G.4 Linear Inequalities in Two Variables Including Systems of Inequalities

In many real-life situations, we are interested in a range of values satisfying certain conditions rather than in one specific value. For example, when exercising, we like to keep the heart rate between 120 and 140 beats per minute. The systolic blood pressure of a healthy person is usually between 100 and 120 mmHg (millimeters of mercury). Such conditions can be described using inequalities. Solving systems of inequalities has its applications in many practical business problems, such as how to allocate resources to achieve a maximum profit or a minimum cost. In this section, we study graphical solutions of linear inequalities and systems of linear inequalities.

Linear Inequalities in Two Variables

Definition 4.1 ► Any inequality that can be written as

$$\boldsymbol{Ax+By<C},\ \ \boldsymbol{Ax+By\le C},\ \ \boldsymbol{Ax+By>C},\ \ \boldsymbol{Ax+By\ge C}, \text{ or } \boldsymbol{Ax+By\neq C},$$

where $A, B, C \in \mathbb{R}$ and A and B are not both 0, is a **linear inequality in two variables**.

To **solve** an inequality in two variables, x and y, means to **find <u>all</u> ordered pairs** $(\boldsymbol{x}, \boldsymbol{y})$ satisfying the inequality.

Inequalities in two variables arise from many situations. For example, suppose that the number of full-time students, f, and part-time students, p, enrolled in upgrading courses at the University of the Fraser Valley is at most 1200. This situation can be represented by the inequality

$$f+p\le 1200.$$

Some of the solutions (f, p) of this inequality are: $(1000, 200)$, $(1000, 199)$, $(1000, 198)$, $(600, 600)$. $(550, 600)$, $(1100, 0)$, and many others.

The solution sets of inequalities in two variables contain infinitely many ordered pairs of numbers which, when graphed in a system of coordinates, fulfill specific regions of the coordinate plane. That is why it is more beneficial to present such solutions in the form of a graph rather than using set notation. To graph the region of points satisfying the inequality $f+p\le 1200$, we may want to solve it first for p,

$$p\le -f+1200,$$

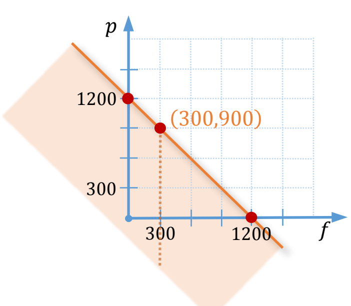


Figure 1a

and then graph the related equation, $p=-f+1200$, called the **boundary line**. Notice, that setting f to, for instance, 300 results in the inequality

$$p\le -300+1200=900.$$

So, any point with the first coordinate of 300 and the second coordinate of 900 or less satisfies the inequality (see the dotted half-line in *Figure 1a*). Generally, observe that any point with the first coordinate f and the second coordinate $-f+1200$ or less satisfies the inequality. Since the union of all half-lines that start from the boundary line and go down is the whole half-plane below the boundary line, we shade it as the solution set to the discussed inequality (see *Figure 1a*). The solution set also includes the points of the boundary line, as the inequality includes equation.

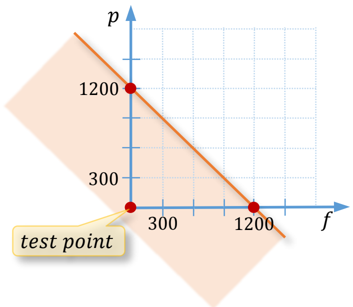


Figure 1b

The above strategy can be applied to any linear inequality in two variables. Hence, one can conclude that the solution set to a given linear inequality in two variables consists of **all points of one of the half-planes** obtained by cutting the coordinate plane by the corresponding boundary line. This fact allows us to find the solution region even faster. After graphing the boundary line, to know which half-plane to shade as the solution set, it is enough to check just one point, called a **test point**, chosen outside of the boundary line. In our example, it was enough to test for example point $(0,0)$. Since $0 \leq -0 + 1200$ is a true statement, then the point $(0,0)$ belongs to the solution set. This means that the half-plane containing this test point must be the solution set to the given inequality, so we shade it.

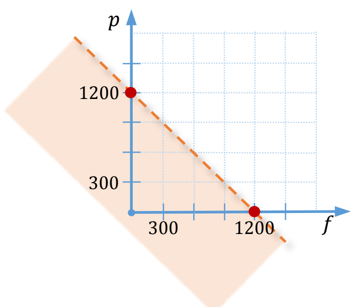


Figure 1c

The solution set of the strong inequality $p < -f + 1200$ consists of the same region as in *Figure 1b*, except for the points on the boundary line. This is because the points of the boundary line satisfy the equation $p = -f + 1200$, but not the inequality $p < -f + 1200$. To indicate this on the graph, we draw the boundary line using a dashed line (see *Figure 1c*).

In summary, to graph the solution set of a linear inequality in two variables, follow the steps:

1. Draw the graph of the corresponding **boundary line**.
   Make the line **solid** if the inequality involves $\leq$ or $\geq$.
   Make the line **dashed** if the inequality involves $<$ or $>$.

2. Choose a **test point** outside of the line and substitute the coordinates of that point into the inequality.

3. If the test point satisfies the original inequality, **shade the half-plane containing the point.**
   If the test point does not satisfy the original inequality, **shade the other half-plane** (the one that does not contain the point).

Example 1 ► **Determining if a Given Ordered Pair of Numbers is a Solution to a Given Inequality**

Determine if the points $(3,1)$ and $(2,1)$ are solutions to the inequality $5x - 2y > 8$.

Solution ► An ordered pair is a solution to the inequality $5x - 2y > 8$ if its coordinates satisfy this inequality. So, to determine whether the pair $(3,1)$ is a solution, we substitute 3 for x and 1 for y. The inequality becomes

$$5 \cdot 3 - 2 \cdot 1 > 8,$$

which simplifies to the true inequality $13 > 8$.
Thus, $(3,1)$ is a solution to $5x - 2y > 8$.

However, replacing x by 2 and y by 1 results in $5 \cdot 2 - 2 \cdot 1 > 8$, or equivalently $8 > 8$. Since 8 is not larger than 8, the point $(2,1)$ does not satisfy the inequality. Thus, $(2,1)$ is not a solution to $5x - 2y > 8$.

Example 2 ► **Graphing Linear Inequalities in Two Variables**

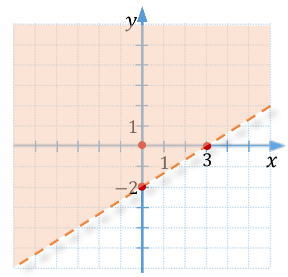
Graph the solution set of each inequality in two variables.

a. $2x - 3y < 6$ **b.** $y \leq 3x - 1$

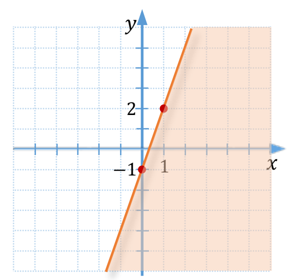
c. $x \geq -3$ **d.** $y \neq x$

Solution ►

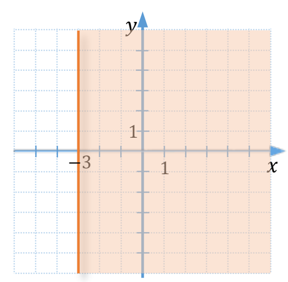
a. First, we graph the boundary line $2x - 3y = 6$, using the x- and y-intercepts: $(3,0)$ and $(0,-2)$. Since the inequality $<$ does not involve an equation, the line is marked as **dashed**, which indicates that the points on the line are not the solutions of the inequality. Then, we choose the point $(0,0)$ for the test point. Since $2 \cdot 0 - 3 \cdot 0 < 6$ is a true statement, then we shade the half-plane containing $(0,0)$ as the solution set.



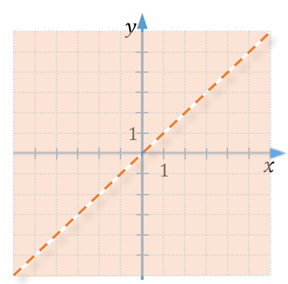
b. First, we graph the boundary line $y \leq 3x - 1$, using the slope and y-intercept. Since the inequality $\leq$ contains an equation, the line is marked as **solid**. This indicates that the points on the line belong to solutions of the inequality. To decide which half-plane to shade as the solution region, we observe that y is lower than or equal to the $3x - 1$, which tells us that the solution points lie below or on the boundary line. So, we shade the half-plane below the line.



c. As before, to graph $x \geq -3$, first, we graph the solid vertical line $x = -3$, and then we shade the half-plane consisting of points with x-coordinates larger or equal to -3. So the solution set is the half-plane to the right of the boundary line, including this line.



d. The solution set of the inequality $y \neq x$ consists of all points that do not satisfy the equation $y = x$. This means that we mark the boundary line as dashed and shade the rest of the points of the coordinate plane.



Systems of Linear Inequalities

Let us refer back to our original problem about the full-time and part-time students that was modelled by the inequality $f + p \leq 1200$. Since f and p represent the number of students, it is reasonable to assume that $f \geq 0$ and $p \geq 0$. This means that we are really interested in solutions to the system of inequalities

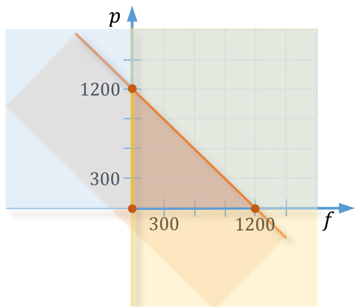


Figure 2

$$\begin{cases} p \leq -f + 1200 \\ f \geq 0 \\ p \geq 0 \end{cases}$$

To find this solution set, we graph each inequality in the same coordinate system. The solutions to the first inequality are marked in orange, the second inequality, in yellow, and the third inequality, in blue (see *Figure 2*). The intersection of the three shadings, orange, yellow, and blue, results in the brown triangular region, including the border lines and the vertices. This is the overall solution set to our system of inequalities. It tells us that the coordinates of any point from the triangular region, including its boundary, could represent the actual number of full-time and part-time students enrolled in upgrading courses during the given semester.

To graph the solution set to a system of inequalities, follow the steps:

1. Using different shadings, graph the solution set to each inequality in the system, drawing the solid or dashed boundary lines, whichever applies.

2. Shade the **intersection** of the solution sets more strongly if the inequalities were connected by the word "**and**". Mark each intersection point of boundary lines with a **filled in** circle if **both** lines are **solid**, or with a **hollow** circle if at least one of the lines is dashed.

or

Shade the **union** of the solution sets more strongly if the inequalities were connected by the word "**or**". Mark each intersection of boundary lines with a **hollow** circle if **both** lines are **dashed**, or with a **filled in** circle if at least one of the lines is **solid**.

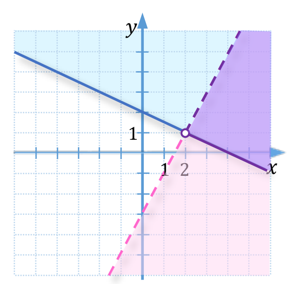
Remember that a brace indicates the "*and*" connection!

Example 3 ► **Graphing Systems of Linear Inequalities in Two Variables**

Graph the solution set to each system of inequalities in two variables.

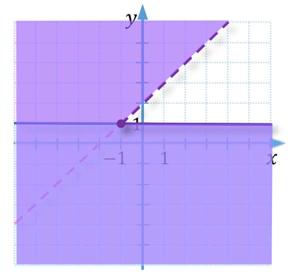
a. $\begin{cases} y < 2x - 3 \\ y \geq -\frac{1}{2}x + 1 \end{cases}$ **b.** $y > x + 2$ or $y \leq 1$

Solution ► **a.** First, we graph the solution set to $y < 2x - 3$ in pink, and the solution set to $y \geq -\frac{1}{2}x + 2$ in blue. Since both inequalities must be satisfied, the solution set of the system is the **intersection** of the solution sets of individual inequalities. So, we shade the overlapping region, in purple, indicating the solid or dashed border lines. Since the



intersection of the boundary lines lies on a dashed line, it does not satisfy one of the inequalities, so it is not a solution to the system. Therefore, we mark it with a hollow circle.

b. As before, we graph the solution set to $y > x + 2$ in pink, and the solution set to $y \leq 1$ in blue. Since the two inequalities are connected with the word "or", we look for the **union** of the two solutions. So, we shade the overall region, in purple, indicating the solid or dashed border lines. Since the intersection of these lines belongs to a solid line, it satisfies one of the inequalities, so it is also a solution of this system. Therefore, we mark it by a filled in circle.



Absolute Value Inequalities in Two Variables

As shown in *section L6*, absolute value linear inequalities can be written as systems of linear inequalities. So we can graph their solution sets, using techniques described above.

Example 4 ► **Graphing Absolute Value Linear Inequalities in Two Variables**

Rewrite the following absolute value inequalities as systems of linear inequalities and then graph them.

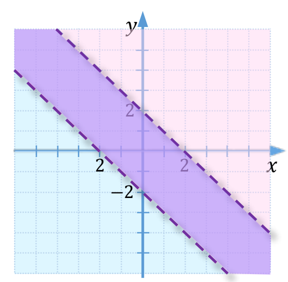
a. $|x + y| < 2$ **b.** $|x + 2| \geq y$ **c.** $|x - 1| \geq 2$

Solution ► **a.** First, we rewrite the inequality $|x + y| < 2$ in the equivalent form of the system of inequalities,

$$-2 < x + y < 2.$$

The solution set to this system is the intersection of the solutions to $-2 < x + y$ and $x + y < 2$. For easier graphing, let us rewrite the last two inequalities in the explicit form

$$\begin{cases} y > -x - 2 \\ y < -x + 2 \end{cases}$$



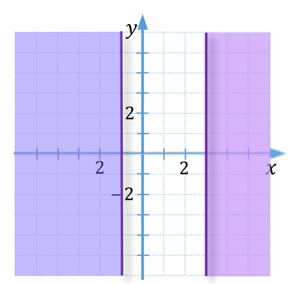
So, we graph $y > -x - 2$ in pink, $y < -x + 2$ in blue, and the final solution, in purple. Since both inequalities are strong (do not contain equation), the boundary lines are dashed.

b. We rewrite the inequality $|x - 1| \geq 2$ in the form of the system of inequalities,

$$x - 1 \geq 2 \text{ or } x - 1 \leq -2,$$

or equivalently as

$$x \geq 3 \text{ or } x \leq -1.$$



Thus, the solution set to this system is the union of the solutions to $x \geq 3$, marked in pink, and $x \leq -1$, marked in blue. The overall solution to the system is marked in purple and includes the boundary lines.

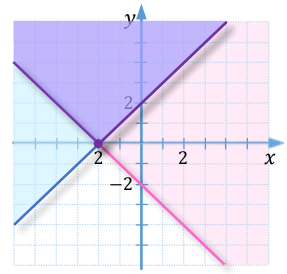
c. We rewrite the inequality $|x+2| \le y$ in the form of the system of inequalities,

$$-y \le x + 2 \le y,$$

or equivalently as

$$y \ge -x - 2 \text{ and } y \ge x + 2.$$

Thus, the solution set to this system is the intersection of the solutions to $y \ge -x - 2$, marked in pink, and $y \ge x + 2$, marked in blue. The overall solution to the system, marked in purple, includes the border lines and the vertex.



G.4 Exercises

Vocabulary Check *Fill in each blank with the most appropriate term from the given list:* ***above, below, boundary, dashed, intersection, satisfies, solid, test, union****.*

1. To graph the solution set to the inequality $y > x + 3$, first, we graph the _______________ line $y = x + 3$. Since equation is not a part of the inequality $>$, the boundary line is marked as a ____________ line.

2. The solution set to the inequality $y > x + 3$ lies _______________ the boundary line.

3. The solution set to the inequality $y \le x + 3$ lies _______________ the boundary line.

4. The boundary line of the solution region to the inequality $y \le x + 3$ is graphed as a ____________ line because the equality is a part of the inequality $\le$.

5. To decide which half-plane to shade when graphing solutions to the inequality $5x - 3y \ge 15$, we use a _______ point that does not lie on the boundary line. We shade the half-plane that includes the test point if it ______________ the inequality. In case the chosen test point doesn't satisfy the inequality, we shade the opposite half-plane.

6. To graph the solution set to a system of inequalities with the connecting word "and" we shade the ______________ of solutions to individual inequalities.

7. To graph the solution set to a system of inequalities with the connecting word "or" we shade the ______________ of solutions to individual inequalities.

Concept Check *For each inequality, determine if the given points belong to the solution set of the inequality.*

8. $y \ge -4x + 3$; $(1,-1)$, $(1,0)$

9. $2x - 3y < 6$; $(3,0)$, $(2,-1)$

10. $y > -2$; $(0,0)$, $(-1,-1)$

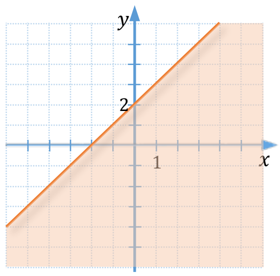
11. $x \ge -2$; $(-2,1)$, $(-3,1)$

Concept Check

12. Match the given inequalities with the graphs of their solution sets.

a. $y \geq x + 2$ **b.** $y < -x + 2$ **c.** $y \leq x + 2$ **d.** $y > -x + 2$

I



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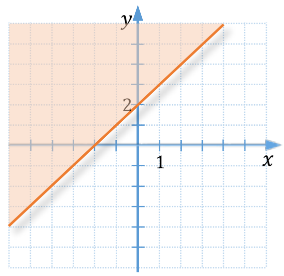

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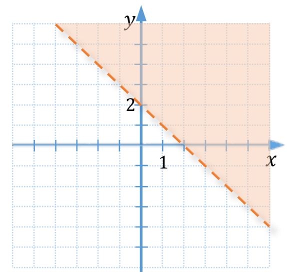

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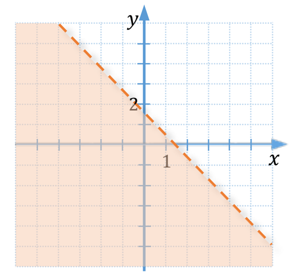

Concept Check *Graph each linear inequality in two variables.*

13. $y \geq -\frac{1}{2}x + 3$

14. $y \leq \frac{1}{3}x - 2$

15. $y < 2x - 4$

16. $y > -x + 3$

17. $y \geq -3$

18. $y < 4.5$

19. $x > 1$

20. $x \leq -2.5$

21. $x + 3y > -3$

22. $5x - 3y \leq 15$

23. $y - 3x \geq 0$

24. $3x - 2y < -6$

25. $3x \leq 2y$

26. $3y \neq 4x$

27. $y \neq 2$

Graph each compound inequality.

28. $\begin{cases} x + y \geq 3 \\ x - y < 4 \end{cases}$

29. $\begin{cases} x \geq -2 \\ y \leq -2x + 3 \end{cases}$

30. $\begin{cases} x - y < 2 \\ x + 2y \geq 8 \end{cases}$

31. $\begin{cases} 2x - y < 2 \\ x + 2y > 6 \end{cases}$

32. $\begin{cases} 3x + y \leq 6 \\ 3x + y \geq -3 \end{cases}$

33. $\begin{cases} y < 3 \\ x + y < 5 \end{cases}$

34. $3x + 2y > 2$ or $x \geq 4$

35. $x + y > 1$ or $x + y < 3$

36. $y \geq -1$ or $2x + y > 3$

37. $y > x + 3$ or $x > 3$

Analytic Skills *For each problem, write a system of inequalities describing the situation and then graph the solution set in the xy-plane.*

38. At a movie theater, tickets cost \$8 and a bag of popcorn costs \$4. Let x be the number of tickets bought and y be the number of bags of popcorn bought. Graph the region in the xy-plane that represents all possible combinations of tickets and bags of popcorn that cost \$32 or less.

39. Suppose that candy costs \$3 per pound and cashews cost \$5 per pound. Let x be the number of pounds of candy bought and y be the number of pounds of cashews bought. Graph the region in the xy-plane that represents all possible weight combinations that cost less than \$15.