\frac{-2}{-3}$, if needed.

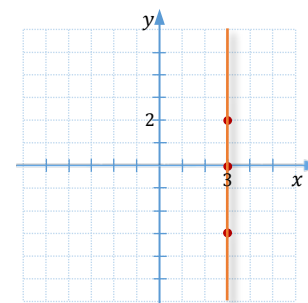
Some linear equations contain just one variable. For example, $x = 3$ or $y = -2$. How would we graph such equations in the xy -plane?

Observe that $y = -2$ can be seen as $y = 0x - 2$, so we can graph it as before, using **slope** of **zero** and **y -intercept** equal to -2 . The graph consists of all points that have y -coordinates equal to -2 . Those are the points of the type $(x, -2)$, where x is any real number. The graph is a **horizontal line** passing through the point $(0, 2)$.



Note: The horizontal line $y = 0$ is the x -axis.

The equation $x = 3$ doesn't have a slope-intercept representation, but it is satisfied by any point with x -coordinate equal to 3. So, by plotting several points of the type $(3, y)$, where y is any real number, we obtain a **vertical line** passing through the point $(3, 0)$. This particular line doesn't have a y -intercept, and its **slope** = $\frac{\text{rise}}{\text{run}}$ is considered to be **undefined**. This is because the "*run*" part calculated between any two points on the line is equal to zero and we can't perform division by zero.



Note: The vertical line $x = 0$ is the y -axis.

In general, the graph of any equation of the type

$$y = b, \text{ where } b \in \mathbb{R}$$

is a **horizontal line** with the y -intercept at b . The **slope** of such line is **zero**.

The graph of any equation of the type

$$x = a, \text{ where } a \in \mathbb{R}$$

is a **vertical line** with the x -intercept at a . The **slope** of such line is **undefined**.

Example 5 ▶ Graphing Special Types of Linear Equations

Graph each equation and state its slope.

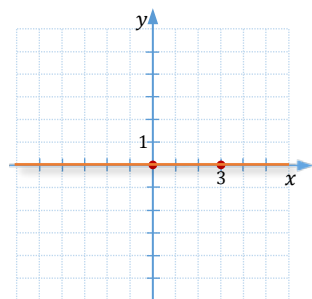
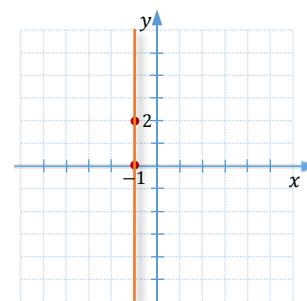
a. $x = -1$

b. $y = 0$

c. $y = x$

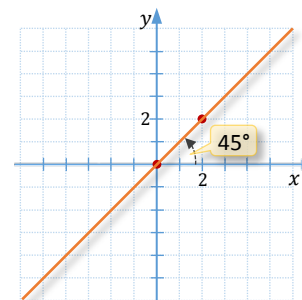
Solution ▶

- a. The solutions to the equation $x = -1$ are all pairs of the type $(-1, y)$, so after plotting points like $(-1, 0)$, $(-1, 2)$, etc., we observe that the graph is a **vertical line** intercepting x -axis at $x = -1$. So the **slope** of this line is **undefined**.



- b. The solutions to the equation $y = 0$ are all pairs of the type $(x, 0)$, so after plotting points like $(0, 0)$, $(0, 3)$, etc., we observe that the graph is a **horizontal line** following the x -axis. The **slope** of this line is **zero**.

- c. The solutions to the equation $y = x$ are all pairs of the type (x, x) , so after plotting points like $(0, 0)$, $(2, 2)$, etc., we observe that the graph is a **diagonal line**, passing through the origin and making 45° with the x -axis. The **slope** of this line is **1**.

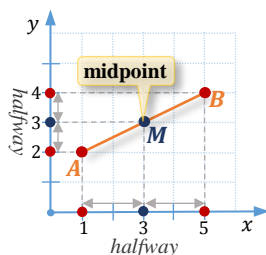
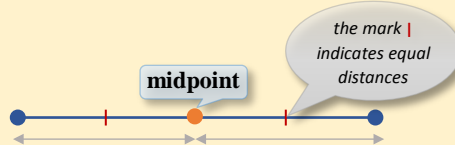


Observation: A graph of any equation of the type $y = mx$ is a line passing through the origin, as the point $(0, 0)$ is one of the solutions.

Midpoint Formula

To find a representative value of a list of numbers, we often calculate the average of these numbers. Particularly, to find an average of, for example, two test scores, 72 and 84, we take half of the sum of these scores. So, the average of 72 and 84 is equal to $\frac{72+84}{2} = \frac{156}{2} = 78$. Observe that 78 lies on a number line exactly halfway between 72 and 84. The idea of taking an average is employed in calculating coordinates of the midpoint of any line segment.

Definition 1.3 ▶ The **midpoint** of a line segment is the point of the segment that is equidistant from both ends of this segment.



Suppose $A = (x_1, y_1)$, $B = (x_2, y_2)$, and M is the **midpoint** of the line segment \overline{AB} . Then the x -coordinate of M lies halfway between the two end x -values, x_1 and x_2 , and the y -coordinate of M lies halfway between the two end y -values, y_1 and y_2 . So, the coordinates of the midpoint are **averages** of corresponding x -, and y -coordinates:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \quad (1)$$

Example 6 ▶ **Finding Coordinates of the Midpoint**

Find the midpoint M of the line segment connecting $P = (-3, 7)$ and $Q = (5, -12)$.

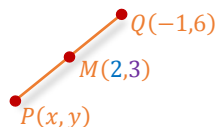
Solution ▶ The coordinates of the midpoint M are averages of the x - and y -coordinates of the endpoints. So,

$$M = \left(\frac{-3 + 5}{2}, \frac{7 + (-12)}{2} \right) = \left(1, -\frac{5}{2} \right).$$

Example 7 ▶ **Finding Coordinates of an Endpoint Given the Midpoint and the Other Endpoint**

Suppose segment PQ has its midpoint M at $(2, 3)$. Find the coordinates of point P , knowing that $Q = (-1, 6)$.

Solution ▶ Let $P = (x, y)$ and $Q = (-1, 6)$. Since $M = (2, 3)$ is the midpoint of \overline{PQ} , by formula (1), the following equations must hold:



$$\frac{x + (-1)}{2} = 2 \quad \text{and} \quad \frac{y + 6}{2} = 3$$

Multiplying these equations by 2, we obtain

$$x + (-1) = 4 \quad \text{and} \quad y + 6 = 6,$$

which results in

$$x = 5 \quad \text{and} \quad y = 0.$$

Hence, the coordinates of point P are **(5, 0)**.

G.1 Exercises

Vocabulary Check Fill in each blank with the most appropriate term or phrase from the given list: *averages, graph, horizontal, linear, line, ordered, origin, root, slope, slope-intercept, solution, undefined, vertical, x, x-axis, x-intercept, y, y-axis, y-intercept, zero.*

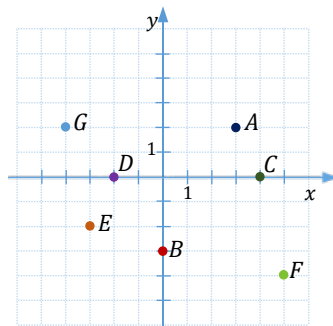
- The point with coordinates $(0, 0)$ is called the _____ of a rectangular coordinate system.
- Each point P of a plane in a rectangular coordinate system is identified with an _____ pair of numbers (x, y) , where x measures the _____ displacement of the point P from the origin and y measures the _____ displacement of the point P from the origin.
- Any point on the _____ has the x -coordinate equal to 0.
- Any point on the _____ has the y -coordinate equal to 0.
- A _____ of an equation consists of all points (x, y) satisfying the equation.
- To find the x -intercept of a line, we let _____ equal 0 and solve for _____. To find the y -intercept, we let _____ equal 0 and solve for _____.
- Any equation of the form $Ax + By = C$, where $A, B, C \in \mathbb{R}$, and A and B are not both 0, is called a _____ equation in two variables. The graph of such equation is a _____.
- In the _____ form of a line, $y = mx + b$, the coefficient m represents the _____ and the free term b represents the _____ of this line.
- The slope of a vertical line is _____ and the slope of a horizontal line is _____.
- A point where the graph of an equation crosses the x -axis is called the _____ of this graph. This point is also referred to as the _____ or _____ of the equation.
- The coordinates of the midpoint of a line segment are the _____ of the x - and y -coordinates of the endpoints of this segment.

Concept Check

12. Plot each point in a rectangular coordinate system.

a. $(1, 2)$ b. $(-2, 0)$ c. $(0, -3)$ d. $(4, -1)$ e. $(-1, -3)$

13. State the coordinates of each plotted point.



Concept Check Determine if the given ordered pair is a solution of the given equation.

14. $(-2, 2)$; $y = \frac{1}{2}x + 3$ 15. $(4, -5)$; $3x - 2y = 2$ 16. $(5, 4)$; $4x - 5y = 1$

Concept Check Graph each equation using the suggested table of values.

17. $y = 2x - 3$ 18. $y = -\frac{1}{3}x + 2$ 19. $x + y = 3$ 20. $4x - 5y = 20$

x	y
0	
1	
2	
3	

x	y
-3	
0	
3	
6	

x	y
0	
	0
-1	
	1

x	y
0	
	0
2	
	-3

Discussion Point

21. What choices of x -values would be helpful to find points on the graph of $y = \frac{5}{3}x + 4$?

Concept Check Graph each equation using a table of values.

22. $y = \frac{1}{3}x$ 23. $y = \frac{1}{2}x + 2$ 24. $6x - 3y = -9$ 25. $6x + 2y = 8$
 26. $y = \frac{2}{3}x - 1$ 27. $y = -\frac{3}{2}x$ 28. $3x + y = -1$ 29. $2x = -5y$
 30. $-3x = -3$ 31. $6y - 18 = 0$ 32. $y = -x$ 33. $2y - 3x = 12$

Concept Check Determine the x - and y -intercepts of each line and then graph it. Find additional points, if needed.

34. $5x + 2y = 10$ 35. $x - 3y = 6$ 36. $8y + 2x = -4$ 37. $3y - 5x = 15$
 38. $y = -\frac{2}{5}x - 2$ 39. $y = \frac{1}{2}x - \frac{3}{2}$ 40. $2x - 3y = -9$ 41. $2x = -y$

Concept Check Determine the slope and y -intercept of each line and then graph it.

42. $y = 2x - 3$ 43. $y = -3x + 2$ 44. $y = -\frac{4}{3}x + 1$ 45. $y = \frac{2}{5}x + 3$
 46. $2x + y = 6$ 47. $3x + 2y = 4$ 48. $-\frac{2}{3}x - y = 2$ 49. $2x - 3y = 12$
 50. $2x = 3y$ 51. $y = \frac{3}{2}$ 52. $y = x$ 53. $x = 3$

Concept Check Find the midpoint of each segment with the given endpoints.

54. $(-8, 4)$ and $(-2, -6)$ 55. $(4, -3)$ and $(-1, 3)$ 56. $(-5, -3)$ and $(7, 5)$
 57. $(-7, 5)$ and $(-2, 11)$ 58. $(\frac{1}{2}, \frac{1}{3})$ and $(\frac{3}{2}, -\frac{5}{3})$ 59. $(\frac{3}{5}, -\frac{1}{3})$ and $(\frac{1}{2}, -\frac{5}{2})$

Analytic Skills Segment PQ has the given coordinates for one endpoint P and for its midpoint M . Find the coordinates of the other endpoint Q .

60. $P(-3,2), M(3,-2)$

61. $P(7, 10), M(5,3)$

62. $P(5,-4), M(0,6)$

63. $P(-5,-2), M(-1,4)$