G.2 Slope of a Line and Its Interpretation

Slope (steepness) is a very important concept that appears in many branches of mathematics as well as statistics, physics, business, and other areas. In algebra, slope is used when graphing lines or analysing linear equations or functions. In calculus, the concept of slope is used to describe the behaviour of many functions. In statistics, slope of a regression line explains the general trend in the analysed set of data. In business, slope plays an important role in linear programming. In addition, slope is often used in many practical ways, such as the slope of a road (*grade*), slope of a roof (*pitch*), slope of a ramp, etc.
In this section, we will define, calculate, and provide some interpretations of slope.

Slope

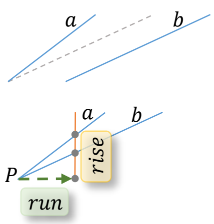


Figure 1a

Given two lines, a and b, how can we tell which one is steeper? One way to compare the steepness of these lines is to move them closer to each other, so that a point of intersection, P, can be seen, as in *Figure 1a*. Then, after running horizontally a few steps from the point P, draw a vertical line to observe how high the two lines have risen. The line that crosses this vertical line at a higher point is steeper. So, for example in *Figure 1a*, line a is steeper than line b. Observe that because we run the same horizontal distance for both lines, we could compare the steepness of the two lines just by looking at the vertical rise. However, since the *run* distance can be chosen arbitrarily, to represent the steepness of any line, we must look at the *rise* (vertical change) in respect to the *run* (horizontal change). This is where the concept of slope as a ratio of *rise* to *run* arises.

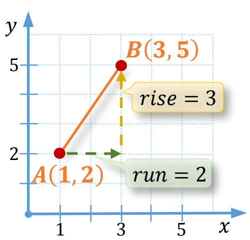


Figure 1b

To measure the slope of a line or a line segment, we choose any two distinct points of such a figure and calculate the ratio of the **vertical change** (***rise***) to the **horizontal change** (***run***) between the two points. For example, the slope between points $A(1,2)$ and $B(3,5)$ equals

$$\frac{rise}{run} = \frac{3}{2},$$

as in *Figure 1a*. If we rewrite this ratio so that the denominator is kept as one,

$$\frac{3}{2} = \frac{1.5}{1} = 1.5,$$

we can think of slope as of the **rate of change in y-values with respect to x-values**. So, a slope of 1.5 tells us that the y-value increases by 1.5 units per every increase of one unit in x-value.

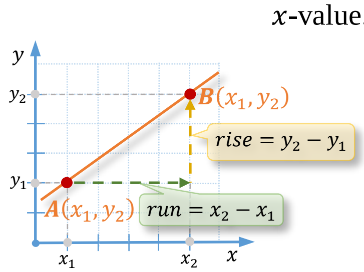


Figure 1c

Generally, the slope of a line passing through two distinct points, $(\boldsymbol{x_1}, \boldsymbol{y_1})$ and $(\boldsymbol{x_2}, \boldsymbol{y_2})$, is the **ratio** of the change in y-values, $y_2 - y_1$, to the change in x-values, $x_2 - x_1$, as presented in *Figure 1c*. Therefore, the formula for calculating slope can be presented as

$$\frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x},$$

where the Greek letter Δ (delta) is used to denote the change in a variable.

Definition 2.1 ► Suppose a line passes through two distinct points $(\boldsymbol{x_1}, \boldsymbol{y_1})$ and $(\boldsymbol{x_2}, \boldsymbol{y_2})$.

If $x_1 \neq x_2$, then the **slope** of this line, often denoted by $\boldsymbol{m}$, is equal to

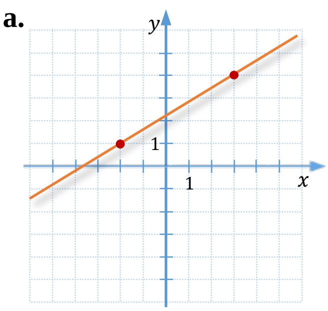
$$\boxed{\boldsymbol{m} = \frac{\boldsymbol{rise}}{\boldsymbol{run}} = \frac{\boldsymbol{change\ in\ y}}{\boldsymbol{change\ in\ x}} = \frac{\boldsymbol{y_2 - y_1}}{\boldsymbol{x_2 - x_1}} = \frac{\boldsymbol{\Delta y}}{\boldsymbol{\Delta x}}.}$$

If $x_1 = x_2$, then the **slope** of the line is said to be **undefined**.

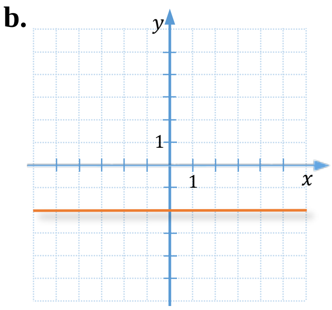
Example 1 ► Determining Slope of a Line, Given Its Graph

Determine the slope of each line.

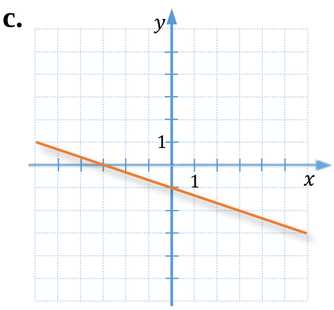
a.



b.

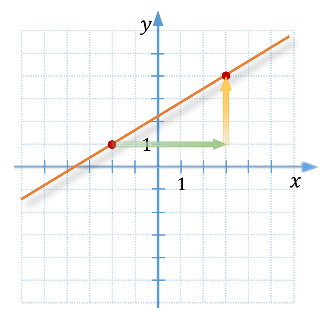


c.



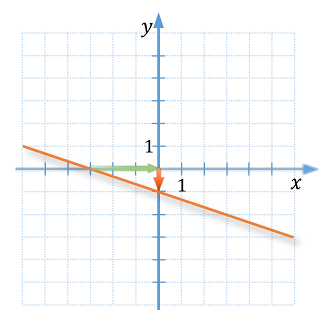
Solution ►

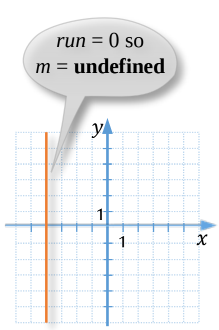
a. To read the slope we choose two distinct points with integral coefficients (often called ***lattice points***), such as the points suggested in the graph. Then, starting from the first point $(-2,1)$ we *run* 5 units and *rise* 3 units to reach the second point $(3,4)$. So, the slope of this line is $m = \frac{5}{3}$.



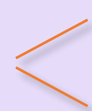
b. This is a horizontal line, so the *rise* between any two points of this line is zero. Therefore the slope of such a line is also **zero**.

c. If we refer to the lattice points $(-3,0)$ and $(0,-1)$, then the *run* is 3 and the *rise* (or rather ***fall***) is -1. Therefore the slope of this line is $m = -\frac{1}{3}$.





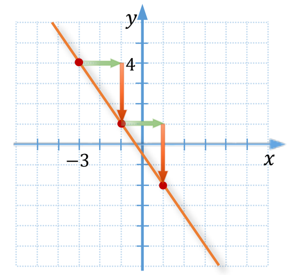
Observation:
A line that **increases** from left to right has a **positive slope**.
A line that **decreases** from left to right has a **negative slope**.
The slope of a **horizontal** line is **zero**.
The slope of a **vertical** line is **undefined**.



Example 2 ► **Graphing Lines, Given Slope and a Point**

Graph the line with slope $-\frac{3}{2}$ that passes through the point $(-3,4)$.

Solution ► First, plot the point $(-3,4)$. To find another point that belongs to this line, start at the plotted point and run 2 units, then fall 3 units. This leads us to point $(-1,1)$. For better precision, repeat the movement (two across and 3 down) to plot one more point, $(1,-2)$. Finally, draw a line connecting the plotted points.



Example 3 ► **Calculating Slope of a Line, Given Two Points**

Determine the slope of a line passing through the points $(-3,5)$ and $(7,-11)$.

Solution ► The slope of the line passing through $(-3,5)$ and $(7,-11)$ is the quotient

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-(-11)}{-3-7} = \frac{5+11}{-10} = -\frac{16}{10} = \mathbf{-1.6}$$

Example 4 ► **Determining Slope of a Line, Given Its Equation**

Determine the slope of a line given by the equation $2x - 5y = 7$.

Solution ► To see the slope of a line in its equation, we change the equation to its slope-intercept form, $y = mx + b$. The slope is the coefficient m. When solving $2x - 5y = 7$ for y, we obtain

$$-5y = -2x + 7$$

$$y = \frac{2}{5}x - \frac{7}{5}.$$

So, the slope of this line is equal to $\mathbf{\frac{2}{5}}$.

Example 5 ► **Interpreting Slope as an Average Rate of Change**

On February 11, 2016, the Dow Jones Industrial Average index value was \$15,660.18. On November 11, 2016, this value was \$18,847.66. Using this information, what was the average rate of change in value of the Dow index per month during this period of time?

Solution ► The value of the Dow index has increased by $18{,}847.66 - 15{,}660.18 = 3187.48$ dollars over the 9 months (from February 11 to November 11). So, the slope of the line segment connecting the Dow index values on these two days (as marked on the above chart) equals

$$\frac{3187.48}{9} \cong \mathbf{354.16}\ \$/month$$

This means that the value of the Dow index was increasing on average by 354.16 dollars per month between February 11, 2016 and November 11, 2016.

Observe that the change in value was actually different in each month. Sometimes the change was larger than the calculated slope, but sometimes the change was smaller or even negative. However, the **slope** of the above segment gave us the information about the **average rate of change** in Dow's value during the stated period.

Parallel and Perpendicular Lines

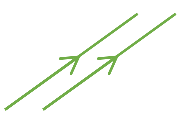


Figure 2

Since slope measures the steepness of lines, and **parallel lines** have the same steepness, then the **slopes** of **parallel lines** are **equal**.

To indicate on a diagram that lines are parallel, we draw on each line arrows pointing in the same direction (see *Figure 2*). To state in mathematical notation that two lines are parallel, we use the $\parallel$ sign.

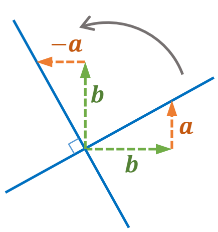


Figure 3

To see how the slopes of perpendicular lines are related, rotate a line with a given slope $\frac{a}{b}$ (where $b \neq 0$) by 90°, as in *Figure 3*. Observe that under this rotation the vertical change $\boldsymbol{a}$ becomes the horizontal change but in opposite direction ($-\boldsymbol{a}$), and the horizontal change $\boldsymbol{b}$ becomes the vertical change. So, the **slope** of the **perpendicular line** is $-\frac{b}{a}$. In other words, **slopes** of **perpendicular lines** are **opposite reciprocals**. Notice that the **product of perpendicular slopes**, $\frac{a}{b} \cdot \left(-\frac{b}{a}\right)$, is equal to $\mathbf{-1}$.

In the case of $b = 0$, the slope is undefined, so the line is vertical. After rotation by 90°, we obtain a horizontal line, with a slope of zero. So a line with a zero slope and a line with an "undefined" slope can also be considered perpendicular.

To indicate on a diagram that two lines are perpendicular, we draw a square at the intersection of the two lines, as in *Figure 3*. To state in mathematical notation that two lines are perpendicular, we use the $\perp$ sign.

In summary, if $\boldsymbol{m_1}$ and $\boldsymbol{m_2}$ are **slopes** of two lines, then the lines are:

- **parallel** iff $\boldsymbol{m_1 = m_2}$, and
- **perpendicular** iff $\boldsymbol{m_1 = -\frac{1}{m_2}}$ (or equivalently, if $\boldsymbol{m_1 \cdot m_2 = -1}$)
  In addition, a **horizontal** line (with a slope of **zero**) is **perpendicular** to a **vertical** line (with **undefined** slope).

Example 6 ► **Determining Whether the Given Lines are Parallel, Perpendicular, or Neither**

For each pair of linear equations, determine whether the lines are parallel, perpendicular, or neither.

a. $3x + 5y = 7$
$5x - 3y = 4$

b. $y = x$
$2x - 2y = 5$

c. $y = 5$
$y = 5x$

Solution ►

a. As seen in *section G1*, the slope of a line given by an equation in standard form, $Ax + By = C$, is equal to $-\frac{A}{B}$. One could confirm this by solving the equation for y and taking the coefficient by x for the slope.
Using this fact, the slope of the line $3x + 5y = 7$ is $-\mathbf{\frac{3}{5}}$, and the slope of $5x - 3y = 4$ is $\mathbf{\frac{5}{3}}$. Since these two slopes are opposite reciprocals of each other, the two lines are **perpendicular**.

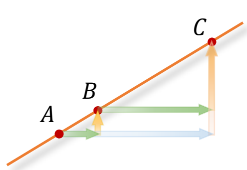
b. The slope of the line $y = x$ is **1** and the slope of $2x - 2y = 5$ is also $\frac{2}{2} = \mathbf{1}$. So, the two lines are parallel.

c. The line $y = 5$ can be seen as $y = 0x + 5$, so its slope is **0**. The slope of the second line, $y = 5x$, is **5**. So, the two lines are neither parallel nor perpendicular.

Collinear Points

Definition 2.2 ► Points that lie on the same line are called **collinear**.

Two points are always collinear because there is only one line passing through these points. The question is how could we check if a third point is collinear with the given two points? If we have an equation of the line passing through the first two points, we could plug in the coordinates of the third point and see if the equation is satisfied. If it is, the third point is collinear with the other two. But, can we check if points are collinear without referring to an equation of a line?



Notice that if several points lie on the same line, the slope between any pair of these points will be equal to the slope of this line. So, these slopes will be the same. One can also show that if the slopes between any two points in the group are the same, then such points lie on the same line. So, they are collinear.

Points are **collinear** iff the **slope** between each pair of points **is the same**.

Example 7 ► **Determine Whether the Given Points are Collinear**

Determine whether the points $A(-3,7)$, $B(-1,2)$, and $C = (3,-8)$ are collinear.

Solution ► Let m_{AB} represent the slope of $\overline{AB}$ and m_{BC} represent the slope of $\overline{BC}$. Since

$$m_{AB} = \frac{2-7}{-1-(-3)} = -\frac{5}{2} \quad \text{and} \quad m_{BC} = \frac{-8-2}{3-(-1)} = -\frac{10}{4} = -\frac{5}{2},$$

Then all points A, B, and C lie on the same line. Thus, they are collinear.

Example 8 ► **Finding the Missing Coordinate of a Collinear Point**

For what value of y are the points $P(2, 2)$, $Q(-1, y)$, and $R(1, 6)$ collinear?

Solution ► For the points P, Q, and R to be collinear, we need the slopes between any two pairs of these points to be equal. For example, the slope m_{PQ} should be equal to the slope m_{PR}. So, we solve the equation

$$m_{PQ} = m_{PR}$$

for y:

$$\frac{y-2}{-1-2} = \frac{6-2}{1-2}$$

$$\frac{y-2}{-3} = -4 \qquad /\cdot(-3)$$

$$y - 2 = 12 \qquad /+2$$

$$y = 14$$

Thus, point Q is collinear with points P and R, if $y = \mathbf{14}$.

G.2 Exercises

Vocabulary Check *Fill in each blank with the most appropriate term or phrase from the given list:* ***slope, undefined, increases, negative, collinear, opposite reciprocals, parallel, zero.***

1. The average rate of change between two points on a graph is measured by the ____________ of the line segment connecting the two points.

2. A vertical line has _______________ slope. The slope of a horizontal line is ________ .

3. A line with a positive slope _______________ from left to right.

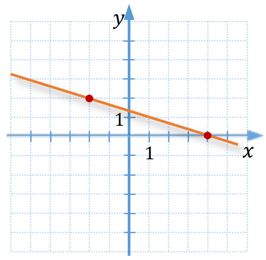
4. A decreasing line has a ______________ slope.

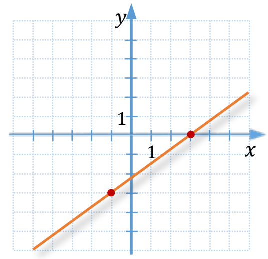
5. If the slope between each pair of points is constantly the same, then the points are _________________.

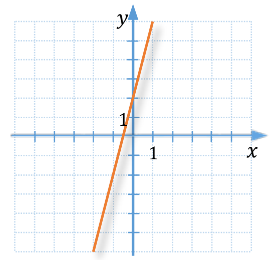
6. _____________ lines have the same slopes.

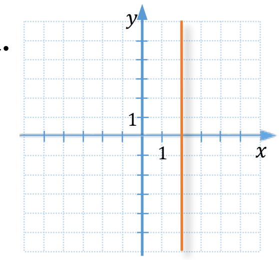
7. The slopes of perpendicular lines are ____________ ______________ .

Concept Check *Given the graph, find the slope of each line.*

8. 

9. 

10. 

11. 

Concept Check *Given the equation, find the slope of each line.*

12. $y = \frac{1}{2}x - 7$

13. $y = -\frac{1}{3}x + 5$

14. $4x - 5y = 2$

15. $3x + 4y = 2$

16. $x = 7$

17. $y = -\frac{3}{4}$

18. $y + x = 1$

19. $-8x - 7y = 24$

20. $-9y - 36 + 4x = 0$

Concept Check *Graph each line satisfying the given information.*

21. passing through $(-2, -4)$ with slope $m = 4$

22. passing through $(-1, -2)$ with slope $m = -3$

23. passing through $(-3, 2)$ with slope $m = \frac{1}{2}$

24. passing through $(-3,4)$ with slope $m = -\frac{2}{5}$

25. passing through $(2, -1)$ with undefined slope

26. passing through $(2, -1)$ with slope $m = 0$

Concept Check

27. Which of the following forms of the slope formula are correct?

a. $m = \frac{y_1 - y_2}{x_2 - x_1}$

b. $m = \frac{y_1 - y_2}{x_1 - x_2}$

c. $m = \frac{x_2 - x_1}{y_2 - y_1}$

d. $m = \frac{y_2 - y_1}{x_2 - x_1}$

Concept Check *Find the slope of the line through each pair of points.*

28. $(-2,2),\ (4,5)$

29. $(8,7),\ (2,-1)$

30. $(9,-4),\ (3,-8)$

31. $(-5,2),\ (-9,5)$

32. $(-2,3),\ (7,-12)$

33. $(3,-1),\ \left(-\frac{1}{2}, \frac{1}{5}\right)$

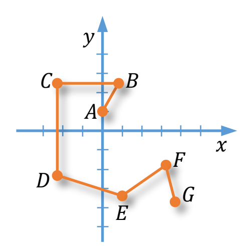
34. $(-5,2),\ (8,2)$

35. $(-3,4),\ (-3,10)$

36. $\left(\frac{1}{2}, 6\right),\ \left(-\frac{2}{3}, \frac{5}{2}\right)$

Concept Check

37. List the line segments in the accompanying figure with respect to their slopes, from the smallest to the largest slope. List the segment with an undefined slope as last.



38. ***Concept Check*** *Match each situation in **a–d** with the most appropriate graph in **A–D**.*

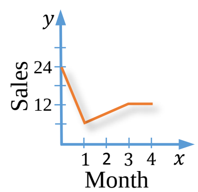
a. Sales rose sharply during the first month and then leveled off during the second month before falling steadily to the original size by the end of the fourth month.

b. Sales fell sharply during the first month and then rose slowly during the second and third months before leveling off during the fourth month.

c. Sales rose sharply during the first month and then fell to the original level during the second month before rising steadily for the next two months.

d. Sales fell during the first two months, leveled off during the third month, and came back to the original level during the fourth month.

A.



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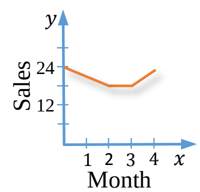
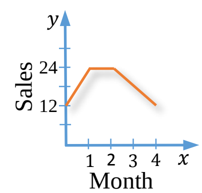
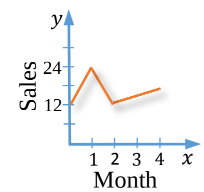


C.

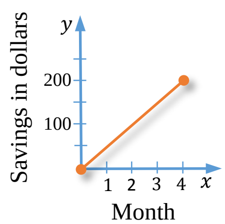


D.



Find and interpret the average rate of change illustrated in each graph.

39.



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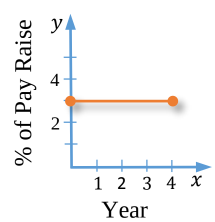

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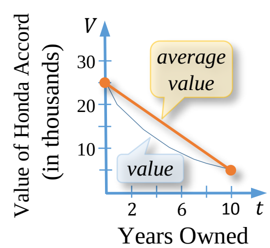

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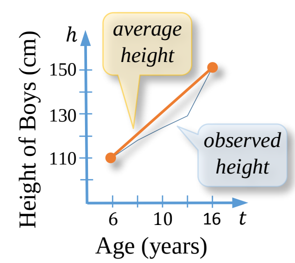

Analytic Skills *Sketch a graph that models the given situation.*

43. The distance that a cyclist is from home if he is initially 20 miles away from home and arrives home after riding at a constant speed for 2 hours.

44. The distance that an athlete is from home if the athlete runs away from home at 8 miles per hour for 30 minutes and then walks back home at 4 miles per hour.

45. The distance that a person is from home if this individual drives (at a constant speed) to a mall, stays 2 hours, and then drives home, assuming that the distance to the mall is 20 miles and that the trip takes 30 minutes.

46. The amount of water in a 10,000-gallon swimming pool that is filled at the rate of 1000 gallons per hour, left full for 10 hours, and then drained at the rate of 2000 gallons per hour.

Analytic Skills *Solve each problem.*

47. A 80,000-liters swimming pool is being filled at a constant rate. Over a 5-hour period, the water in the pool increases from $\frac{1}{4}$ full to $\frac{5}{8}$ full. At what rate is water entering the pool?

48. An airplane on a 1,800-kilometer trip is flying at a constant rate. Over a 2-hour period, the location of the plane changes from covering $\frac{1}{3}$ of the distance to covering $\frac{3}{4}$ of the distance. What is the speed of the airplane?

Discussion Point

49. Suppose we see a road sign informing that a road grade is 7% for the next 1.5 miles. In meters, what would be the expected change in elevation 1.5 miles down the road? (*Recall:* 1 mile ≈ 1.61 kilometers)

Concept Check *Decide whether each pair of lines is parallel, perpendicular, or neither.*

50. $y = x$
$y = -x$

51. $y = 3x - 6$
$y = -\frac{1}{3}x + 5$

52. $2x + y = 7$
$-6x - 3y = 1$

53. $x = 3$
$x = -2$

54. $3x + 4y = 3$
$3x - 4y = 5$

55. $5x - 2y = 3$
$2x - 5y = 1$

56. $y - 4x = 1$
$x + 4y = 3$

57. $y = \frac{2}{3}x - 2$
$-2x + 3y = 6$

Concept Check *Solve each problem.*

58. Check whether or not the points $(-2, 7)$, $(1, 5)$, and $(3, 4)$ are collinear.

59. The following points, $(2, 2)$, $(-1, k)$, and $(1, 6)$ are collinear. Find the value of k.