G.3		Forms of Linear Equations in Two Variables						
	L g o a	inear equations in two variables can take different forms. Some forms are easier to use for raphing, while others are more suitable for finding an equation of a line given two pieces of information. In this section, we will take a closer look at various forms of linear equations nd their utilities.						
Forms of Linea	ar Equ	lations						
	T fe	The form of a linear equation that is most useful for graphing lines is the slope-intercept orm, as introduced in <i>section G1</i> .						
Definition 3.1	Г	The slope-intercept form of the equation of a line with slope <i>m</i> and <i>y</i> -intercept (0, <i>b</i>) is						
		y = mx + b.						
Example 1	V	Vriting and Graphing Equation of a Line in Slope-Intercept Form						
	V tl	Vrite the equation in slope-intercept form of the line satisfying the given conditions, and hen graph this line.						
	a	slope $-\frac{4}{5}$ and y-intercept (0, -2)						
	b	slope $\frac{1}{2}$ and passing through (2, -5)						
Solution	a	 To write this equation, we substitute m = -⁴/₅ and b = -2 into the slope-intercept form. So, we obtain y = -⁴/₅x - 2. To graph this line, we start with plotting the y-intercept (0, -2). To find the second point, we follow the slope, as in <i>Example 2, section G2</i>. According to the slope -⁴/₅ = ⁻⁴/₅, starting from (0, -2), we could run 5 units to the right and 4 units down, but then we would go out of the grid. So, this time, let the negative sign in the slope be kept in the denominator, ⁴/₋₅. Thus, we run 5 units to the left and 4 units up to reach the point (0, -2). Then we draw the line by connecting the two points. Since m = ¹/₂, our equation has a form y = ¹/₂x + b. To find b, we substitute point 						
		(2, -5) into this equation and solve for b. So $-5 = \frac{1}{2}(2) + b$ gives us -5 = 1 + b and finally b = -6.						

Therefore, our equation of the line is $y = \frac{1}{2}x - 6$.

We graph it, starting by plotting the given point (2, -5) and finding the second point by following the slope of $\frac{1}{2}$, as described in *Example 2, section G2*.



The form of a linear equation that is most useful when writing equations of lines with unknown *y*-intercept is the slope-point form.

Definition 3.2 The slope-point form of the equation of a line with slope m and passing through the point (x_1, y_1) is

 $y-y_1=m(x-x_1).$

This form is based on the defining property of a line. A line can be defined as a set of points with a constant slope m between any two of these points. So, if (x_1, y_1) is a given (fixed) point of the line and (x, y) is any (variable) point of the line, then, since the slope is equal to m for all such points, we can write the equation

$$m = \frac{y - y_1}{x - x_1}.$$

After multiplying by the denominator, we obtain the slope-point formula, as in *Definition* 3.2.

Example 2 • Writing Equation of a Line Using Slope-Point Form

Use the slope-point form to write an equation of the line satisfying the given conditions. Leave the answer in the slope-intercept form and then graph the line.

- **a.** slope $-\frac{2}{3}$ and passing through (1, -3)
- **b.** passing through points (2, 5) and (-1, -2)

Solution

a. To write this equation, we plug the slope $m = -\frac{2}{3}$ and the coordinates of the point (1, -3) into the slope-point form of a line. So, we obtain

$$y - (-3) = -\frac{2}{3}(x - 1)$$

$$y + 3 = -\frac{2}{3}x + \frac{2}{3} \qquad /-3$$

$$y = -\frac{2}{3}x + \frac{2}{3} - \frac{9}{3}$$

$$y = -\frac{2}{3}x - \frac{7}{3}$$

To graph this line, we start with plotting the point (1, -3) and then apply the slope of $-\frac{2}{3}$ to find additional points that belong to the line.



$$m = \frac{\Delta y}{\Delta x} = \frac{-2-5}{-1-2} = \frac{-7}{-3} = \frac{7}{3}$$

Then, using the calculated slope and one of the given points, for example (2, 5), we write the slope-point equation of the line

and solve it for *y*:



To graph this line, it is enough to connect the two given points.

One of the most popular forms of a linear equation is the standard form. This form is helpful when graphing lines based on x- and y-intercepts, as illustrated in *Example 3, section G1*.

Definition 3.3 • The **standard form** of a linear equation is

Ax + By = C,

Where $A, B, C \in \mathbb{R}$, A and B are not both 0, and $A \ge 0$.

When writing linear equations in standard form, the expectation is to use a **nonnegative** coefficient *A* and clear any fractions, if possible. For example, to write $-x + \frac{1}{2}y = 3$ in standard form, we multiply the equation by (-2), to obtain 2x - y = -6. In addition, we prefer to write equations in simplest form, where the greatest common factor of *A*, *B*, and *C* is 1. For example, we prefer to write 2x - y = -6 rather than any multiple of this equation, such as 4x - 2y = -12, or 6x - 3y = -18.

Observe that if $B \neq 0$ then the **slope** of the line given by the equation Ax + By = C is $-\frac{A}{B}$. This is because after solving this equation for y, we obtain $y = -\frac{A}{B}x + \frac{C}{B}$. If B = 0, then the slope is **undefined**, as we are unable to divide by zero. The form of a linear equation that is most useful when writing equations of lines based on their x- and y-intercepts is the intercept form.

Definition 3.4 **•** The intercept form of a linear equation is $\frac{x}{a} + \frac{y}{b} = 1$, where *a* is the *x*-intercept and *b* is the *y*-intercept of the line. We should be able to convert a linear equation from one form to another. Example 3 **Converting a Linear Equation to a Different Form** Write the equation 3x + 7y = 2 in slope-intercept form. a. Write the equation $y = \frac{3}{5}x + \frac{7}{2}$ in standard form. Write the equation $\frac{x}{4} - \frac{y}{3} = 1$ in standard form. b. c. Solution To write the equation 3x + 7y = 2 in slope-intercept form, we solve it for y. a. 3x + 7y = 2/-3x7v = -3x + 2 /÷ 7 $y = -\frac{3}{7}x + \frac{2}{7}$ **b.** To write the equation $y = \frac{3}{5}x + \frac{7}{2}$ in standard form, we bring the *x*-term to the left side of the equation and multiply the equation by the LCD, with the appropriate sign. $y = \frac{3}{5}x + \frac{7}{2} \qquad \qquad /-\frac{3}{5}x$ $-\frac{3}{5}x + y = \frac{7}{2}$ /· (-10) 6x - 10y = -35To write the equation $\frac{x}{4} - \frac{y}{3} = 1$ in standard form, we multiply it by the LCD, with the c. appropriate sign. $\frac{x}{4} - \frac{y}{3} = 1$ / 12 3x - 4y = 12Example 4 Writing Equation of a Line Using Intercept Form

Write an equation of the line passing through points (0, -2) and (7, 0). Leave the answer in standard form.

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(a,b)

х

Since point (0, -2) is the *y*-intercept and point (7, 0) is the *x*-intercept of our line, to write the equation of the line we can use the intercept form with a = -2 and b = 7. So, we have

$$\frac{x}{-2} + \frac{y}{7} = 1.$$

To change this equation to standard form, we multiply it by the LCD = -14. Thus,

$$7x - 2y = -14.$$

Equations representing horizontal or vertical lines are special cases of linear equations in standard form, and as such, they deserve special consideration.

The horizontal line passing through the point (a, b) has equation y = b, while the vertical line passing through the same point has equation x = a.

The equation of a **horizontal line**, y = b, can be shown in standard form as 0x + y = b. Observe, that the slope of such a line is $-\frac{0}{1} = 0$.

The equation of a vertical line, x = a, can be shown in standard form as x + 0y = a. Observe, that the slope of such a line is $-\frac{1}{a} =$ undefined.

Example 5 • Writing Equations of Horizontal and Vertical Lines

Find equations of the vertical and horizontal lines that pass through the point (3, -2). Then, graph these two lines.

Solution

Since x-coordinates of all points of the vertical line, including (3, -2), are the same, then these x-coordinates must be equal to 3. So, the equation of the vertical line is x = 3.

Since y-coordinates of all points of a horizontal line, including (3, -2), are the same, then these y-coordinates must be equal to -2. So, the equation of the horizontal line is y = -2.



Here is a summary of the various forms of linear equations.

Forms of Linear Equations								
Equation	Description	When to Use						
y = mx + b	Slope-Intercept Form slope is <i>m</i> <i>y</i> -intercept is (0, <i>b</i>)	This form is ideal for graphing by using the <i>y</i> -intercept and the slope.						
$y - y_1 = m(x - x_1)$	Slope-Point Form slope is m the line passes through (x_1, y_1)	This form is ideal for finding the equation of a line if the slope and a point on the line, or two points on the line, are known.						

Forms of Linear Equations in Two Variables

Ax + By = C	Standard Form slope is $-\frac{A}{B}$, if $B \neq 0$ <i>x</i> -intercept is $\left(\frac{C}{A}, 0\right)$, if $A \neq 0$. <i>y</i> -intercept is $\left(0, \frac{C}{B}\right)$, if $B \neq 0$.	This form is useful for graphing, as the x - and y -intercepts, as well as the slope, can be easily found by dividing appropriate coefficients.
$\frac{x}{a} + \frac{y}{b} = 1$	Intercept Form slope is $-\frac{b}{a}$ <i>x</i> -intercept is (<i>a</i> , 0) <i>y</i> -intercept is (0, <i>b</i>)	This form is ideal for graphing, using the x - and y -intercepts.
y = b	Horizontal Line slope is 0 y-intercept is (0, b)	This form is used to write equations of, for example, horizontal asymptotes.
x = a	Vertical Line slope is undefined x-intercept is (a , 0)	This form is used to write equations of, for example, vertical asymptotes.

Note: Except for the equations for a horizontal or vertical line, all of the above forms of linear equations can be converted into each other via algebraic transformations.

Writing Equations of Parallel and Perpendicular Lines

Recall that the slopes of parallel lines are the same, and slopes of perpendicular lines are opposite reciprocals. See *section* G2.

Example 6 • Writing Equations of Parallel Lines Passing Through a Given Point

Find the slope-intercept form of a line parallel to y = -2x + 5 that passes through the point (-4,5). Then, graph both lines on the same grid.

Solution Since the line is parallel to y = -2x + 5, its slope is -2. So, we plug the slope of -2 and the coordinates of the point (-4,5) into the slope-point form of a linear equation.

$$y - 5 = -2(x + 4)$$

This can be simplified to the slope-intercept form, as follows:

$$y - 5 = -2x - 8$$
$$y = -2x - 3$$



As shown in the accompanying graph, the line y = -2x - 3 (in orange) is parallel to the line y = -2x + 5 (in green) and passes through the given point (-4,5).

Example 7 • Writing Equations of Perpendicular Lines Passing Through a Given Point

Find the slope-intercept form of a line perpendicular to 2x - 3y = 6 that passes through the point (1,4). Then, graph both lines on the same grid.

Solution The slope of the given line, 2x - 3y = 3, is $\frac{2}{3}$. To find the slope of a perpendicular line, we take the opposite reciprocal of $\frac{2}{3}$, which is $-\frac{3}{2}$. Since we already know the slope and the point, we can plug these pieces of information into the slope-point formula. So, we have



As shown in the accompanying graph, the line 2x - 3y = 6 (in orange) is indeed perpendicular to the line $y = -\frac{3}{2}x + \frac{11}{2}$ (in green) and passes through the given point (1,4).

Linear Equations in Applied Problems

Linear equations can be used to model a variety of applications in sciences, business, and other areas. Here are some examples.

Example 8 🕨	Given the Rate of Change and the Initial Value, Determine the Linear Model Relating the Variables
	A young couple buys furniture for \$2000, agreeing to pay \$200 down and \$100 at the end of each month until the entire debt is paid off.
	a. Write an equation to express the amount paid off, P , in terms of the number of monthly payments, m .
	b. Graph the equation found in part a .
	c. Use the graph to estimate how long it will take to pay off the debt.
Solution >	a. Since each month the couple pays \$100, after m months, the amount paid off by the monthly installments is 100 m . If we add the initial payment of \$200, the equation representing the amount paid off can be written as

$$P = 100m + 200$$

b. To graph this equation, we use the slope-intercept method. Starting with the *P*-intercept of 200, we run 1 and rise 100, repeating this process as many times as needed to hit a lattice point on the chosen scale. So, as shown in the accompanying graph, the line passes through points (6, 800) and (18, 2000).



c. As shown in the graph, \$2000 will be paid off in 18 months.

Example 9 Finding a Linear Equation that Fits the Data Given by Two Ordered Pairs

Gabriel Daniel Fahrenheit invented the mercury thermometer in 1717. The thermometer shows that water freezes at 32°F and boils at 212°F. In 1742, Anders Celsius invented the Celsius temperature scale. On this scale, water freezes at 0°C and boils at 100°C. Determine a linear equation that can be used to predict the Celsius temperature, C, when the Fahrenheit temperature, F, is known.

Solution

50- -120

40-100

30- -80

20-10--60

To predict the Celsius temperature, C, knowing the Fahrenheit temperature, F, we treat the variable C as dependent on the variable F. So, we consider C as the second coordinate when setting up the ordered pairs, (F, C), of given data. The corresponding freezing temperatures give us the pair (32,0) and the boiling temperatures give us the pair (212,100). To find the equation of a line passing through these two points, first, we calculate the slope, and then, we use the slope-point formula. So, the slope is

$$m = \frac{100 - 0}{212 - 32} = \frac{100}{180} = \frac{5}{9}$$

and using the point (32,0), the equation of the line is

$$C=\frac{5}{9}(F-32)$$

Example 10

Determining if the Given Set of Data Follows a Linear Pattern

Determine whether the data given in each table follow a linear pattern. If they do, find the slope-intercept form of an equation of the line passing through all the given points.

a.	x	1	3	5	7	9	b.	x	10	20	30	40	5(
	y	12	16	20	24	28		y	15	21	26	30	35

Solution

a. The set of points follows a linear pattern if the slopes between consecutive pairs of these points are the same. These slopes are the ratios of increments in *y*-values to increments in *x*-values. Notice that the increases between successive *x*-values of the given points are constantly equal to 2. So, to check if the points follow a linear pattern, it is enough to check if the increases between successive *y*-values are also constant. Observe that the numbers in the list 12, 16, 20, 24, 28 steadily increase by 4. Thus, the given set of data follow a linear pattern.

To find an equation of the line passing through these points, we use the slope, which is $\frac{4}{2} = 2$, and one of the given points, for example (1,12). By plugging these pieces of information into the slope-point formula, we obtain

y - 12 = 2(x - 1),

which after simplifying becomes

y - 12 = 2x - 2 /+12 y = 2x + 10

b. Observe that the increments between consecutive *x*-values of the given points are constantly equal to 10, while the increments between consecutive *y*-values in the list 15, 21, 26, 30, 35 are 6, 5, 4, 5. So, they are not constant. Therefore, the given set of data does not follow a linear pattern.

Example 11 🕨	Finding a Linear Model Relating the Number of Items Bought at a Fixed Amount							
	 A manager for a country market buys apples at \$0.25 each and pears at \$0.50 each. Write a linear equation in standard form relating the number of apples, <i>a</i>, and pears, <i>p</i>, she can buy for \$80. Then, a. graph the equation and b. using the graph, find at least 3 points (<i>a</i>, <i>p</i>) satisfying the equation, and interpret their meanings in the context of the problem. 							
Solution >	It costs $0.25a$ dollars to buy <i>a</i> apples. Similarly, it costs $0.50p$ dollars to buy <i>p</i> pears. Since the total charge is \$80, we have							
	0.25a + 0.50p = 80							
	We could convert the coefficients into integers by multiplying the equation by a hundred. So, we obtain							
	25a + 50p = 8000,							
	which, after dividing by 25, turns into							
	a+2p=320.							
	 a. To graph this equation, we will represent the number of apples, a, on the horizontal axis and the number of pears, p, on the vertical axis, respecting the alphabetical order of labelling the axes. Using the intercept method, we connect points (320,0) and (0,160). 							
	 b. Aside of the intercepts, (320,0) and (0,160), the graph shows us a few more points that satisfy the equation. In particular, (80, 120) and (160, 80) are points of the graph. If a point (a, p) of the graph has integral coefficients, it tells us that for \$80, the manager could buy a apples and p pears. For example, the point (80, 120) tells us that the manager can buy 80 apples and 120 pears for \$80. 							

G.3 Exercises

Vocabulary Check Fill in each blank with the most appropriate term or phrase from the given list: **b**, coefficients, intercept, parallel, slope-point, standard, x-intercept, x = a, y-intercept, y = b.

- 1. When graphing a linear equation written in the slope-intercept form, we first plot the ______.
- 2. To write a linear equation when two points on the line are given, we usually use the ______ form.
- **3.** When writing a linear equation in ______ form, we start with a positive *x*-term followed by the *y*-term. Also, if possible, we clear all the fractional ______.
- 4. The equation of a vertical line passing through the point (*a*, *b*) is _____.
- 5. The equation of a horizontal line passing through the point (*a*, *b*) is _____.
- 6. The linear equation $\frac{x}{a} + \frac{y}{b} = 1$ is written in the ______ form. In this form, the value *a* represents the ______, while the value ______ represents the *y*-intercept.
- 7. Two lines that have no points in common are ______.

Concept Check Write each equation in standard form.

8. $y = -\frac{1}{2}x - 7$ 9. $y = \frac{1}{3}x + 5$ 10. $\frac{x}{5} + \frac{y}{-4} = 1$ 11. $y - 7 = \frac{3}{2}(x - 3)$ 12. $y - \frac{5}{2} = -\frac{2}{3}(x + 6)$ 13. 2y = -0.21x + 0.35

Concept Check Write each equation in slope-intercept form.

14. $3y = \frac{1}{2}x - 5$ **15.** $\frac{x}{3} + \frac{y}{5} = 1$ **16.** 4x - 5y = 10**17.** 3x + 4y = 7**18.** $y + \frac{3}{2} = \frac{2}{5}(x + 2)$ **19.** $y - \frac{1}{2} = -\frac{2}{3}\left(x - \frac{1}{2}\right)$

Concept Check Write an equation in slope-intercept form of the line shown in each graph.



Find an equation of the line that satisfies the given conditions. Write the equation in **slope-intercept** and **standard** form.

- **24.** through (-3,2), with slope $m = \frac{1}{2}$
- **25.** through (-2,3), with slope m = -4

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26.	with slope $m = \frac{3}{2}$ and y-intercept at -1	27.	with slope $m = -\frac{1}{5}$ and <i>y</i> -intercept at 2
28.	through $(-1, -2)$, with <i>y</i> -intercept at -3	29.	through (-4,5), with <i>y</i> -intercept at $\frac{3}{2}$
30.	through $(2, -1)$ and $(-4, 6)$	31.	through $(3,7)$ and $(-5,1)$
32.	through $\left(-\frac{4}{3}, -2\right)$ and $\left(\frac{4}{5}, \frac{2}{3}\right)$	33.	through $\left(\frac{4}{3}, \frac{3}{2}\right)$ and $\left(-\frac{1}{2}, \frac{4}{3}\right)$
г.		1	

Find an equation of the line that satisfies the given conditions.

34.	through $(-5,7)$, with slope 0	35.	through $(-2, -4)$, with slope 0
36.	through $(-1, -2)$, with undefined slope	37.	through $(-3,4)$, with undefined slope
38.	through $(-3,6)$ and horizontal	39.	through $\left(-\frac{5}{3},-\frac{7}{2}\right)$ and horizontal
40.	through $\left(-\frac{3}{4},-\frac{3}{2}\right)$ and vertical	41.	through $(5, -11)$ and vertical

Concept Check Write an equation in standard form for each of the lines described. In each case make a sketch of the given line and the line satisfying the conditions.

42.	through (7,2) and parallel to $3x - y = 4$	43.	through (4,1) and parallel to $2x + 5y = 10$
44.	through $(-2,3)$ and parallel to $-x + 2y = 6$	45.	through $(-1, -3)$ and parallel to $-x + 3y = 12$
46.	through $(-1,2)$ and parallel to $y = 3$	47.	through $(-1,2)$ and parallel to $x = -3$
48.	through (6,2) and perpendicular to $2x - y = 5$	49.	through (0,2) and perpendicular to $5x + y = 15$
50.	through $(-2,4)$ and perpendicular to $3x + y = 6$	51.	through $(-4, -1)$ and perpendicular to $x - 3y = 9$
52.	through $(3, -4)$ and perpendicular to $x = 2$	53.	through $(3, -4)$ and perpendicular to $y = -3$

Analytic Skills For each situation, write an equation in the form y = mx + b, and then answer the question of the problem.

- 54. Membership in the Midwest Athletic Club costs 99, plus 41 per month. Let x represent the number of months and y represent the cost. How much does one-year membership cost?
- 55. A cell phone plan includes 900 anytime minutes for \$60 per month, plus a one-time activation fee of \$36. Acell phone is included at no additional charge. Let *x* represent the number of months of service and *y* represent the cost. If you sign a 1-yr contract, how much will this cell phone plan cost?
- 56. There is a \$30 fee to rent a chainsaw, plus \$6 per day. Let x represent the number of days the saw is rented and y represent the total charge to the renter, in dollars. If the total charge is \$138, for how many days is the saw rented?
- **57.** A rental car costs \$50 plus \$0.12 per kilometer. Let x represent the number of kilometers driven and y represent the total charge to the renter, in dollars. How many kilometers was the car driven if the renter paid \$84.20?

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Analytic Skills Solve each problem.

- **58.** At its inception, a professional organization had 26 members. Three years later, the organization had grown to 83 members. If membership continues to grow at the same rate, find an equation that represents the number n of members in the organization after t years.
- **59.** Thirty minutes after a truck driver passes the 142-km marker on a freeway, he passes the 170-km marker. Find an equation that shows the distance d he drives in t hr.
- **60.** The average annual cost of a private college or university is shown in the table. This cost includes tuition, fees, room, and board.

Year y	2007	2016		
Cost C	\$37000	\$72000		

- **a.** Find the slope-intercept form of a line that passes through these two data points.
- **b.** Interpret the slope in the context of the problem.
- c. To the nearest thousand, estimate the cost of private college or university in 2020.
- **61.** The life expectancy for a person born in 1900 was 48 years, and in 2000 it was 77 years. To the nearest year, estimate the life expectancy for someone born in 1970.
- 62. After 2 years, the amount in a savings account earning simple interest was \$1070. After 5 years, the amount in the account was \$1175. Find an equation that represents the amount A in the account after t years.
- **63.** A real-estate agent receives a flat monthly salary plus a 0.5% commission on her monthly home sales. In a particular month, her home sales were \$500,000, and her total monthly income was \$4300.
 - **a.** Write an equation in slope-intercept form that shows the real-estate agent's total monthly income I in terms of her monthly home sales s.
 - **b.** Graph the equation on the coordinate plane.
 - **c.** What does the *I*-intercept represent in the context of the problem?
 - **d.** What does the slope represent in the context of the problem?
- **64.** A taxi company charges a flat meter fare of \$1.25 plus an additional fee for each kilometer (or part thereof) driven. A passenger pays \$10.25 for a 6-kilometer taxi ride.
 - **a.** Find an equation in slope-intercept form that models the total meter fare f in terms of the number k of kilometers driven.
 - **b.** Graph the equation on the coordinate plane.
 - c. What does the slope of the graph of the equation in part **a**. represent in this situation?
 - d. How many kilometers were driven if a passenger pays \$20.75?
- **65.** Fold a string like this:





Figure 3.1

Count how many pieces of string you would have after cutting the string as shown in *Figure 3.1*. Predict how many pieces of string you would have if you made 2, 3, or more such cuts. Complete the table below and determine whether or not the data in the table follow a linear pattern. Can you find an equation that predicts the number of pieces if you know the number of cuts?

# of cuts	0	1	2	3	4	5
# of pieces						



