Linear Equations and Inequalities

One of the main concepts in Algebra is solving equations or inequalities. This is because solutions to most application problems involve setting up and solving equations or inequalities that describe the situation presented in the problem. In this unit, we will study techniques of solving linear equations and inequalities in one variable, linear forms of absolute value equations and inequalities, and applications of these techniques in word problems.

L.1 Linear Equations in One Variable

When two algebraic expressions are compared by an equal sign (=), an **equation** is formed. An equation can be interpreted as a scale that balances two quantities. It can also be seen as a mathematical sentence with the verb "equals" or the verb phrase "is equal to". For example, the equation $3x - 1 = 5$ corresponds to the sentence:

One less than three times an unknown number equals five.

Unless we know the value of the unknown number (the variable x), we are unable to determine whether or not the above sentence is a true or false statement. For example, if $x = 1$, the equation $3x - 1 = 5$ becomes a false statement, as $3 \cdot 1 - 1 \neq 5$ (the "scale" is not in balance); while, if $x = 2$, the equation $3x - 1 = 5$ becomes a true statement, as $3 \cdot 2 - 1 = 5$ (the "scale" is in balance). For this reason, such sentences (equations) are called **open sentences**. Each variable value that satisfies an equation (i.e., makes it a true statement) is a **solution** (i.e., a ***root***, or a ***zero***) of the equation. An equation is **solved** by finding its **solution set**, the set of all solutions.

Attention:

* ***Equations*** *can be* ***solved*** *by finding the variable value(s) satisfying the equation.*

Example: $\overbrace{2x + x - 1}^{left\ side} \underset{equal\ sign}{=} \overbrace{5}^{right\ side}$ *can be solved for x*

* ***Expressions*** *can only be* ***simplified*** *or* ***evaluated***

Example: $2x + x - 1$ *can be simplified to* $3x - 1$
or evaluated for a particular x-value

Example 1 ► **Distinguishing Between Expressions and Equations**

Decide whether each of the following is an expression or an equation.

a. $4x - 16$ **b.** $4x - 16 = 0$

Solution ► **a.** $4x - 16$ is an **expression** as it does not contain any symbol of equality.

This expression can be ***evaluated*** *(for instance, if $x = 4$, the expression assumes the value 0), or it can be written in a different form. For example, we could* ***factor*** *it. So, we could write*

$$4x - 16 = 4(x - 4).$$

Notice that the equal symbol (=) in the above line does not indicate an equation, but rather an equivalency between the two expressions, $4x - 16$ and $4(x - 4)$.

b. $4x - 16 = 0$ is an **equation** as it contains an equal symbol $(=)$ that connects two sides of the equation.

To solve this equation we could factor the left-hand side expression,

$$4(x - 4) = 0,$$

and then from the zero product property, we have

$$x - 4 = 0,$$

which leads us to the solution

$$x = \mathbf{4}.$$

Attention: *Even though the two algebraic forms,* $4x - 16$ *and* $4x - 16 = 0$ *are related to each other, it is important that we neither* ***voluntarily add*** *the "*$= 0$*" part when we want to change the form of the expression, nor* ***voluntarily drop*** *the "*$= 0$*" part when we solve the equation.*

Example 2 ► **Determining if a Given Number is a Solution to an Equation**

Determine whether the number -2 is a solution to the given equation.

a. $4x = 10 + x$ **b.** $x^2 - 4 = 0$

Solution ►

a. To determine whether -2 is a solution to the equation $4x = 10 + x$, we substitute -2 in place of the variable x and find out whether the resulting equation is a true statement. This gives us

$$4(-2) \stackrel{?}{=} 10 + (-2)$$

$$-8 \stackrel{?}{=} 8$$

Since the resulting equation is not a true statement, the number $\mathbf{-2}$ **is not a solution** to the given equation.

b. After substituting -2 for x in the equation $x^2 - 4 = 0$, we obtain

$$(-2)^2 - 4 \stackrel{?}{=} 0$$

$$4 - 4 \stackrel{?}{=} 0,$$

which becomes $0 = 0$, a true statement. Therefore, the number $\mathbf{-2}$ **is a solution** to the given equation.

Equations can be classified with respect to the number of solutions. There are **identities**, **conditional** equations, and **contradictions** (or ***inconsistent*** equations).

Definition 1.1 ► An **identity** is an equation that is satisfied by every real number for which the expressions on both sides of the equation are defined. Some examples of identities are

$$2x + 5x = 7x, \quad x^2 - 4 = (x+2)(x-2), \quad \text{or} \quad \frac{x}{x} = 1.$$

The solution set of the first two identities is the set of all real numbers, $\mathbb{R}$. *However, since the expression* $\frac{x}{x}$ *is undefined for* $x = 0$, *the solution set of the equation* $\frac{x}{x} = 1$ *is the set of all nonzero real numbers,* $\{\boldsymbol{x} | \boldsymbol{x} \neq \mathbf{0}\}$.

A **conditional** equation is an equation that is satisfied by at least one real number, but is not an identity. This is the most commonly encountered type of equation. Here are some examples of conditional equations:

$$3x - 1 = 5, \quad x^2 - 4 = 0, \quad \text{or} \quad \sqrt{x} = 2.$$

The solution set of the first equation is $\{2\}$; *of the second equation is* $\{-2,2\}$; *and of the last equation is* $\{4\}$.

A **contradiction** (an ***inconsistent*** equation) is an equation that has **no solution**. Here are some examples of contradictions:

$$5 = 1, \quad 3x - 3x = 8, \quad \text{or} \quad 0x = 1.$$

The solution set of any contradiction is the empty set, $\emptyset$.

Example 3 ► **Recognizing Conditional Equations, Identities, and Contradictions**

Determine whether the given equation is *conditional*, an *identity*, or a *contradiction*.

a. $x = x$ **b.** $x^2 = 0$ **c.** $\frac{1}{x} = 0$

Solution ►

a. This equation is satisfied by any real number. Therefore, it is an **identity**.

b. This equation is satisfied by $x = 0$, as $0^2 = 0$. However, any nonzero real number when squared becomes a positive number. So, the left side of the equation $x^2 = 0$ does not equal to zero for a nonzero x. That means that a nonzero number does not satisfy the equation. Therefore, the equation $x^2 = 0$ has exactly one solution, $x = 0$. So, the equation is **conditional**.

c. A fraction equals zero only when its numerator equals to zero. Since the numerator of $\frac{1}{x}$ does not equal to zero, then no matter what the value of x would be, the left side of the equation will never equal zero. This means that there is no x-value that would satisfy the equation $\frac{1}{x} = 0$. Therefore, the equation has no solution, which means it is a **contradiction**.

Attention: *Do not confuse the solution* $x = 0$ *to the equation in Example 3b with an empty set* $\emptyset$. *An empty set means that there is no solution.* $x = 0$ *means that there is one solution equal to zero.*

Solving Linear Equations in One Variable

In this section, we will focus on solving linear (up to the first degree) equations in one variable. Before introducing a formal definition of a linear equation, let us recall the definition of a term, a constant term, and a linear term.

Definition 1.2 ► A **term** is a **<u>product</u>** of numbers, letters, and possibly other algebraic expressions.
Examples of terms: $2,\ -3x,\ \frac{2}{3}(x+1),\ 5x^2y,\ -5\sqrt{x}$
A **constant term** is a number or a product of numbers.
Examples of constant terms: $2,\ -3,\ \frac{2}{3},\ 0,\ -5\pi$
A **linear term** is a product of numbers and the first power of a single variable.
Examples of linear terms: $-3x,\ \frac{2}{3}x,\ x,\ -5\pi x$

Definition 1.3 ► A **linear** equation is an equation with only **constant** or **linear terms**. A linear equation in one variable can be written in the form $\boldsymbol{Ax + B = 0}$, for some real numbers A and B, and a variable x.

Here are some examples of *linear* equations: $2x+1=0,\quad 2=5,\quad 3x-7=6+2x$

Here are some examples of *nonlinear* equations: $x^2=16,\quad x+\sqrt{x}=-1,\quad 1+\frac{1}{x}=\frac{1}{x+1}$

So far, we have been finding solutions to equations mostly by guessing a value that would make the equation true. To find a methodical way of solving equations, observe the relations between equations with the same solution set. For example, equations

$$3x-1=5,\ \ 3x=6,\ \ and\ \ x=2$$

all have the same solution set $\{2\}$. While the solution to the last equation, $x=2$, is easily "seen" to be 2, the solution to the first equation, $3x-1=5$, is not readily apparent. Notice that the second equation is obtained by adding 1 to both sides of the first equation. Similarly, the last equation is obtained by dividing the second equation by 3. This suggests that to solve a linear equation, it is enough to write a sequence of simpler and simpler equations that preserve the solution set, and eventually result in an equation of the form:

$$x = constant \quad \text{or} \quad 0 = constant.$$

If the resulting equation is of the form $x = constant$, the solution is this constant.
If the resulting equation is $\mathbf{0=0}$, then the original equation is an **identity**, as it is true for **all real values $\boldsymbol{x}$**.
If the resulting equation is $\mathbf{0} = \boldsymbol{constant\ other\ than\ zero}$, then the original equation is a **contradiction**, as there is **no real values $\boldsymbol{x}$** that would make it true.

Definition 1.4 ► **Equivalent equations** are equations with the same solution set.

How can we transform an equation to obtain a simpler but equivalent one?

We can certainly simplify expressions on both sides of the equation, following properties of operations listed in *section R3*. Also, recall that an equation works like a scale in balance.

Therefore, adding (or subtracting) the same quantity to (from) both sides of the equation will preserve this balance. Similarly, multiplying (or dividing) both sides of the equation by a nonzero quantity will preserve the balance.

Suppose we work with an equation $A = B$, where A and B represent some algebraic expressions. In addition, suppose that C is a real number (or another expression).
Here is a summary of the basic equality operations that can be performed to produce equivalent equations:

Equality Operation	General Rule	Example
Simplification	Write each expression in a simpler but equivalent form	$2(x-3) = 1+3$ can be written as $2x - 6 = 4$
Addition	$A + C = B + C$	$2x - 6 + 6 = 4 + 6$
Subtraction	$A - C = B - C$	$2x - 6 - 4 = 4 - 4$
Multiplication	$CA = CB, \quad if\ C \neq 0$	$\frac{1}{2} \cdot 2x = \frac{1}{2} \cdot 10$
Division	$\frac{A}{C} = \frac{B}{C}, \quad if\ C \neq 0$	$\frac{2x}{2} = \frac{10}{2}$

Example 4 ► **Using Equality Operations to Solve Linear Equations in One Variable**

Solve each equation.

a. $4x - 12 + 3x = 3 + 5x - 2x$ **b.** $2[3(x-6) - x] = 3x - 2(5 - x)$

Solution ► **a.** First, simplify each side of the equation and then isolate the linear terms (terms containing x) on one side of the equation. Here is a sequence of equivalent equations that leads to the solution:

Each equation is written underneath the previous one, with the **"=" symbol aligned in a column**

There is only **one "=" symbol** in each line

$4x - 12 + 3x = 3 + 6x - 2x$	collect like terms	(1)
$7x - 12 = 3 + 4x$	move the x-terms to the left side by subtracting $4x$ and the constant terms to the right side by adding 12	(2)
$7x - \cancel{12} - \cancel{4x} + \cancel{12} = 3 + \cancel{4x} - \cancel{4x} + 12$		(3)
$7x - 4x = 3 + 12$	collect like terms	(4)
$3x = 15$	isolate x by dividing both sides by 3	(5)
$\frac{\cancel{3}x}{\cancel{3}} = \frac{15}{3}$	Simplify each side	(6)
$x = \mathbf{5}$		(7)

Let us analyze the relation between line (2) and (4).

$$7x \boxed{-12} = 3 \boxed{+4x} \qquad (2)$$

$$7x \boxed{-4x} = 3 \boxed{+12} \qquad (4)$$

By subtracting $4x$ from both sides of the equation (2), we actually 'moved' the term $+4x$ to the left side of equation (4) as $-4x$. Similarly, the addition of 12 to both sides of equation (2) caused the term -12 to 'move' to the other side as $+12$, in equation (4). This shows that the addition and subtraction property of equality allows us to change the position of a term from one side of an equation to another, by simply changing its sign. Although line (3) is helpful when explaining why we can move particular terms to another side by changing their signs, it is often cumbersome, especially when working with longer equations. So, in practice, we will avoid writing lines such as (3). Since it is important to indicate what operation is applied to the equation, we will record the operations performed in the right margin, after a slash symbol (/). Here is how we could record the solution to equation (1) in a concise way.

$$4x - 12 + 3x = 3 + 6x - 2x$$

$$7x - 12 = 3 + 4x$$

$$7x - 4x = 3 + 12$$

$$3x = 15$$

$$x = \mathbf{5}$$

b. First, release all the brackets, starting from the inner-most brackets. If applicable, remember to collect like terms after releasing each bracket. Finally, isolate x by applying appropriate equality operations. Here is our solution:

$$2[3(x - 6) - x] = 3x - 2(5 - x)$$ release red brackets

$$2[3x - 18 - x] = 3x - 10 + 2x$$ collect like terms

$$2[2x - 18] = 5x - 10$$ release the blue bracket

$$4x - 36 = 5x - 10$$ $/\ -5x, +36$

$$-x = 26$$ $/\ \div(-1)$

$$x = \mathbf{-26}$$

multiplication by -1 works as well

Note: *Notice that we could choose to collect x-terms on the right side of the equation as well. This would shorten the solution by one line and save us the division by* -1. *Here is the alternative ending of the above solution.*

$$4x - 36 = 5x - 10$$ $/\ -4x, +10$

$$-26 = x$$

Example 5 ► **Solving Linear Equations Involving Fractions**

Solve

$$\frac{x-4}{4}+\frac{2x+1}{6}=5.$$

Solution ► First, clear the fractions and then solve the resulting equation as in *Example 4*. To clear fractions, multiply both sides of the equation by the LCD of 4 and 6, which is 12.

$$\frac{x-4}{4}+\frac{2x+1}{6}=5 \qquad / \cdot 12 \qquad (8)$$

$$\overset{3}{\cancel{12}}\left(\frac{x-4}{\cancel{4}}\right)+\overset{2}{\cancel{12}}\left(\frac{2x+1}{\cancel{6}}\right)=12\cdot 5 \qquad (9)$$

When multiplying each term by the $LCD = 12$, **simplify** it with the **denominator** before **multiplying** the result by the **numerator**.

$$3(x-4)+2(2x+1)=60 \qquad (10)$$

$$3x-12+4x+2=60 \qquad (11)$$

$$7x-10=60 \qquad (12)$$

$$7x=70 \qquad (13)$$

$$x=\frac{70}{7}=\mathbf{10} \qquad (14)$$

So the solution to the given equation is $x=\mathbf{10}$.

Note: *Notice, that if the division of 12 by 4 and then by 6 can be performed fluently in our minds, writing equation* (9) *is not necessary. One could write equation* (10) *directly after the original equation* (8). *One could think: 12 divided by 4 is 3 so I multiply the resulting 3 by the numerator* $(x-4)$. *Similarly, 12 divided by 6 is 2 so I multiply the resulting 2 by the numerator* $(2x+1)$. *It is important though that each term, including the free term 5, gets multiplied by 12.*

Also, notice that the reason we multiply equations involving fractions by LCD's is to clear the denominators of those fractions. That means that if the multiplication by an appropriate LCD is performed correctly, the resulting equation should not involve any denominators!

Example 6 ► **Solving Linear Equations Involving Decimals**

Solve $0.07x-0.03(15-x)=0.05(14)$.

Solution ► To solve this equation, it is convenient (although not necessary) to clear the decimals first. This is done by multiplying the given equation by 100.

Each **term** (product of numbers and variable expressions) needs to be multiplied by 100.

$$0.07x-0.03(15-x)=0.05(14) \qquad / \cdot 100$$

$$7x-3(15-x)=5(14)$$

$$7x - 45 + 3x = 70$$

$$10x = 70 + 45$$

$$x = \frac{115}{10} = \mathbf{11.5}$$

So the solution to the given equation is $x = \mathbf{11.5}$.

Note: *In general, if n is the highest number of decimal places to clear in an equation, we multiply it by 10^n.*

Attention: *To multiply a product AB by a number C, we multiply just one factor of this product, either A or B, but not both! For example,*

$$10 \cdot 0.3(0.5 - x) = (10 \cdot 0.3)(0.5 - x) = 3(0.5 - x) \quad ✓$$

or

$$10 \cdot 0.3(0.5 - x) = 0.3 \cdot [10(0.5 - x)] = 0.3(5 - 10x) \quad ✓$$

but

$$10 \cdot 0.3(0.5 - x) \neq (10 \cdot 0.3)[10(0.5 - x)] = 3(5 - 10x) \quad ✗$$

Summary of Solving a Linear Equation in One Variable

1. **Clear fractions or decimals**. Eliminate fractions by multiplying each side by the least common denominator (LCD). Eliminate decimals by multiplying by a power of 10.

2. **Clear brackets (**starting from the inner-most ones) by applying the distributive property of multiplication. **Simplify** each side of the equation by **combining like terms**, as needed.

3. **Collect** and **combine variable terms** on one side and free terms on the other side of the equation. Use the addition property of equality to collect all variable terms on one side of the equation and all free terms (numbers) on the other side.

4. **Isolate the variable** by dividing the equation by the linear coefficient (coefficient of the variable term).

L.1 Exercises

Vocabulary Check *Fill in each blank with the most appropriate term or phrase from the given list:* ***conditional, empty set, equations, equivalent, evaluated, identity, linear, solution.***

1. A _____ equation can be written in the form $Ax + B = 0$, for some $A, B \in \mathbb{R}$.

2. A _____ of an equation is a value of the variable that makes the equation a true statement.

3. Two equations are _____ if they have exactly the same solution sets.

4. An _____ is an equation that is satisfied by all real numbers for which both sides are defined.

5. The solution set of any contradiction is the _____ _____ .

6. A _____ equation has at least one solution but is not an identity.

7. _____ can be solved while expressions can only be simplified or _____.

Concept Check *True or False? Explain.*

8. The equation $5x - 1 = 9$ is equivalent to $5x - 5 = 5$.

9. The equation $x + \sqrt{x} = -1 + \sqrt{x}$ is equivalent to $x = -1$.

10. The solution set to $12x = 0$ is $\emptyset$.

11. The equation $x - 0.3x = 0.97x$ is an identity.

12. To solve $-\frac{2}{3}x = \frac{3}{5}$, we could multiply each side by the reciprocal of $-\frac{2}{3}$.

13. If a and b are real numbers, then $ax + b = 0$ has a solution.

Concept Check *Decide whether each of the following is an equation to solve or an expression to simplify.*

14. $3x + 2(x - 6) - 1$

15. $3x + 2(x - 6) = 1$

16. $-5x + 19 = 3x - 5$

17. $-5x + 19 - 3x + 5$

Concept Check *Determine whether or not the given equation is linear.*

18. $4x + 2 = x - 3$

19. $12 = x^2 + x$

20. $x + \frac{1}{x} = 1$

21. $2 = 5$

22. $\sqrt{16} = x$

23. $\sqrt{x} = 9$

Discussion Point

24. If both sides of the equation $2 = \frac{1}{x}$ are multiplied by x, we obtain $2x = 1$. Since the last equation is linear, does this mean that the original equation $2 = \frac{1}{x}$ is linear as well? Are these two equations equivalent?

Concept Check *Determine whether the given value is a solution of the equation.*

25. $2,\ 3x - 4 = 2$

26. $-2,\ \frac{1}{x} - \frac{1}{2} = -1$

27. $6,\ \sqrt{2x + 4} = -4$

28. $-4,\ (x - 1)^2 = 25$

Solve each equation. If applicable, tell whether the equation is an ***identity*** *or a* ***contradiction****.*

29. $6x - 5 = 0$

30. $-2x + 5 = 0$

31. $-3x + 6 = 12$

32. $5x - 3 = -13$

33. $3y - 5 = 4 + 12y$

34. $9y - 4 = 14 + 15y$

35. $2(2a - 3) - 7 = 4a - 13$

36. $3(4 - 2b) = 4 - (6b - 8)$

37. $-3t + 5 = 4 - 3t$

38. $5p - 3 = 11 + 4p + p$

39. $13 - 9(2n + 3) = 4(6n + 1) - 15n$

40. $5(5n - 7) + 40 = 2n - 3(8n + 5)$

41. $3[1 - (4x - 5)] = 5(6 - 2x)$

42. $-4(3x + 7) = 2[9 - (7x + 10)]$

43. $3[5 - 3(4 - t)] - 2 = 5[3(5t - 4) + 8] - 16$

44. $6[7 - 4(8 - t)] - 13 = -5[3(5t - 4) + 8]$

45. $\frac{2}{3}(9n - 6) - 5 = \frac{2}{5}(30n - 25) - 7n$

46. $\frac{1}{2}(18 - 6n) + 5n = 10 - \frac{1}{4}(16n + 20)$

47. $\frac{8x}{3} - \frac{5x}{4} = -17$

48. $\frac{7x}{2} - \frac{5x}{10} = 5$

49. $\frac{3x-1}{4} + \frac{x+3}{6} = 3$

50. $\frac{3x+2}{7} - \frac{x+4}{5} = 2$

51. $\frac{2}{3}\left(\frac{7}{8} + 4x\right) - \frac{5}{8} = \frac{3}{8}$

52. $\frac{3}{4}\left(3x - \frac{1}{2}\right) - \frac{2}{3} = \frac{1}{3}$

53. $x - 2.3 = 0.08x + 3.5$

54. $x + 1.6 = 0.02x - 3.6$

55. $0.05x + 0.03(5000 - x) = 0.04 \cdot 5000$

56. $0.02x + 0.04 \cdot 3000 = 0.03(x + 3000)$

L.2 Formulas and Applications

In the previous section, we studied how to solve linear equations. Those skills are often helpful in problem solving. However, the process of solving an application problem has many components. One of them is the ability to construct a mathematical model of the problem. This is usually done by observing the relationship between the variable quantities in the problem and writing an equation that describes this relationship.

Definition 2.1 ► An equation that represents or models a relationship between two or more quantities is called a **formula**.

To model real situations, we often use well-known formulas, such as $R \cdot T = D$, or $a^2 + b^2 = c^2$. However, sometimes we need to construct our own models.

Data Modelling

Example 1 ► **Constructing a Formula to Model a Set of Data Following a Linear Pattern**

The table below shows the cost C of driving n miles in a taxi cab in Santa Barbara, CA.

number of miles driven, $\boldsymbol{n}$	1	2	3	4	5
cost $\boldsymbol{C}$ in dollars	5.50	8.50	11.50	14.50	17.50

a. Construct a formula relating the cost (C) and the number of miles driven (n) for this taxi cab.
b. Use the constructed formula to find the cost of a 16-mile trip by this taxi.
c. If a customer paid a fare of \$29.50, what was the distance driven by the taxi?

Solution ► **a.** Observe that the increase in cost when driving each additional mile after the first is constantly \$3.00. This is because

$$17.5 - 14.5 = 14.5 - 11.5 = 11.5 - 8.5 = 8.5 - 5.5 = 3$$

If n represents the number of miles driven, then the number of miles after the first can be represented by $(n - 1)$. The cost of driving n miles is the cost of driving the first mile plus the cost of driving the additional miles, after the first one. So, we can write

$$\textbf{\textit{total cost}}\ C = \begin{pmatrix} \textit{cost of the} \\ \textbf{\textit{first mile}} \end{pmatrix} + \begin{pmatrix} \textbf{\textit{cost increase}} \\ \textit{per mile} \end{pmatrix} \cdot \begin{pmatrix} \textbf{\textit{number of}} \\ \textit{additional}\ \textbf{\textit{miles}} \end{pmatrix}$$

or symbolically,

$$C = 5.5 + 3(n - 1)$$

The above equation can be simplified to

$$C = 5.5 + 3n - 3 = 3n + 2.5.$$

Therefore, $\boldsymbol{C = 3n + 2.5}$ is the formula that models the given data.

b. Since the number of driven miles is $n = 16$, we evaluate

$$C = 3 \cdot 16 + 2.5 = 50.5$$

Therefore, the cost of a 16-mile trip is **$50.50**.

c. This time, we are given the cost $C = 29.50$ and we are looking for the corresponding number of miles n. To find n, we substitute 31.7 for C in our formula $C = 3n + 2.5$ and then solve the resulting equation for n. We obtain

$$\begin{aligned} 29.5 &= 3n + 2.5 && /-2.5 \\ 27 &= 3n && /\div 3 \\ n &= 9 \end{aligned}$$

So, the taxi drove **9 miles**.

Notice that in the solution to *Example 1c*, we could first solve the equation $C = 3n + 2.5$ for n:

$$\begin{aligned} C &= 3n + 2.5 && /-2.5 \\ C - 2.5 &= 3n && /\div 3 \\ \boldsymbol{n} &= \boldsymbol{\frac{C - 2.5}{3}}, \end{aligned}$$

and then use the resulting formula to evaluate n at $C = 29.50$.

$$n = \frac{29.5 - 2.5}{3} = \frac{27}{3} = 9.$$

The advantage of solving the formula $C = 3n + 2.5$ for the variable n first is such that the resulting formula $n = \frac{C-2.5}{3}$ makes evaluations of n for various values of C easier. For example, to find the number of miles n driven for the fare of $35.5, we could evaluate directly using $n = \frac{35.5-2.5}{3} = \frac{33}{3} = 11$ rather than solving the equation $35.5 = 3n + 2.5$ again.

Solving Formulas for a Variable

If a formula is going to be used for repeated evaluation of a specific variable, it is convenient to rearrange this formula in such a way that the desired variable is **isolated on one side** of the equation and it does not appear on the other side. Such a formula may also be called a function.

Definition 2.2 ► A **function** is a rule for determining the value of one variable from the values of one or more other variables, in a unique way. We say that the first variable **is a function** of the other variable(s).

For example, consider the uniform motion relation between distance, rate, and time.
To evaluate rate when distance and time is given, we use the formula

$$\boldsymbol{R = \frac{D}{T}}.$$

This formula describes **rate as a function of distance and time**, as rate can be uniquely calculated for any possible input of distance and time.
To evaluate time when distance and rate is given, we use the formula

$$\boldsymbol{T = \frac{D}{R}}.$$

This formula describes **time as a function of distance and rate**, as time can be uniquely calculated for any possible input of distance and rate.
Finally, to evaluate distance when rate and time is given, we use the formula

$$\boldsymbol{D = R \cdot T}.$$

Here, the **distance is** presented as **a function of rate and time**, as it can be uniquely calculated for any possible input of rate and time.

To **solve a formula for a given variable** means to rearrange the formula so that the desired **variable equals to an expression that contains only <u>other</u> variables** but not the one that we solve for. This can be done the same way as when solving equations.

Here are some hints and guidelines to keep in mind when solving formulas:

- **Highlight** the variable of interest and solve the equation as if the other variables were just numbers (think of easy numbers), without actually performing the given operations.

  Example: To solve $mx + b = c$ for m,

  we pretend to solve, for example:

  $m \cdot 2 + 3 = 1 \qquad /-3$
  $m \cdot 2 = 1 - 3 \qquad /\div 2$
  $m = \frac{1-3}{2}$

  so we write:

  $mx + b = c \qquad /-b$
  $mx = c - b \qquad /\div x$
  $\boldsymbol{m = \frac{c-b}{x}}$

- **Reverse** (undo) **operations** to isolate the desired variable.

  Example: To solve $2L + 2W = P$ for W, first, observe the operations applied to W:

  $$W \xrightarrow{\cdot 2} 2W \xrightarrow{+2L} 2L + 2W$$

  Then, reverse these operations, starting from the last one first.

  $$W \xleftarrow[\div 2]{} 2W \xleftarrow[-2L]{} 2L + 2W$$

  So, we solve the formula as follows:

  $$2L + 2W = P \qquad /-2L$$
  $$2W = P - 2L \qquad /\div 2$$
  $$\boldsymbol{W = \frac{P - 2L}{2}}$$

  Notice that the last equation can also be written in the equivalent form $\boldsymbol{W = \frac{P}{2} - L}$.

- **Keep the** desired **variable in the numerator**.

Example: To solve $R = \frac{D}{T}$ for T, we could take the reciprocal of each side of the equation to keep T in the numerator,

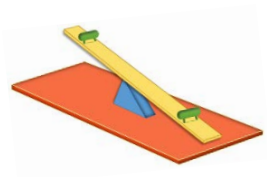
$$\frac{T}{D} = \frac{1}{R},$$

and then multiply by D to "undo" the division. Therefore, $\boldsymbol{T = \frac{D}{R}}$.

Observation: *Another way of solving $R = \frac{D}{T}$ for T is by multiplying both sides by T and dividing by R.*

$$\frac{T}{\cancel{R}} \cdot \cancel{R} = \frac{D}{\cancel{T}} \cdot \frac{\cancel{T}}{R}$$

This would also result in $\boldsymbol{T = \frac{D}{R}}$. Observe, that no matter how we solve this formula for T, the result differs from the original formula by interchanging (swapping) the variables T and R.



Note: *When working only with multiplication and division, by applying inverse operations, any factor of the numerator can be moved to the other side into the denominator, and likewise, any factor of the denominator can be moved to the other side into the numerator. Sometimes it helps to think of this movement of variables as the movement of a "teeter-totter".*

For example, the formula $\frac{bh}{2} = A$ can be solved for h by dividing by b and multiplying by 2. So, we can write directly $\boldsymbol{h = \frac{2A}{b}}$.

2 was down so now goes up
and
b was up so now goes down

- **Keep the** desired **variable in one place.**

Example: To solve $A = P + Prt$ for P, we can factor P out,

$$A = P(1 + rt)$$

and then divide by the bracket. Thus,

$$\boldsymbol{P = \frac{A}{1 + rt}}.$$

Example 2 ► **Solving Formulas for a Variable**

Solve each formula for the indicated variable.

a. $a_n = a_1 + (n - 1)d$ for n **b.** $\frac{PV}{T} = \frac{P_0V_0}{T_0}$ for T

Solution ► **a.** To solve $a_n = a_1 + (n - 1)d$ for n we use the reverse operations strategy, starting with reversing the addition, then multiplication, and finally the subtraction.

$$a_n = a_1 + (n - 1)d \qquad /-a_1$$

$$a_n - a_1 = (n-1)d \qquad /\div d$$

$$\frac{a_n - a_1}{d} = n - 1 \qquad /+1$$

$$\boldsymbol{n = \frac{a_n - a_1}{d} + 1}$$

The last equation can also be written in the equivalent form $\boldsymbol{n = \frac{a_n - a_1 + d}{d}}$.

b. To solve $\frac{PV}{T} = \frac{P_0V_0}{T_0}$ for T, first, we can take the reciprocal of each side of the equation to keep T in the numerator,

$$\frac{T}{PV} = \frac{T_0}{P_0V_0},$$

and then multiply by PV to "undo" the division. So,

$$\boldsymbol{T = \frac{T_0PV}{P_0V_0}}.$$

Attention: *Taking reciprocal of each side of an equation is a good strategy only if both sides are in the form of a* ***single fraction****. For example, to use the reciprocal property when solving* $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$ *for a, first, we perform the addition to create a single fraction,* $\frac{1}{a} = \frac{c+b}{bc}$. *Then, taking reciprocals of both sides will give us an instant result of* $a = \frac{bc}{c+b}$.

Warning! *The reciprocal of* $\frac{1}{b} + \frac{1}{c}$ *is* ***not*** *equal to* $b + c$.

Example 3 ► **Using Formulas in Application Problems**

Body mass index I can be used to determine whether an individual has a healthy weight for his or her height. An index in the range 18.5–24.9 indicates a normal weight. Body mass index is given by the formula $\boldsymbol{I = \frac{W}{H^2}}$, where W represents weight, in kilograms, and H represents height, in meters.

a. What is the body index of 182 cm tall Allen, who weighs 89 kg ?

b. Katia has a body mass index of 24.5 and a height of 1.7 meters. What is her weight?

Solution ►

a. Since the formula calls for height H is in meters, first, we convert 182 cm to 1.82 meters, and then we substitute $H = 1.82$ and $W = 89$ into the formula. So,

$$\boldsymbol{I} = \frac{89}{1.82^2} \approx \boldsymbol{26.9}$$

When rounded to one decimal place. Thus Allen is overweighted.

b. To find Katia's weight, we may want to solve the given formula for W first, and then plug in the given data. So, Katia's weight is

$$\boldsymbol{W} = I\,H^2 = 24.5 \cdot 1.7^2 \approx \boldsymbol{70.8\ kg}$$

L.2 Exercises

1. ***Concept Check*** *When a formula is solved for a particular variable, several different equivalent forms may be possible. When solving formula $A = \frac{a+b}{2}h$ for b, one possible correct answer is $b = \frac{2A}{h} - a$. Which of the following are equivalent to this?*

A. $b = \frac{A}{\frac{1}{2}h} - a$ **B.** $b = \frac{2A-a}{h}$ **C.** $b = \frac{2A-ah}{h}$ **D.** $b = \frac{A-ah}{\frac{1}{2}h}$

2. ***Concept Check*** When *a group of students solved the formula $A = P + Prt$ for P, they obtained several different answers. Which of the following answers are* ***not*** *correct and why?*

A. $P = \frac{A}{rt}$ **B.** $P = \frac{A}{1+rt}$ **C.** $P = A - Prt$ **D.** $P = \frac{A-P}{rt}$

Solve each formula for the specified variable.

3. $I = Prt$ for r (*simple interest*)

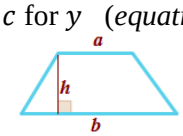
4. $C = 2\pi r$ for r (*circumference of a circle*)

5. $E = mc^2$ for m (*mass-energy relation*)

6. $F = \frac{mv^2}{r}$ for m (*force in a circular motion*)

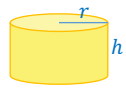
7. $A = \frac{(a+b)}{2}$ for b (*average*)

8. $Ax + By = c$ for y (*equation of a line*)



9. $P = 2l + 2w$ for l (*perimeter of a rectangle*)

10. $A = \frac{h}{2}(a + b)$ for a (*area of a trapezoid*)



11. $S = \pi rs + \pi r^2$ for π (*surface area of a cone*)

12. $S = 2\pi rh + 2\pi r^2$ for h (*surface area of a cylinder*)

13. $F = \frac{9}{5}C + 32$ for C (*Celsius to Fahrenheit*)

14. $C = \frac{9}{5}(F - 32)$ for F (*Fahrenheit to Celsius*)

15. $Q = \frac{p-q}{2}$ for p

16. $Q = \frac{p-q}{2}$ for q

17. $T = B + Bqt$ for q

18. $d = R - Rst$ for t

19. $d = R - Rst$ for R

20. $T = B + Bqt$ for B

Analytic Skills *Solve each problem.*

21. The table below shows the cost C of driving n kilometers in a taxi cab in Abbotsford, BC.

number of kilometers driven, n	1	2	3	4	5
cost C in dollars	5.10	7.00	8.90	10.80	12.70

a. Construct a formula relating the cost (C) and the number of kilometers driven (n) for this taxi cab.
b. Use the constructed formula to find the cost of a 10-km trip by this taxi.
c. If a customer paid the fare of \$29.80, what was the distance driven by the taxi?

22. The table below shows the cost C of driving n kilometers in a taxi cab in Vancouver, BC.

number of kilometers driven, $\boldsymbol{n}$	1	2	3	4	5
cost $\boldsymbol{C}$ in dollars	5.35	7.20	9.05	10.90	12.75

a. Construct a formula relating the cost (C) and the number of kilometers driven (n) for this taxi cab.
b. Use the constructed formula to find the cost of a 20-km trip by this taxi.
c. If a customer paid the fare of \$25.70, what was the distance driven by the taxi?

23.

Assume that the rule for determining the amount of a medicine dosage for a child is given by the formula

$$c = \frac{ad}{a + 12},$$

where a represents the child's age, in years, and d represents the usual adult dosage, in milligrams.

a. If the usual adult dosage of a particular medication is 250 mg. Find the dosage for a child of age 3.
b. Solve the formula for d.
c. Find the corresponding adult dosage, if an eight year old child uses 30 ml of a certain medication.

24. Colleges accommodate students who need to take different total-credit-hour loads. They determine the number of "full-time-equivalent" students, F, using the formula $F = \frac{n}{15}$, where n is the total number of credits students enroll in for a given semester.
a. Determine the number of full-time-equivalent students on a campus in which students register for 42,690 credits.
b. Solve the formula for n.
c. Find the total number of credits students enroll in a particular semester if the number of full-time-equivalent students in this semester is 2854.

25. During a strenuous skating workout, an athlete can burn 530 calories in 40 minutes.
a. Write a formula that calculates the calories C burned from skating 40 minutes a day for x days.
b. How many calories could be burned in 30 days?

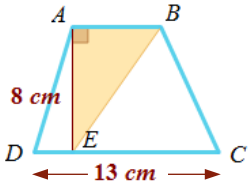
26. *Refer to information given in problem 25.*
a. If a person loses 1 pound for every 3500 calories burned, write a formula that gives the number of pounds P lost in x days from skating 40 minutes per day.
b. How many pounds could be lost in 100 days? *Round the answer to the nearest pound.*

27. Write a formula that expresses the width W of a rectangle as a function of its area A and length L.

28. Write a formula that expresses the area A of a circle as a function of its diameter d.

29. **a.** Solve the formula $I = Prt$ for t.
b. Using this formula, determine how long it will take a deposit of \$75 to earn \$6 interest when invested at 4% simple interest.

30. *Refer to information given in the accompanying figure.*
If the area of the shaded triangle ABE is 20 cm^2, what is the area of the trapezoid $ABCD$?



L.3 Applications of Linear Equations

In this section, we study some strategies for solving problems with the use of linear equations, or well-known formulas. While there are many approaches to problem-solving, the following steps prove to be helpful.

Five Steps for Problem Solving
1. **Familiarize** yourself with the problem.
2. **Translate** the problem to a symbolic representation (usually an **equation** or an **inequality**).
3. **Solve** the equation(s) or the inequality(s).
4. **Check** if the answer makes sense in the original problem.
5. **State the answer** to the original problem clearly.

Here are some hints of how to **familiarize** yourself with the problem:

- **Read** the problem carefully a few times. In the first reading focus on the general setting of the problem. See if you can identify this problem as one of a motion, investment, geometry, age, mixture or solution, work, or a number problem, and draw from your experiences with these types of problems. During the second reading, focus on the specific information given in the problem, skipping unnecessary words, if possible.
- **List the information** given, including **units**, and check **what the problem asks for**.
- If applicable, **make a diagram** and label it with the given information.
- **Introduce a variable**(s) for the unknown quantity(ies). Make sure that the variable(s) is/are clearly defined (including units) by writing a "let" statement or labeling appropriate part(s) of the diagram. Choose descriptive letters for the variable(s). For example, let l be the length in centimeters, let t be the time in hours, etc.
- Express **other unknown values** in terms of the already introduced variable(s).
- Write applicable **formulas**.
- **Organize your data** in a meaningful way, for example by filling in a table associated with the applicable formula, inserting the data into an appropriate diagram, or listing them with respect to an observed pattern or rule.
- **Guess** a possible answer and check your guess. Observe the way in which the guess is checked. This may help you translate the problem into an equation.

Translation of English Phrases or Sentences to Expressions or Equations

One of the important phases of problem-solving is **translating** English words into a **symbolic representation**.

Here are the most commonly used **key words** suggesting a particular operation:

ADDITION (+)	SUBTRACTION (–)	MULTIPLICATION (·)	DIVISION (÷)
sum	**difference**	**product**	**quotient**
plus	minus	multiply	divide
add	**subtract from**	times	ratio
total	**less than**	**of**	**out of**
more than	less	half of	per
increase by	decrease by	half as much as	shared
together	diminished	twice, triple	cut into
perimeter	shorter	area	

Example 1 ► **Translating English Words to an Algebraic Expression or Equation**

Translate the word description into an algebraic expression or equation.

a. The sum of half of a number and two
b. The square of a difference of two numbers
c. Triple a number, increased by five, is one less than twice the number.
d. The quotient of a number and seven diminished by the number
e. The quotient of a number and seven, diminished by the number
f. The perimeter of a rectangle is four less than its area.
g. In a package of 12 eggs, the ratio of white to brown eggs is one out of three.
h. Five percent of the area of a triangle whose base is one unit shorter than the height

Solution ► **a.** Let x represents "*a number*". Then

*The **sum of** **half of** a number **and** two* translates to $\frac{1}{2}x + 2$

Notice that the word "***sum***" indicates addition sign at the position of the word "***and***". Since addition is a binary operation (needs two inputs), we reserve space for "*half of a number*" on one side and "*two*" on the other side of the addition sign.

b. Suppose x and y are the "*two numbers*". Then

***The square of** a **difference of** two numbers* translates to $(x - y)^2$

Notice that we are squaring everything that comes after "*the square of*".

c. Let x represents "*a number*". Then

***Triple** a number, increased by five, **is** one **less than twice** the number.*

translates to the equation: $3x + 5 = 2x - 1$

This time, we translated a sentence that results in an equation rather than expression. Notice that the "equal" sign is used in place of the word "is". Also, remember that phrases "less than" or "subtracted from" work "backwards". For example, A **less than** B or A **subtracted from** B translates to $B - A$. However, the word "less" is used in the usual direction, from left to right. For example, A **less** B translates to $A - B$.

d. Let x represent "*a number*". Then

***The quotient of** a number **and** seven **diminished** by the number* translates to $\frac{x}{7-x}$

Notice that "**the** number" refers to the same number x.

e. Let x represent "*a number*". Then

***The quotient of** a number **and** seven, **diminished** by the number* translates to $\frac{x}{7} - x$

Here, the **comma** indicates the end of the "**quotient section**". So, we diminish the quotient rather than diminishing the seven, as in *Example 1d*.

f. Let l and w represent the length and the width of a rectangle. Then

*The **perimeter** of a rectangle **is** four **less than** its **area**.*

translates to the equation: $\boldsymbol{2l + 2w = lw - 4}$

Here, we use a formula for the perimeter ($\boldsymbol{2l + 2w}$) and for the area ($\boldsymbol{lw}$) of a rectangle.

g. Let w represent the number of white eggs in a package of 12 eggs. Then $(12 - w)$ represents the number of brown eggs in this package. Therefore,

*In a package of 12 eggs, **the ratio of the number of white eggs to the number of brown eggs is the same as two to three**.*

translates to the equation: $\boldsymbol{\frac{w}{12-w} = \frac{2}{3}}$

Here, we expressed the unknown number of brown eggs $(12 - w)$ in terms of the number w of white eggs. Also, notice that the order of listing terms in a proportion is essential. Here, the first terms of the two ratios are written in the numerators (in blue) and the second terms (in brown) are written in the denominators.

h. Let h represent the height of a triangle. Since the base is *one unit shorter than the height*, we express it as $(h - 1)$. Using the formula $\frac{1}{2}bh$ for the area of a triangle, we translate

five percent of the area of a triangle** whose **base** **is** **one unit shorter than the height

to the expression: $\boldsymbol{0.05 \cdot \frac{1}{2}(h - 1)h}$

Here, we convert *five percent* to the number 0.05, as *per-cent* means *per hundred*, which tells us to divide 5 by a hundred.
Also, observe that the above word description is not a sentence, even though it contains the word "*is*". Therefore, the resulting symbolic form is an expression, not an equation. The word "*is*" relates the base and the height, which in turn allows us to substitute $(h - 1)$ in place of b, and obtain an expression in one variable.

So far, we provided some hints of how to familiarize ourself with a problem, we worked through some examples of how to translate word descriptions to a symbolic form, and we reviewed the process of solving linear equations (*see section L1*). In the rest of this section, we will show various methods of solving commonly occurring types of problems, using representative examples.

Number Relation Problems

In number relation type of problems, we look for relations between quantities. Typically, we introduce a variable for one quantity and express the other quantities in terms of this variable following the relations given in the problem.

Example 2 ► **Solving a Number Relation Problem with Three Numbers**

The sum of three numbers is forty-two. The second number is twice the first number, and the third number is three less than the second number. Find the three numbers.

Solution ► There are three unknown numbers such that their sum is forty-two. This information allows us to write the equation

$$1^{\text{st}} \text{ number} + 2^{\text{nd}} \text{ number} + 3^{\text{rd}} \text{ number} = 42.$$

To solve such an equation, we wish to express all three unknown numbers in terms of one variable. Since the second number refers to the first, and the third number refers to the second, which in turn refers to the first, it is convenient to introduce a variable for the first number.
So, let n represent the first number.
The second number is twice the first, so $2n$ represents the second number.
The third number is three less than the second number, so $2n - 3$ represents the third number.

Therefore, our equation turns out to be

$$\begin{aligned} n + 2n + (2n - 3) &= 42 \\ 5n - 3 &= 42 && / +3 \\ 5n &= 45 && / \div 5 \\ n &= 9. \end{aligned}$$

Hence, the first number is **9**, the second number is $2n = 2 \cdot 9 = \mathbf{18}$, and the third number is $2n - 3 = 18 - 3 = \mathbf{15}$.

Consecutive Numbers Problems

Since **consecutive numbers** differ by one, we can represent them as $\boldsymbol{n, n+1, n+2}$, and so on.

Consecutive even or **consecutive odd** numbers differ by two, so both types of numbers can be represented by $\boldsymbol{n, n+2, n+4}$, and so on.

Notice that if the first number n is even, then $n + 2, n + 4, \ldots$ are also even; however, if the first number n is odd then $n + 2, n + 4, \ldots$ are also odd.

Example 3 ► **Solving a Consecutive Odd Integers Problem**

Find three consecutive odd integers such that three times the middle integer is seven more than the sum of the first and third integers.

Solution ► Let the three consecutive odd numbers be called $n, n + 2$, and $n + 4$. We translate *three times the middle integer is seven more than the sum of the first and third integers* into the equation

$$3(n + 2) = n + (n + 4) + 7$$

which gives

$$3n + 6 = 2n + 11 \qquad / \ -6, -2n$$

$$n = 5$$

Hence, the first number is **5**, the second number is $n + 2 = \mathbf{7}$, and the third number is $n + 4 = \mathbf{9}$.

Percent Problems

Rules to remember when solving percent problems:

$$1 = 100\% \quad \text{and} \quad \frac{\textit{is a part}}{\textit{of a whole}} = \frac{\%}{\mathbf{100}}$$

Also, remember that

$$\textbf{percent increase}(\textit{decrease}) = \frac{\textbf{last} - \textbf{first}}{\textbf{first}} \cdot 100\%$$

Example 4 ► **Finding the Amount of Tax**

Joe bought a new computer for \$1506.75, including sales tax at 5%. What amount of tax did he pay?

Solution ► Suppose the computer cost p dollars. Then the tax paid for this computer is 5% of p dollars, which can be represented by the expression $0.05p$. Since the total cost of the computer with tax is \$1506.75, we set up the equation

$$p + 0.05p = 1506.75$$

which gives us

$$1.05p = 1506.75 \qquad / \div 1.05$$

$$p = 1435$$

The question calls for the amount of tax, so we calculate $0.05p = 0.05 \cdot 1435 = 71.75$.

Joe paid \$71.75 of tax for his computer.

Example 5 ► **Solving a Percent Increase Problem**

After 1 yr on the job, Garry got a raise from \$10.50 per hour to \$11.34 per hour. What was the percent increase in his hourly wage?

Solution ► We calculate the percent increase by following the rule $\frac{\textbf{last}-\textbf{first}}{\textbf{first}} \cdot 100\%$.

So, Garry's hourly wage was increased by $\frac{11.34-10.50}{10.50} \cdot 100\% = \mathbf{8}\%$.

Investment Problems

When working with investment problems we often use the simple interest formula $\boldsymbol{I = Prt}$, where I represents the amount of interest, P represents the principal (amount of money invested), r represents the interest rate, and t stands for the time in years.

Also, it is helpful to organize data in a diagram like this:

$$\textit{total amount}$$

account ***I*** account ***II***

$$\boldsymbol{interest}\ from\ account\ \boldsymbol{I}\ +\ \boldsymbol{interest}\ from\ account\ \boldsymbol{II}\ =\ \boldsymbol{total\ interest}$$

or

$$\boldsymbol{interest}\ from\ account\ \boldsymbol{I}\ =\ \boldsymbol{interest}\ from\ account\ \boldsymbol{II}$$

Example 6 ► **Solving an Investment Problem**

A student takes out two student loans, one at 6% annual interest and the other at 4% annual interest. The total amount of the two loans is \$5000, and the total interest after 1 year is \$244. Find the amount of each loan.

Solution ► To solve this problem in one equation, we would like to introduce only one variable. Suppose $\boldsymbol{x}$ is the amount of the first student loan. Then the amount of the second student loan is the remaining portion of the \$5000. So, it is $(5000 - \boldsymbol{x})$.

Using the simple interest formula $I = Prt$, for $t = 1$, we calculate the interest obtained from the 6% to be $0.06 \cdot \boldsymbol{x}$ and from the 4% account to be $0.04(5000 - \boldsymbol{x})$. The equation arises from the fact that the total interest equals to \$244, as indicated in the diagram below.

$$\$5000$$

$\boldsymbol{x}$ at 6% $(5000 - \boldsymbol{x})$ at 4%

$$0.06 \cdot \boldsymbol{x}\ +\ 0.04(5000 - \boldsymbol{x})\ =\ 244$$

For easier calculations, we may want to clear decimals by multiplying this equation by 100. This gives us

$$\begin{aligned} 6x + 4(5000 - x) &= 24400 \\ 6x + 20000 - 4x &= 24400 && / -20000 \\ 2x &= 4400 && / \div 2 \end{aligned}$$

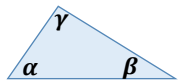
and finally

$$x = \$2200$$

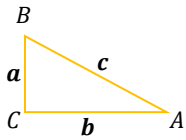
Thus, the first loan is **\$2200** and the second loan is $5000 - x = 5000 - 2200 = \$\mathbf{2800}$.

Geometry Problems

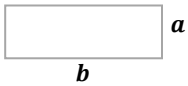
In geometry problems, we often use well-known formulas or facts that pertain to geometric figures. Here is a list of facts and formulas that are handy to know when solving various problems.



- The **sum of angles** in a triangle equals **180°**.

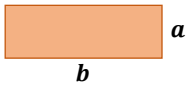


- The lengths of sides in a right-angle triangle ABC satisfy the **Pythagorean equation** $\boldsymbol{a^2 + b^2 = c^2}$, where c is the hypotenuse of the triangle.

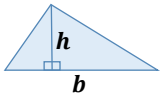


- The **perimeter of a rectangle** with sides a and b is given by the formula $\boldsymbol{2a + 2b}$.

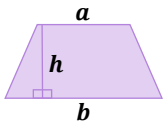
- The **circumference** of a circle with radius r is given by the formula $\boldsymbol{2\pi r}$.



- The **area of** a **rectangle** or a **parallelogram** with base b and height h is given by the formula $\boldsymbol{bh}$.



- The **area of** a **triangle** with base b and height h is given by the formula $\boldsymbol{\frac{1}{2}bh}$.



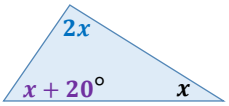
- The **area of** a **trapezoid** with bases a and b, and height h is given by the formula $\boldsymbol{\frac{1}{2}(a + b)h}$.

- The **area of** a **circle** with radius r is given by the formula πr^2.

Example 7 ► **Finding the Measure of Angles in a Triangle**

In a triangular cross section of a roof, the second angle is twice as large as the first. The third angle is 20° greater than the first angle. Find the measures of the angle

Solution ►



Observe that the size of the second and the third angle is compared to the size of the first angle. Therefore, it is convenient to introduce a variable, $\boldsymbol{x}$, for the measure of the first angle. Then, the expression for the measure of the second angle, which *is twice as large as the first*, is $\boldsymbol{2x}$ and the expression for the measure of the third angle, which is *20° greater than the first*, is $\boldsymbol{x + 20°}$. To visualize the situation, it might be helpful to draw a triangle and label the three angles.

Since the sum of angles in any triangle is equal to 180°, we set up the equation

$$x + 2x + x + 20° = 180°$$

This gives us

$$4x + 20° = 180° \qquad / -20°$$

$$4x = 160° \qquad / \div 4$$

$$x = 40°$$

So, the measure of the first angle is $\boldsymbol{40°}$,
the measure of the second angle is $2x = 2 \cdot 40° = \boldsymbol{80°}$, and
the measure of the third angle is $x + 20° = 40° + 20° = \boldsymbol{60°}$.

Total Value Problems

When solving total value types of problems, it is helpful to organize the data in a table that compares the number of items and the value of these items. For example:

	item ***A***	item ***B***	total
number of items			
value of items			

Example 8 ► **Solving a Coin Problem**

A collection of twenty-two coins has a value of \$4.75. The collection contains dimes and quarters. Find the number of quarters in the collection.

Solution ► Suppose the number of quarters is n. Since the whole collection contains 22 coins, then the number of dimes can be represented by $22 - n$. Also, in cents, the value of n quarters is $25n$, while the value of $22 - n$ dimes is $10(22 - n)$. We can organize this information as in the table below.

	dimes	quarters	Total
number of coins	$22 - n$	n	22
value of coins (in cents)	$10(22 - n)$	$25n$	475

The value is written in cents!

Using the last row of this table, we set up the equation

$$10(22 - n) + 25n = 475$$

and then solve it for n.

$$220 - 10n + 25n = 475 \qquad / \ -220$$
$$15n = 255 \qquad / \ \div 15$$
$$n = 17$$

So, there are **17** quarters in the collection of coins.

Mixture-Solution Problems

When solving total mixture or solution problems, it is helpful to organize the data in a table that follows one of the formulas

$$unit\ price \cdot number\ of\ units = total\ value \quad \text{or} \quad percent \cdot volume = content$$

	unit price ·	# of units	= value
type I			
type II			
mix			

	% ·	volume	= content
type I			
type II			
solution			

Example 9 ► **Solving a Mixture Problem**

A mixture of nuts was made from peanuts that cost \$3.60 per pound and cashews that cost \$9.00 per pound. How many of each type of nut were used to obtain a 60 pounds of mixture that costs \$5.40 per pound?

Solution ► In this problem, we mix two types of nuts: peanuts and cashews. Let x represent the number of pounds of peanuts. Since there are 60 pounds of the mixture, we will express the number of pounds of cashews as $60 - x$.

The information given in the problem can be organized as in the following table.

	unit price ·	# of units	= value (in \$)
peanuts	3.60	x	$3.6x$
cashews	9.00	$60 - x$	$9(60 - x)$
mix	5.40	60	324

To complete the last column, multiply the first two columns.

Using the last column of this table, we set up the equation

$$3.6x + 9(60 - x) = 324$$

and then solve it for x.

$$3.6x + 540 - 9x = 360 \qquad / -540$$
$$-5.4x = -216 \qquad / \div (-5.4)$$
$$x = 40$$

So, there were **40** pounds of peanuts and $60 - x = 60 - 40 = \mathbf{20}$ pounds of cashews used for the mix.

Example 10 ► **Solving a Solution Problem**

How many milliliters of pure acid must be added to 60 ml of an 8% acid solution to make a 20% acid solution?

Solution ► Let x represent the volume of the pure acid, in milliliters. The 20% solution is made by combining x ml of the pure acid with 60 ml of an 8% acid, so the volume of the solution can be expressed as $x + 60$.

Now, let us organize this information in the table below.

	% ·	volume	= acid
pure acid	1	x	x
8% acid	0.08	60	4.8
solution	0.20	$x + 60$	$0.2(x + 60)$

To complete the last column, multiply the first two columns.

Using the last column of this table, we set up the equation

$$x + 4.8 = 0.2(x + 60)$$

and then solve it for x.

$$x + 4.8 = 0.2x + 12 \qquad / \ -0.2x, -4.8$$
$$0.8x = 7.2 \qquad / \ \div 0.8$$
$$x = 9$$

So, there must be added **9** milliliters of pure acid.

Motion Problems

When solving motion problems, refer to the formula

$$\boldsymbol{Rate \cdot Time = Distance}$$

and organize date in a table like this:

	R ·	T	= D
motion I			
motion II			
total			

Some boxes in the "total" row are often left empty. For example, in motion problems, we usually do not add rates. Sometimes, the "total" row may not be used at all.

If two moving object (or two components of a motion) are analyzed, we usually encounter the following situations:

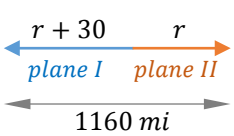
1. The two objects A and B move apart, approach each other, or move successively in the same direction (see the diagram below). In these cases, it is likely we are interested in the **total distance** covered. So, the last row in the above table will be useful to record the total values.

2. Both objects follow the same pathway. Then the **distances** corresponding to the two motions are the same and we may want to **equal** them. I such cases, there may not be any total values to consider, so the last row in the above table may not be used at all.

Example 11 ► **Solving a Motion Problem where Distances Add**

Two small planes start from the same point and fly in opposite directions. The first plane is flying 30 mph faster than the second plane. In 4 hours, the planes are 2320 miles apart. Find the rate of each plane.

Solution ►



The rates of both planes are unknown. However, since the rate of the first plane is 30 mph faster than the rate of the second plane, we can introduce only one variable. For example, suppose r represents the rate of the second plane. Then the rate of the first plane is represented by the expression $r + 30$.

In addition, notice that 1160 miles is the **total distance** covered by both planes, and 4 hours is the flight time of each plane.

Now, we can complete a table following the formula $R \cdot T = D$.

	R $\cdot$	T	$=$ D
plane I	$r + 30$	4	$4(r + 30)$
plane II	r	4	$4r$
total			1160

Notice that neither the total rate nor the total time was included here. This is because these values are not relevant to this particular problem. The equation that relates distances comes from the last column:

$$4(r + 30) + 4r = 1160$$

After solving it for r,

$$4r + 120 + 4r = 1160 \qquad / -120$$

$$8r = 1040 \qquad / \div 8$$

we obtain

$$r = 130$$

Therefore, the speed of the first plane is $r + 30 = 130 + 30 =$ **160 mph** and the speed of the second plane is **130 mph**.

Example 12 ► **Solving a Motion Problem where Distances are the Same**

A speeding car traveling at 80 mph passes a police officer. Ten seconds later, the police officer chases the speeding car at a speed of 100 mph. How long, in minutes, does it take the police officer to catch up with the car?

Solution ► Let t represent the time, in minutes, needed for the police officer to catch up with the car. The time that the speeding car drives is 10 seconds longer than the time that the police officer drives. To match the denominations, we convert 10 seconds to $\frac{10}{60} = \frac{1}{6}$ of a minute. So, the time used by the car is $t + \frac{1}{6}$.

In addition, the rates are given in miles per hour, but we need to express them as miles per minute. We can convert $\frac{80\ mi}{1\ h}$ into, for example, $\frac{80\ mi}{60\ min} = \frac{4}{3}\frac{mi}{min}$, and similarly, $\frac{100\ mi}{1\ h}$ to $\frac{100\ mi}{60\ min} = \frac{5}{3}\frac{mi}{min}$.
Now, we can complete a table that follows the formula $R \cdot T = D$.

	R $\cdot$	T	$=$ D
car	$\frac{4}{3}$	$t + \frac{1}{6}$	$\frac{4}{3}(t + \frac{1}{6})$
police	$\frac{5}{3}$	t	$\frac{5}{3}t$

Notice that this time there is no need for the "total" row.

Since distances covered by the car and the police officer are the same, we set up the equation

$$\frac{4}{3}\left(t + \frac{1}{6}\right) = \frac{5}{3}t \qquad / \cdot 3$$

To solve it for t, we may want to clear some fractions first. After multiplying by 3, we obtain

$$4\left(t+\frac{1}{6}\right)=5t$$

which becomes

$$4t+\frac{2}{3}=5t \qquad / -4t$$

and finally

$$\frac{2}{3}=t$$

So, the police officer needs $\mathbf{\frac{2}{3}}$ of a minute (which is 40 seconds) to catch up with the car.

Even though the above examples show a lot of ideas and methods used in solving specific types of problems, we should keep in mind that the best way to learn problem-solving is to **solve a lot of problems**. This is because every problem might present slightly different challenges than the ones that we have seen before. The more problems we solve, the more experience we gain, and with time, problem-solving becomes easier.

L.3 Exercises

Concept Check *Translate each word description into an algebraic expression or equation.*

1. A number less seven

2. A number less than seven

3. Half of the sum of two numbers

4. Two out of all apples in the bag

5. The difference of squares of two numbers

6. The product of two consecutive numbers

7. The sum of three consecutive integers is 30.

8. Five more than a number is double the number.

9. The quotient of three times a number and 10

10. Three percent of a number decreased by a hundred

11. Three percent of a number, decreased by a hundred

12. The product of 8 more than a number and 5 less than the number

13. A number subtracted from the square of the number

14. The product of six and a number increased by twelve

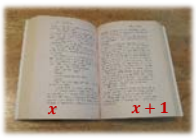
Concept Check *Solve each problem.*

15. If the quotient of a number and 6 is added to twice the number, the result is 8 less than the number. Find the number.

16. When 75% of a number is added to 6, the result is 3 more than the number. Find the number.

17. The sum of the numbers on two adjacent post-office boxes is 697. What are the numbers?

18. The sum of the page numbers on a pair of facing pages of a book is 543. What are the page numbers?



19. The enrollment at a community college declined from 12,750 during one school year to 11,350 the following year. Find the percent decrease to the nearest percent.

20. Between 2010 and 2015, the population of Alaska grew from 710,231 to 737,625. What was the percent increase to the nearest tenth of a percent?

Analytic Skills *Solve each problem.*

21. Find three consecutive odd integers such that the sum of the first, two times the second, and three times the third is 70.

22. Find three consecutive even integers such that the sum of the first, five times the second, and four times the third is 1226.

23. Twice the sum of three consecutive odd integers is 150. Find the three integers.

24. Jeff knows that his neighbour Sarah paid $40,230, including sales tax, for a new Buick Park Avenue. If the sales tax rate is 8%, then what is the cost of the car before tax?

25. After a salesman receives a 5% raise, his new salary is $40,530. What was his old salary?

26. Clayton bought a ticket to a rock concert at a discount. The regular price of the ticket was $70.00, but he only paid $59.50. What was the percent discount?

27. About 182,900 patents were issued by the U.S. government in 2007. This was a decrease of about 7% from the number of patents issued in 2006. To the nearest hundred, how many patents were issued in 2006?

28. A person invests some money at 5% and some money at 6% simple interest. The total amount of money invested is $2400 and the total interest after 1 year is $130.50. Find the amount invested at each rate.

29. Thomas Flanagan has $40,000 to invest. He will put part of the money in an account paying 4% simple interest and the remainder into stocks paying 6% simple interest. The total annual income from these investments should be $2040. How much should he invest at each rate?

30. Jennifer Siegel invested some money at 4.5% simple interest and $1000 less than twice this amount at 3%. Her total annual income from the interest was $1020. How much was invested at each rate?

31. Piotr Galkowski invested some money at 3.5% simple interest, and $5000 more than three times this amount at 4%. He earned $1440 in annual interest. How much did he invest at each rate?

32. Dan Abbey has invested $12,000 in bonds paying 6%. How much additional money should he invest in a certificate of deposit paying 3% simple interest so that the total return on the two investments will be 4%?

33. Mona Galland received a year-end bonus of $17,000 from her company and invested the money in an account paying 6.5%. How much additional money should she deposit in an account paying 5% so that the return on the two investments will be 6%?

34. A piece of wire that is 100 cm long is to be cut into two pieces, each to be bent to make a square. The length of a side of one square is to be twice the length of a side of the other. How should the wire be cut?

35. The measure of the largest angle in a triangle is twice the measure of the smallest angle. The third angle is 10° less than the largest angle. Find the measure of each angle.

36. A carpenter used 30 ft of molding in three pieces to trim a garage door. If the long piece was 2 ft longer than twice the length of each shorter piece, then how long was each piece?

37. Clint is constructing two adjacent rectangular dog pens. Each pen will be three times as long as it is wide, and the pens will share a common long side. If Clint has 65 ft of fencing, what are the dimensions of each pen?

38. The width of a standard tennis court used for playing doubles is 42 feet less than the length. The perimeter of the court is 228 feet. Find the dimensions of the court.

39. Dana inserted eight coins, consisting of dimes and nickels, into a vending machine to purchase a *Snickers* bar for 55 cents. How many coins of each type did she use?

40. Ravi took eight coins from his pocket, which contained only dimes, nickels, and quarters, and bought the Sunday Edition of *The Daily Star* for 75 cents. If the number of nickels he used was one more than the number of dimes, then how many of each type of coin did he use?

41. Pecans that cost $28.50 per kilogram were mixed with almonds that cost $22.25 per kilogram. How many kilograms of each were used to make a 25-kilogram mixture costing $24.25 per kilogram?

42. A manager bought 12 pounds of peanuts for $30. He wants to mix $5 per pound cashews with the peanuts to get a batch of mixed nuts that is worth $4 per pound. How many pounds of cashews are needed?

43. Adult tickets for a play cost $10.00, and children's tickets cost $4.00. For one performance, 460 tickets were sold. Receipts for the performance were $3760. Find the number of adult tickets sold.

44. Tickets for a school play sold for $7.50 for each adult and $3.00 for each child. The total receipts for 113 tickets sold were $663. Find the number of adult tickets sold.

45. Find the cost per ounce of a sunscreen made from 100 ounces of lotion that cost $3.46 per ounce and 60 ounces of lotion that cost $12.50 per ounce.

46. A tea mixture was made from 40 lb of tea costing $5.40 per pound and 60 lb of tea costing $3.25 per pound. Find the cost of the tea mixture.

47. A pharmacist has 200 milliliters of a solution that is 40% active ingredient. How much pure water should she add to the solution to get a solution that is 25% active ingredient?

48. How many pounds of a 15% aluminum alloy must be mixed with 500 lb of a 22% aluminum alloy to make a 20% aluminum alloy?

49. A silversmith mixed 25 g of a 70% silver alloy with 50 g of a 15% silver alloy. What is the percent concentration of the resulting alloy?

50. How many milliliters of alcohol must be added to 200 ml of a 25% iodine solution to make a 10% iodine solution?

51. A car radiator contains 9 liters of a 40% antifreeze solution. How many liters will have to be replaced with pure antifreeze if the resulting solution is to be 60% antifreeze?

52. Two planes are 1620 mi apart and are traveling toward each other. One plane is traveling 120 mph faster than the other plane. The planes meet in 1.5 h. Find the speed of each plane.

53. An airplane traveling 390 mph in still air encounters a 65-mph headwind. To the nearest minute, how long will it take the plane to travel 725 mi into the wind?

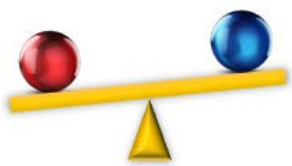
54. Angela leaves from Jocelyn's house on her bicycle traveling at 12 mph. Ten minutes later, Jocelyn leaves her house on her bicycle traveling at 15 mph to catch up with Angela. How long, in minutes, does it take Jocelyn to reach Angela?

55. Hana walked from her home to a bicycle repair shop at a rate of 3.5 mph and then rode her bicycle back home at a rate of 14 mph. If the total time spent traveling was one hour, how far from Hana's home is the repair shop?

56. A jogger and a cyclist set out at 9 A.M. from the same point headed in the same direction. The average speed of the cyclist is four times the average speed of the jogger. In 2 h, the cyclist is 33 mi ahead of the jogger. How far did the cyclist ride?

57. If a 2 mi long parade is proceeding at 3 mph, how long will it take a runner jogging at 6 mph to travel from the front of the parade to the end of the parade?

L.4 Linear Inequalities and Interval Notation



Mathematical inequalities are often used in everyday life situations. We observe speed limits on highways, minimum payments on credit card bills, maximum smartphone data usage per month, the amount of time we need to get from home to school, etc. When we think about these situations, we often refer to limits, such as "a speed limit of 100 kilometers per hour" or "a limit of 1 GB of data per month." However, we don't have to travel at exactly 100 kilometers per hour on the highway or use exactly 1 GB of data per month. The limit only establishes a boundary for what is allowable. For example, a driver traveling x kilometers per hour is obeying the speed limit of 100 kilometers per hour if $x \leq 100$ and breaking the speed limit if $x > 100$. A speed of $x = 100$ represents the boundary between obeying the speed limit and breaking it. Solving linear inequalities is closely related to solving linear equations because equality is the boundary between *greater than* and *less than*. In this section, we discuss techniques needed to solve linear inequalities and ways of presenting these solutions.

Linear Inequalities

Definition 4.1 ► A **linear inequality** is an inequality with only **constant** or **linear terms**.
A linear inequality in one variable can be written in one of the following forms:

$$\boldsymbol{Ax + B > 0}, \quad \boldsymbol{Ax + B \geq 0}, \quad \boldsymbol{Ax + B < 0}, \quad \boldsymbol{Ax + B \leq 0}, \quad \boldsymbol{Ax + B \neq 0},$$

for some real numbers A and B, and a variable x.

A variable value that makes an inequality true is called a **solution** to this inequality. We say that such variable value **satisfies** the inequality.

Example 1 ► **Determining if a Given Number is a Solution of an Inequality**

Determine whether each of the given values is a solution of the inequality.

a. $3x - 7 > -2$; 2, 1 **b.** $\frac{y}{2} - 6 \geq -3$; 8, 6

Solution ►

a. To check if 2 is a solution of $3x - 7 > -2$, replace x by 2 and determine whether the resulting inequality $3 \cdot 2 - 7 > -2$ is a true statement. Since $6 - 7 = -1$ is indeed larger than -2, then 2 satisfies the inequality. So 2 is a solution of $3x - 7 > -2$.

After replacing x by 1, we obtain $3 \cdot 1 - 7 > -2$, which simplifies to the false statement $-4 > -2$. This shows that 1 is not a solution of the given inequality.

b. To check if 8 is a solution of $\frac{y}{2} - 6 \geq -3$, substitute $y = 8$. The inequality becomes $\frac{8}{2} - 6 \geq -3$, which simplifies to $-2 \geq -3$. Since this is a true statement, 8 is a solution of the given inequality.
Similarly, after substituting $y = 6$, we obtain a true statement $\frac{6}{2} - 6 \geq -3$, as the left side of this inequality equals to -3. This shows that -3 is also a solution to the original inequality.

Usually, an inequality has an infinite number of solutions. For example, one can check that the inequality

$$2x - 10 < 0$$

is satisfied by -5, 0, 1, 3, 4, 4.99, and generally by any number that is smaller than 5. So in the above example, the set of all solutions, called the **solution set**, is infinite. Generally, the solution set to a linear inequality in one variable can be stated either using **set-builder notation**, or **interval notation**. Particularly, the solution set of the above inequality could be stated as $\{x|x > 5\}$, or as $(-\infty, 5)$.

In addition, it is often beneficial to visualize solution sets of inequalities in one variable as graphs on a number line. The solution set to the above example would look like this:

For more information about presenting solution sets of inequalities in the form of a graph or interval notation, refer to *Example 3* and the subsection on "*Interval Notation*" in *Section R2* of the *Review* chapter.

To solve an inequality means to find all the variable values that satisfy the inequality, which in turn means to find its solution set. Similarly as in the case of equations, we find these solutions by producing a sequence of simpler and simpler inequalities preserving the solution set, which eventually result in an inequality of one of the following forms:

$$x > constant, \quad x \geq constant, \quad x < constant, \quad x \leq constant, \quad x \neq constant.$$

Definition 4.2 ► **Equivalent inequalities** are inequalities with the same solution set.

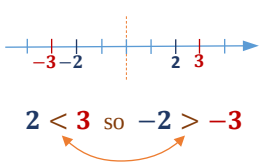


Figure 1

Generally, we create equivalent inequalities in the same way as we create equivalent equations, **except for *multiplying* or *dividing* an inequality by a *negative* number**. Then, we ***reverse the inequality symbol***, as illustrated in *Figure 1*.

So, if we multiply (or divide) the inequality

$$-x \geq 3$$

by -1, then we obtain an equivalent inequality

$$x \leq -3.$$

***multiplying* or *dividing* by a *negative* reverses the inequality sign**

We encourage the reader to confirm that the solution set to both of the above inequalities is $(-\infty, -3]$.

Multiplying or dividing an inequality by a positive number, leaves the inequality sign unchanged.

The table below summarizes the basic inequality operations that can be performed to produce equivalent inequalities, starting with $A < B$, where A and B are any algebraic expressions. Suppose C is a real number or another algebraic expression. Then, we have:

Inequality operation	General Rule	Example
Simplification	Write each expression in a simpler but equivalent form	$2(x-3) < 1+3$ can be written as $2x-6<4$
Addition	if $A<B$ then $A+C<B+C$	if $2x-6<4$ then $2x-6+6<4+6$
Subtraction	if $A<B$ then $A-C<B-C$	if $2x<x+4$ $2x-x<x+4-x$
Multiplication **when multiplying by a <u>negative</u> value, <u>reverse</u> the inequality sign**	if $C>0$ and $A<B$ then $CA<CB$ if $C<0$ and $A<B$ then $CA>CB$	if $2x<10$ then $\frac{1}{2}\cdot 2x<\frac{1}{2}\cdot 10$ if $-x<-5$ then $x>5$
Division **when dividing by a <u>negative</u> value, <u>reverse</u> the inequality sign**	if $C>0$ and $A<B$ then $\frac{A}{C}<\frac{B}{C}$ if $C<0$ and $A<B$ then $\frac{A}{C}>\frac{B}{C}$	if $2x<10$ then $\frac{2x}{2}<\frac{10}{2}$ if $-2x<10$ then $x>-5$

Example 2 ► **Using Inequality Operations to Solve Linear Inequalities in One Variable**

Solve the inequalities. Graph the solution set on a number line and state the answer in interval notation.

a. $\frac{3}{4}x+3>15$ **b.** $-2(x+3)>10$

c. $\frac{1}{2}x-3\le\frac{1}{4}x+2$ **d.** $-\frac{2}{3}(x-3)-\frac{1}{2}\ge\frac{1}{2}(5-x)$

Solution ► **a.** To isolate x, we apply inverse operations in reverse order. So, first we subtract the 3, and then we multiply the inequality by the reciprocal of the leading coefficient. Thus,

$$\frac{3}{4}x+3>15 \qquad / -3$$

$$\frac{3}{4}x>12 \qquad / \cdot\frac{4}{3}$$

$$x>\frac{\overset{4}{\cancel{12}}\cdot 4}{\cancel{3}}=16$$

To visualize the solution set of the inequality $x > 16$ on a number line, we graph the interval of all real numbers that are greater than 16.

Finally, we give the answer in interval notation by stating $\boldsymbol{x \in (16, \infty)}$. This tells us that any x-value greater than 16 satisfies the original inequality.

Note: The answer can be stated as $x \in (16, \infty)$, or simply as $(16, \infty)$. Both forms are correct.

b. Here, we will first simplify the left-hand side expression by expanding the bracket and then follow the steps as in *Example 2a.* Thus,

$$-2(x+3) > 10$$

$$-2x - 6 > 10 \qquad / -3$$

$$-2x > 16 \qquad / \div(-2)$$

$$x < -8$$

The corresponding graph looks like this:

The solution set in interval notation is $(-\infty, \boldsymbol{-8})$.

c. To solve this inequality, we will collect and combine linear terms on the left-hand side and free terms on the right-hand side of the inequality.

$$\frac{1}{2}x - 3 \le \frac{1}{4}x + 2 \qquad / -\frac{1}{4}x, +3$$

$$\frac{1}{2}x - \frac{1}{4}x \le 5$$

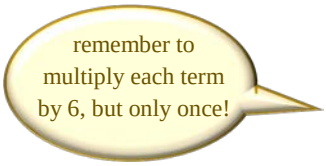
$$\frac{1}{4}x \le 5 \qquad / \cdot 4$$

$$x \le 20$$

This can be graphed as

and stated in interval notation as $(-\infty, \boldsymbol{20}]$.

d. To solve this inequality, it would be beneficial to clear the fractions first. So, we will multiply the inequality by the LCD of 3 and 2, which is 6.



$$-\frac{2}{3}(x-3) - \frac{1}{2} \le \frac{1}{2}(5-x) \qquad / \cdot 6$$

$$-\frac{2 \cdot \overset{2}{\cancel{6}}}{\cancel{3}}(x-3) - \frac{1 \cdot \overset{3}{\cancel{6}}}{\cancel{2}} \le \frac{1 \cdot \overset{3}{\cancel{6}}}{\cancel{2}}(5-x)$$

$$-4(x-3) - 3 \le 3(5-x)$$

$$-4x + 12 - 3 \leq 15 - 3x$$

$$-4x + 9 \leq 15 - 3x \qquad / \ +4x, -15$$

At this point, we could collect linear terms on the left or on the right-hand side of the inequality. Since it is easier to work with a positive coefficient by the x-term, let us move the linear terms to the right-hand side this time.

So, we obtain

$$-6 \leq 15x \qquad / \ \div 15$$

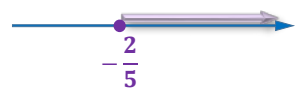
$$\frac{-6}{15} \leq x,$$

which after simplifying to $-\frac{2}{5} \leq x$ and writing the inequality from right to left, gives us the final result

$a \leq b$ can be written as $b \geq a$

$$x \geq -\frac{2}{5}$$

The solution set can be graphed as



which means that all real numbers $x \in \left[-\frac{2}{3}, \infty\right)$ satisfy the original inequality.

Example 3 ► **Solving Special Cases of Linear Inequalities**

Solve each inequality.

a. $-2(x-3) > 5 - 2x$ **b.** $-12 + 2(3 + 4x) < 3(x - 6) + 5x$

Solution ► **a.** Solving the inequality

$$-2(x-3) > 5 - 2x$$

$$-2x + 6 > 5 - 2x \qquad / \ +2x$$

$$6 > 5,$$

leads us to a true statement that does not depend on the variable x. This means that any real number x satisfies the inequality. Therefore, the solution set of the original inequality is equal to all real numbers $\mathbb{R}$. This could also be stated in interval notation as $(-\infty, \infty)$.

b. Solving the inequality

$$-12 + 2(3 + 4x) < 3(x - 6) + 5x$$

$$-12 + 6 + 8x < 3x - 18 + 5x$$

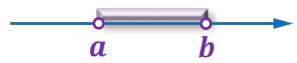
$$-6 + 8x < 8x - 18 \qquad / \ -8x$$

$$-6 < -18$$

leads us to a false statement that does not depend on the variable x. This means that no real number x would satisfy the inequality. Therefore, the solution set of the original inequality is an empty set $\emptyset$. We say that the inequality has **no solution**.

Three-Part Inequalities

The fact that an unknown quantity x lies between two given quantities a and b, where $a < b$, can be recorded with the use of the three-part inequality $a < x < b$. We say that x is enclosed by the values (or oscillates between the values) a and b. For example, the systolic high blood pressure p oscillates between 120 and 140 mm Hg. It is convenient to record this fact using the three-part inequality $120 < p < 140$, rather than saying that p < 140 and at the same time p > 120. The solution set of the three-part inequality $a < x < b$ or $b > x > a$ is a **bounded** interval (a, b) that can be graphed as



The hollow (open) dots indicate that thy endpoints do not belong to the solution set. Such interval is called **open**.

If the inequality symbol includes equation ($\leq$ or $\geq$), the corresponding endpoint of the interval is included in the solution set. On a graph, this is reflected as a solid (closed) dot. For example, the solution set of the three-part inequality $a \leq x < b$ is the interval $[a, b)$, which is graphed as

Such interval is called **half-open** or **half-closed**.

An interval with both endpoints included is referred to as **closed** interval. For example, $[a, b]$ is a closed interval and its graph looks like this

Any three-part inequality of the form

$$constant\ \boldsymbol{a} < (\leq)\ one\ variable\ linear\ \boldsymbol{expression} < (\leq)\ constant\ \boldsymbol{b},$$

where $a \leq b$ can be solved similarly as a single inequality, by applying inequality operations to all of the three parts. When solving such inequality, the goal is to isolate the variable in the middle part by moving all constants to the outside parts.

Example 4 ► **Solving Three-Part Inequalities**

Solve each three-part inequality. Graph the solution set on a number line and state the answer in interval notation.

a. $-2 \leq 1 - 3x \leq 3$ **b.** $-3 < \frac{2x-3}{4} \leq 6$

Solution ► **a.** To isolate x from the expression $1 - 3x$, subtract 1 first, and then divide by -3. These operations must be applied to all three parts of the inequality. So, we have

Remember to **reverse** both inequality symbols when **dividing by a negative** number!

$$-2 \le 1 - 3x \le 3 \qquad / -1$$

$$-3 \le -3x \le 2 \qquad / \div (-3)$$

$$1 \ge x \ge -\frac{2}{3}$$

The result can be graphed as

The inequality is satisfied by all $x \in \left[-\frac{2}{3}, 1\right]$.

b. To isolate x from the expression $\frac{2x-3}{4}$, we first multiply by 4, then add 3, and finally divide by 2.

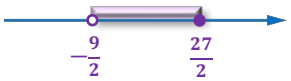
$$-3 < \frac{2x-3}{4} \le 6 \qquad / \cdot 4$$

$$-12 < 2x - 3 \le 24 \qquad / +3$$

$$-9 < 2x \le 27 \qquad / \div 2$$

$$-\frac{9}{2} < x \le \frac{27}{2}$$

The result can be graphed as



The inequality is satisfied by all $x \in \left(-\frac{9}{2}, \frac{27}{2}\right]$.

Inequalities in Application Problems

Linear inequalities are often used to solve problems in areas such as business, construction, design, science, or linear programming. The solution to a problem involving an inequality is generally an interval of real numbers. We often ask for the range of values that solve the problem.

Below is a list of common words and phrases indicating the use of particular inequality symbols.

Word expression	Interpretation
a is less (smaller) than ***b***	$a < b$
a is less than or equal to ***b***	$a \le b$
a is greater (more, bigger) than ***b***	$a > b$
a is greater than or equal to ***b***	$a \ge b$
a is at least ***b***	$a \ge b$
a is at most ***b***	$a \le b$
a is no less than ***b***	$a \ge b$
a is no more than ***b***	$a \le b$
a is exceeds ***b***	$a > b$
a is different than ***b***	$a \ne b$
x is between ***a*** and ***b***	$a < x < b$
x is between ***a*** and ***b*** inclusive	$a \le x \le b$

Example 5 ► **Translating English Words to an Inequality**

Translate the word description into an inequality and then solve it.

a. Twice a number, increased by 3 is at most 9.
b. Two diminished by five times a number is between -4 and 7

Solution ►

a. Twice a number, increased by 3 translates to $2x + 3$. Since "at most" corresponds to the symbol "$\leq$", the inequality to solve is

$$2x + 3 \leq 9 \qquad / -3$$

$$2x \leq 6 \qquad / \div 2$$

$$x \leq 3$$

So, all $\boldsymbol{x} \in (-\infty, \mathbf{3}]$ satisfy the condition of the problem.

b. Two more than five times a number translates to $2 - 5x$. The phrase "between -4 and 7" tells us that the expression $2 - 5x$ is enclosed by the numbers -4 and 7, but not equal to these numbers. So, the inequality to solve is

$$-4 < 2 - 5x < 7 \qquad / -2$$

$$-6 < -5x < 5 \qquad / \div (-5)$$

$$\frac{6}{5} > x > -1$$

Therefore, the solution set to this problem is the interval of numbers $\left(-\mathbf{1}, \frac{\mathbf{6}}{\mathbf{5}}\right)$.

Remember: To record an interval, list its endpoints in ***increasing order*** (from the smaller to the larger number.)

Example 6 ► **Using a Linear Inequality to Compare Cellphone Charges**

A cellular phone company advertises two pricing plans. The first plan costs \$19.95 per month with 20 free minutes and \$0.39 per minute thereafter. The second plan costs \$23.95 per month with 20 free minutes and \$0.30 per minute thereafter. How many minutes can you talk per month for the first plan to cost less than the second?

Solution ►

Let n represent the number of cellphone minutes used per month. Since 20 is the number of free-minutes for both plans, then $n - 20$ represents the number of paid-minutes. Hence, the expression representing the charge according to the first plan is $19.95 + 0.39(n - 20)$ and according to the second plan is $23.95 + 0.3(n - 20)$.

Since we wish for the first plan to be cheaper, than the inequality to solve is

$$19.95 + 0.39(n - 20) < 23.95 + 0.3(n - 20).$$

To work with 'nicer' numbers, such as integers, we may want to eliminate the decimals by multiplying the above inequality by 100 first. Then, after removing the brackets via distribution, we obtain

$$1995 + 39n - 780 < 2395 + 30n - 600 \qquad / \ -30n, -1215$$

$$1215 + 39n < 1795 + 30n \qquad / \ \div 9$$

$$9n < 580$$

$$n < \frac{580}{9} \approx 64.4$$

So, the first plan is cheaper if less than 65 minutes are used during a given month.

Example 7 ► **Finding the Test Score Range of the Missing Test**

Ken scored 74% on his midterm exam. If he wishes to get a B in this course, the average of his midterm and final exam must be between 80% and 86%, inclusive. What range of scores on his final exam guarantee him a B in this course?

Solution ► Let n represent Ken's score on his final exam. Then, the average of his midterm and final exam is represented by the expression

$$\frac{74 + n}{2}.$$

Since this average must be between 80% and 89% inclusive, we need to solve the three-part inequality

$$80 \le \frac{74 + n}{2} \le 86 \qquad / \cdot 2$$

$$160 \le 74 + n \le 172 \qquad / \ -74$$

$$86 \le n \le 98$$

To attain a final grade of a B, Ken's score on his final exam should fall between 86% and 98%, inclusive.

L.4 Exercises

Vocabulary Check *Complete each blank with one of the suggested words, or with the most appropriate term or phrase from the given list:* ***bounded, closed, interval, open, reversed, satisfy, true, unbounded.***

1. We say that a number satisfies one variable inequality if, after substituting this value for the variable, the inequality becomes a _____ statement.

2. The solution set of an inequality is the set of all numbers that _____ the inequality.

3. When multiplying or dividing an inequality by a negative value, the inequality symbol must be _____.

4. The solution set of a single inequality is an _____ interval of real numbers.

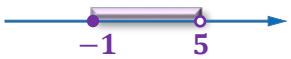
5. The solution set of a three-part inequality is a _____ interval of real numbers.

6. An _____ interval does not include its endpoints.

7. A _____ interval includes its endpoints.

Concept Check *Using interval notation, record the set of numbers presented on the graph. (Refer to the part "Interval Notation" in section R2 of the Review chapter, if needed.)*

8. 

9. 

10. 

11. 

Concept Check *Graph each solution set. For each interval write the corresponding inequality (or inequalities), and for each inequality, write the solution set in interval notation. (Refer to the part "Interval Notation" in section R2 of the Review chapter, if needed.)*

12. $(3, \infty)$

13. $(-\infty, 2]$

14. $[-7,5]$

15. $[-1, 4)$

16. $x \geq -5$

17. $x > 6$

18. $x < -2$

19. $x \leq 0$

20. $-4 < x < 1$

21. $3 \leq x < 7$

22. $-5 < x \leq -2$

23. $0 \leq x \leq 1$

Concept Check *Determine whether the given value is a solution of the inequality.*

24. $4n + 15 > 6n + 20; \quad -5$

25. $16 - 5a > 2a + 9; \quad 1$

26. $\frac{x}{4} + 7 \geq 5; \quad -8$

27. $6y - 7 \leq 2 - y; \quad \frac{2}{3}$

Concept Check *Solve each inequality. Graph the solution set and write the solution using interval notation.*

28. $2 - 3x \geq -4$

29. $4x - 6 > 12 - 10x$

30. $\frac{3}{5}x > 9$

31. $-\frac{2}{3}x \leq 12$

32. $5(x + 3) - 2(x - 4) \geq 2(x + 7)$

33. $5(y + 3) + 9 < 3(y - 2) + 6$

34. $2(2x - 4) - 4x \leq 2x + 3$

35. $7(4 - x) + 5x > 2(16 - x)$

36. $\frac{4}{5}(7x + 6) > 40$

37. $\frac{2}{3}(4x - 3) \leq 30$

38. $\frac{5}{2}(2a - 3) < \frac{1}{3}(6 - 2a)$

39. $\frac{2}{3}(3x - 1) \geq \frac{3}{2}(2x - 3)$

40. $\frac{5-2x}{2} \geq \frac{2x+1}{4}$

41. $\frac{3x-2}{-2} \geq \frac{x-4}{-5}$

42. $0.05 + 0.08x < 0.01x - 0.04(3 - 3x)$

43. $-0.2(5x + 2) > 0.4 + 1.5x$

44. $-\frac{1}{4}(p+6)+\frac{3}{2}(2p-5) \leq 10$

45. $\frac{3}{5}(t-2)-\frac{1}{4}(2t-7) \leq 3$

46. $-6 \leq 5x-7 \leq 4$

47. $-10 < 3b-5 < -1$

48. $2 \leq -3m-7 \leq 4$

49. $4 < -9x+5 < 8$

50. $-\frac{1}{2} < \frac{1}{4}x-3 < \frac{1}{2}$

51. $-\frac{2}{3} \leq 4-\frac{1}{4}x \leq \frac{2}{3}$

52. $-3 \leq \frac{7-3x}{2} < 5$

53. $-7 < \frac{3-2x}{3} \leq -2$

Concept Check *Give, in interval notation, the unknown numbers in each description.*

54. The sum of a number and 5 exceeds 12.

55. 5 times a number, decreased by 6, is smaller than -16.

56. 2 more than three times a number is at least 8.

57. Triple a number, subtracted from 5, is at most 7.

58. Half of a number increased by 3 is no more than 12.

59. Twice a number increased by 1 is different than 14.

60. Double a number is between -6 and 8.

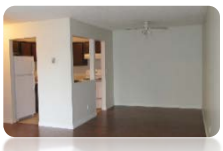
61. Half a number, decreased by 3, is between 1 and 12.

Analytic Skills *Solve each problem.*

62. Amber earned scores of 90 and 82 on her first two tests in Algebra. What score must she make on her third test to keep an average of 84 or greater?

63. An average of 70 to 79 in a mathematics class receives a C grade. A student has scores of 59, 91, 85, and 62 on four tests. Find the range of scores on the fifth test that will give the student a C for the course.

64. To run an advertisement on a certain website, the website owner charges a setup fee of \$250 and \$12 per day to display the advertisement. If a marketing group has a budget of \$1500 for an advertisement, what is the maximum number of days the advertisement can run on the site?



65. A homeowner has a budget of \$150 to paint a room that has 340 ft^2 of wall space. Drop cloths, masking tape, and paint brushes cost \$32. If 1 gal of paint will cover 100 ft^2 of wall space and the paint is sold only in gallons, what is the maximum cost per gallon of paint that the homeowner can pay?

66. The temperature range for a week was between 14°F and 77°F. Using the formula $F = \frac{9}{5}C + 32$, find the temperature range in degrees Celsius.

67. The temperature range for a week in a mountain town was between 0°C and 30°C. Using the formula $C = \frac{5}{9}(F-32)$, find the temperature range in degrees Fahrenheit.

68. George earns $1000 per month plus 5% commission on the amount of sales. George's goal is to earn a minimum of $3200 per month. What amount of sales will enable George to earn $3200 or more per month?

69. Heritage National Bank offers two different chequing accounts. The first charges $3 per month and $0.50 per cheque after the first 10 cheques. The second account charges $8 per month with unlimited cheque writing. How many cheques can be written per month if the first account is to be less expensive than the second account?

70. Glendale Federal Bank offers a chequing account to small businesses. The charge is $8 per month plus $0.12 per cheque after the first 100 cheques. A competitor is offering an account for $5 per month plus $0.15 per cheque after the first 100 cheques. If a business chooses the first account, how many cheques does the business write monthly if it is assumed that the first account will cost less than the competitor's account?

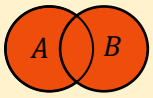
L.5 Operations on Sets and Compound Inequalities

In the previous section, it was shown how to solve a three-part inequality. In this section, we will study how to solve systems of inequalities, often called **compound** inequalities, that consist of two linear inequalities joined by the words "*and*" or "*or*". Studying compound inequalities is an extension of studying three-part inequalities. For example, the three-part inequality, $2 < x \leq 5$, is in fact a system of two inequalities, $2 < x$ *and* $x \leq 5$. The solution set to this system of inequalities consists of all numbers that are larger than 2 *and* at the same time are smaller or equal to 5. However, notice that the system of the same two inequalities connected by the word "*or*", $2 < x$ *or* $x \leq 5$, is satisfied by any real number. This is because any real number is either larger than 2 *or* smaller than 5. Thus, to find solutions to compound inequalities, aside for solving each inequality individually, we need to pay attention to the joining words, "*and*" or "*or*". These words suggest particular operations on the sets of solutions.

Operations on Sets

Sets can be added, subtracted, or multiplied.

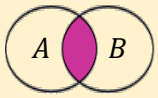
Definition 5.1 ►



The result of the addition of two sets $\boldsymbol{A}$ and $\boldsymbol{B}$ is called a **union** (or sum), symbolized by $\boldsymbol{A} \cup \boldsymbol{B}$, and defined as:

$$\boldsymbol{A} \cup \boldsymbol{B} = \{\boldsymbol{x} | \boldsymbol{x} \in \boldsymbol{A} \textbf{ or } \boldsymbol{x} \in \boldsymbol{B}\}$$

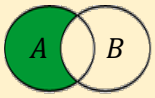
This is the set of all elements that belong to either $\boldsymbol{A}$ ***or*** $\boldsymbol{B}$.



The result of the multiplication of two sets $\boldsymbol{A}$ and $\boldsymbol{B}$ is called an **intersection** (or product, or **common part**), symbolized by $\boldsymbol{A} \cap \boldsymbol{B}$, and defined as:

$$\boldsymbol{A} \cap \boldsymbol{B} = \{\boldsymbol{x} | \boldsymbol{x} \in \boldsymbol{A} \textbf{ and } \boldsymbol{x} \in \boldsymbol{B}\}$$

This is the set of all elements that belong to both $\boldsymbol{A}$ ***and*** $\boldsymbol{B}$.



The result of the subtraction of two sets $\boldsymbol{A}$ and $\boldsymbol{B}$ is called a **difference**, symbolized by $\boldsymbol{A} \setminus \boldsymbol{B}$, and defined as:

$$\boldsymbol{A} \setminus \boldsymbol{B} = \{\boldsymbol{x} | \boldsymbol{x} \in \boldsymbol{A} \textbf{ and } \boldsymbol{x} \notin \boldsymbol{B}\}$$

This is the set of all elements that belong to $\boldsymbol{A}$ ***and*** do not belong to $\boldsymbol{B}$.

Example 1 ► **Performing Operations on Sets**

Suppose $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6\}$, and $C = \{6\}$. Find the following sets:

a. $A \cap B$

b. $A \cup B$

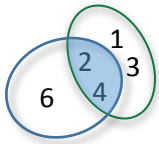
c. $A \setminus B$

d. $B \cup C$

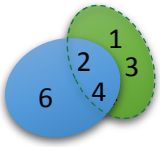
e. $B \cap C$

f. $A \cap C$

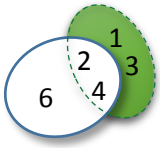
Solution ►



a. The intersection of sets $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$ consists of numbers that belong to both sets, so $\boldsymbol{A \cap B = \{2, 4\}}$.



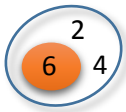
b. The union of sets $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$ consists of numbers that belong to at least one of the sets, so $\boldsymbol{A \cup B = \{1, 2, 3, 4, 6\}}$.



c. The difference $A \setminus B$ of sets $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$ consists of numbers that belong to the set A but do not belong to the set B, so $\boldsymbol{A \setminus B = \{1, 3\}}$.

d. The union of sets $B = \{2, 4, 6\}$, and $C = \{6\}$ consists of numbers that belong to at least one of the sets, so $\boldsymbol{B \cup C = \{2, 4, 6\}}$.

Notice that $B \cup C = B$. This is because C is a **subset** of B.

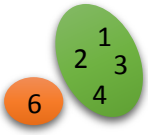


e. The intersection of sets $B = \{2, 4, 6\}$ and $C = \{6\}$ consists of numbers that belong to both sets, so $\boldsymbol{B \cap C = \{6\}}$.

Notice that $B \cap C = C$. This is because C is a **subset** of B.

f. Since the sets $A = \{1, 2, 3, 4\}$ and $C = \{6\}$ do not have any common elements, then $\boldsymbol{A \cap C = \emptyset}$.

Recall: Two sets with no common part are called **disjoint**.
Thus, the sets A and C are disjoint.



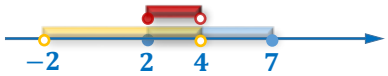
Example 2 ► **Finding Intersections and Unions of Intervals**

Write the result of each set operation as a single interval, if possible.

a. $(-2,4) \cap [2,7]$
b. $(-\infty, 1] \cap (-\infty, 3)$
c. $(-1,3) \cup (1, 6]$
d. $(3, \infty) \cup [5, \infty)$
e. $(-\infty,3) \cup (4, \infty)$
f. $(-\infty,3) \cap (4, \infty)$
g. $(-\infty,5) \cup (4, \infty)$
h. $(-\infty,3] \cap [3, \infty)$

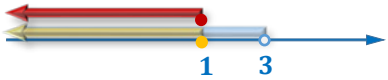
Solution ►

a. The interval of points that belong to both, the interval $(-2,4)$, in yellow, and the interval $[2,7]$, in blue, is marked in red in the graph below.



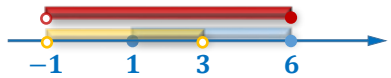
So, we write $(-2,4) \cap [2,7] = [2, 4)$.

b. As in problem **a.**, the common part $(-\infty, 1] \cap (-\infty, 3)$ is illustrated in red on the graph below.



So, we write $(-\infty, 1] \cap (-\infty, 3) = (-\infty, 1]$.

c. This time, we take the union of the interval $(-1,3)$, in yellow, and the interval $(1,6]$, in blue. The result is illustrated in red on the graph below.

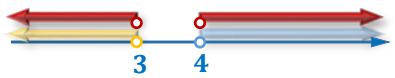


So, we write $(-1,3) \cup (1,6] = (-1,6]$.

d. The union $(3,\infty) \cup [5,\infty)$ is illustrated in red on the graph below.

So, we write $(3,\infty) \cup [5,\infty) = (3,\infty)$.

e. As illustrated in the graph below, this time, the union $(-\infty,3) \cup (4,\infty)$ can't be written in the form of a single interval.



So, the expression $(-\infty,3) \cup (4,\infty)$ cannot be simplified.

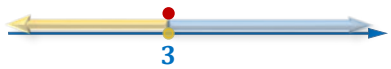
f. As shown in the graph below, the interval $(-\infty,3)$ has no common part with the interval $(4,\infty)$.

Therefore, $(-\infty,3) \cap (4,\infty) = \emptyset$

g. This time, the union $(-\infty,5) \cup (4,\infty)$ covers the entire number line.

Therefore, $(-\infty,5) \cup (4,\infty) = (-\infty,\infty)$

h. As shown in the graph below, there is only one point common to both intervals, $(-\infty,3]$ and $[3,\infty)$. This is the number 3.



Since a single number is not considered to be an interval, we should use a set rather than interval notation when recording the answer. So, $(-\infty,3] \cap [3,\infty) = \{3\}$.

Compound Inequalities

The solution set to a system of two inequalities joined by the word *and* is the intersection of solutions of each inequality in the system. For example, the solution set to the system

$$\begin{cases} x > 1 \\ x \leq 4 \end{cases}$$

is the intersection of the solution set for $x > 1$ and the solution set for $x \leq 4$. This means that the solution to the above system equals $(1, \infty) \cap (-\infty, 4] = (1, 4]$, as illustrated in the graph below.

The solution set to a system of two inequalities joined by the word *or* is the union of solutions of each inequality in the system. For example, the solution set to the system

$$x \leq 1 \text{ or } x > 4$$

is the union of the solution of $x \leq 1$ and, the solution of $x > 4$. This means that the solution to the above system equals $(-\infty, 1) \cup [4, \infty)$, as illustrated in the graph below.

Example 4 ► **Solving Compound Linear Inequalities**

Solve each compound inequality. Pay attention to the joining word *and* or *or* to find the overall solution set. Give the solution set in both interval and graph form.

a. $3x + 7 \geq 4$ *and* $2x - 5 < -1$

b. $-2x - 5 \geq 1$ *or* $x - 5 \geq -3$

c. $3x - 11 < 4$ *or* $4x + 9 \geq 1$

d. $-2 < 3 - \frac{1}{4}x < \frac{1}{2}$

e. $\begin{cases} 4x - 7 < 1 \\ 7 - 3x > -8 \end{cases}$

f. $4x - 2 < -8$ *or* $5x - 3 < 12$

Solution ► **a.** To solve this system of inequalities, first, we solve each individual inequality, keeping in mind the joining word *and*. So, we have

$$\begin{array}{lcl} 3x + 7 \geq 4 \quad / -7 & \textit{and} & 2x - 5 < -1 \quad / +5 \\ 3x \geq -3 \quad / \div 3 & \textit{and} & 2x < 4 \quad / \div 2 \\ x \geq -1 & \textit{and} & x < 2 \end{array}$$

The joining word *and* indicates that we look for the intersection of the obtained solutions. These solutions (in yellow and blue) and their intersection (in red) are shown in the graphed below.

Therefore, the system of inequalities is satisfied by all $x \in [\mathbf{-1}, \mathbf{2})$.

b. As in the previous example, first, we solve each individual inequality, except this time we keep in mind the joining word *or*. So, we have

$$-2x - 5 \geq 1 \quad / +5 \qquad \textit{or} \qquad x - 5 \geq -3 \quad / +5$$

reverse the signs

$$-2x \geq 6 \quad / \div (-2) \quad \textit{or} \quad x \geq 2$$

$$x \leq -3$$

The joining word *or* indicates that we look for the union of the obtained solutions. These solutions (in yellow and blue) and their union (in red) are indicated in the graph below.

Therefore, the system of inequalities is satisfied by all $x \in (-\infty, \mathbf{-3}] \cup [\mathbf{2}, \infty)$.

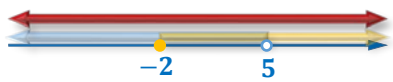
c. As before, we solve each individual inequality, keeping in mind the joining word *or*. So, we have

$$3x - 11 < 4 \quad / +11 \quad \textit{or} \quad 4x + 9 \geq 1 \quad / -9$$

$$3x < 15 \quad / \div 3 \quad \textit{or} \quad 4x \geq -8 \quad / \div 4$$

$$x < 5 \quad \textit{or} \quad x \geq -2$$

The joining word *or* indicates that we look for the union of the obtained solutions. These solutions (in yellow and blue) and their union (in red) are indicated in the graph below.



Therefore, the system of inequalities is satisfied by all real numbers. The solution set equals to $\mathbb{R}$.

d. Any three-part inequality is a system of inequalities with the joining word *and*. The system $-2 < 3 - \frac{1}{4}x < \frac{1}{2}$ could be written as

$$-2 < 3 - \tfrac{1}{4}x \quad \textit{and} \quad 3 - \tfrac{1}{4}x < \tfrac{1}{2}$$

and solved as in *Example 4a*. Alternatively, it could be solved in the three-part form, similarly as in *Section L4, Example 4*. Here is the three-part form solution.

$$-2 < 3 - \frac{1}{4}x < \frac{1}{2} \qquad / -3$$

$$-5 < -\frac{1}{4}x < \frac{1}{2} - \frac{3 \cdot 2}{2}$$

$$-5 < -\frac{1}{4}x < -\frac{5}{2} \qquad / \cdot (-4)$$

reverse the signs

$$20 > x > \frac{5 \cdot 4}{2}$$

$$20 > x > 10$$

So the solution set is the interval $(10, 20)$, visualized in the graph below.

Remark: Solving a system of inequalities in three-part form has its benefits. First, the same operations are applied to all three parts, which eliminates the necessity of repeating the solving process for the second inequality. Second, the solving process of a three-part inequality produces the final interval of solutions rather than two intervals that need to be intersected to obtain the final solution set.

e. The system $\begin{cases} 4x - 7 < 1 \\ 7 - 3x > -8 \end{cases}$ consists of two inequalities joined by the word *and*. So, we solve it similarly as in *Example 4a*.

$$\begin{array}{ccccc} 4x - 7 < 1 & / +7 & \textit{and} & 7 - 3x > -8 & / -7 \\ 4x < 8 & / \div 4 & \textit{and} & -3x > -15 & / \div (-3) \\ x < 2 & & \textit{and} & x < 5 & \end{array}$$

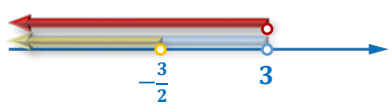
These solutions of each individual inequality (in yellow and blue) and the intersection of these solutions (in red) are indicated in the graph below.

Therefore, the interval $(-\infty, 2)$ is the solution to the whole system.

f. As in *Example 4b* and *4c*, we solve each individual inequality, keeping in mind the joining word *or*. So, we have

$$\begin{array}{ccccc} 4x - 2 < -8 & / +2 & \textit{or} & 5x - 3 < 12 & / +3 \\ 4x \geq -6 & / \div 4 & \textit{or} & 5x \geq 15 & / \div 5 \\ x < -\frac{3}{2} & & \textit{or} & x < 3 & \end{array}$$

The joining word *or* indicates that we look for the union of the obtained solutions. These solutions (in yellow and blue) and their union (in red) are indicated in the graph below.



Therefore, the interval $(-\infty, 3)$ is the solution to the whole system.

Compound Inequalities in Application Problems

Compound inequalities are often used to solve problems that ask for a range of values satisfying certain conditions.

Example 5 ► **Finding the Range of Values Satisfying Conditions of a Problem**

The equation $P = 1 + \frac{d}{33}$ gives the pressure P, in atmospheres (atm), at a depth of d feet in the ocean. Atlantic cod occupy waters with pressures between 1.6 and 7 atmospheres. To the nearest foot, what is the depth range at which Atlantic cod should be searched for?

Solution ► The pressure P suitable for Atlantic cod is between 1.6 to 7 atmospheres. We record this fact in the form of the three-part inequality $1.6 \le P \le 7$. To find the corresponding depth d, in feet, we substitute $P = 1 + \frac{d}{33}$ and solve the three-part inequality for d. So, we have

$$1.6 \le 1 + \frac{d}{33} \le 7 \qquad / -1$$

$$0.6 \le \frac{d}{33} \le 6 \qquad / \cdot 33$$

$$19.8 \le d \le 198$$

Thus, Atlantic cod should be searched for between 20 and 198 feet below the surface.

Example 6 ► **Using Set Operations to Solve Applied Problems Involving Compound Inequalities**

Given the information in the table,

Film	Admissions (in millions)	Lifetime Gross Income (in millions)
Gone With the Wind	226	3440
Star Wars: The Force Awakens	194	2825
The Sound of Music	156	2366
Titanic	128	2516
Avatar	78	3020

list the films that belong to each set.

a. The set of films with admissions greater than 180,000,000 *and* a lifetime gross income greater than 3,000,000,000.

b. The set of films with admissions greater than 150,000,000 *or* a lifetime gross income greater than 3,000,000,000.

c. The set of films with admissions smaller than 180,000,000 *and* a lifetime gross income greater than 2,500,000,000.

Solution ►

a. The set of films with admissions greater than 180,000,000 consists of *Gone With the Wind* and *Star Wars: The Force Awakens*. The set of films with a lifetime gross income greater than 3,000,000,000 consists of *Gone With the Wind* and *Avatar*. Therefore, the set of films satisfying both of these properties contains just one film, which is ***Gone With the Wind***.

b. The set of films with admissions greater than 150,000,000 consists of *Gone With the Wind*, *Star Wars: The Force Awakens*, and *The Sound of Music*. The set of films with lifetime gross income greater than 3,000,000,000 includes *Gone With the Wind* and *Avatar*. Therefore, the set of films satisfying at least one of these properties consists of ***Gone With the Wind, Star Wars: The Force Awakens, The Sound of Music***, and ***Avatar***.

c. The set of films with admissions greater than 180,000,000 includes *Gone With the Wind* and *Star Wars: The Force Awakens*. The set of films with a lifetime gross income greater than 2,500,000,000 consists of *Gone With the Wind*, *Star Wars: The Force*

Awakens, *Titanic*, and *Avatar*. Therefore, the set of films satisfying both of these properties consists of ***Gone With the Wind*** *and* ***Star Wars: The Force Awakens***.

L.5 Exercises

Vocabulary Check *Complete each blank with the most appropriate term from the given list:* ***bounded, compound, intersection, unbounded, union.***

1. The _____ of sets A and B is the set of all elements that are in both A and B.

2. The _____ of sets A and B is the set of all elements that are either in A or in B.

3. Two inequalities joined by the words *and* or *or* is a _____ inequality.

4. The solution set of a compound inequality involving the word *or* usually consists of two _____ intervals.

5. The solution set of a compound inequality involving the word *and* usually consists of one _____ interval.

Concept Check *Let* $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 5\}$, $C = \{5\}$. *Find each set.*

6. $A \cap B$

7. $A \cup B$

8. $B \cup C$

9. $A \setminus B$

10. $A \cap C$

11. $A \cup B \cup C$

12. $B \cap C$

13. $A \cup C$

Concept Check *Write the result of each set operation as a single interval, if possible.*

14. $(-7, 3] \cap [1,6]$

15. $(-8, 5] \cap (-1, 13)$

16. $(0,3) \cup (1, 7]$

17. $[-7, 2] \cup (1, 10)$

18. $(-\infty,13) \cup (1, \infty)$

19. $(-\infty,1) \cap (2, \infty)$

20. $(-\infty,1] \cap [1, \infty)$

21. $(-\infty,-1] \cup [1, \infty)$

22. $(-2, \infty) \cup [3, \infty)$

23. $(-2, \infty) \cap [3, \infty)$

Concept Check *Solve each compound inequality. Give the solution set in both interval and graph form.*

24. $x + 1 > 6$ *or* $1 - x > 3$

25. $-3x \geq -6$ *and* $-2x \leq 12$

26. $4x + 1 < 5$ *and* $4x + 7 > -1$

27. $3y - 11 > 4$ *or* $4y + 9 \leq 1$

28. $3x - 7 < -10$ *and* $5x + 2 \leq 22$

29. $\frac{1}{4}y - 2 < -3$ *or* $1 - \frac{3}{2}y \geq 4$

30. $\begin{cases} 1 - 7x \leq -41 \\ 3x + 1 \geq -8 \end{cases}$

31. $\begin{cases} 2(x + 1) < 8 \\ -2(x - 4) > -2 \end{cases}$

32. $-\frac{2}{3} \leq 3 - \frac{1}{2}a < \frac{2}{3}$

33. $-4 \leq \frac{7-3a}{5} \leq 4$

34. $5x + 12 > 2$ *or* $7x - 1 < 13$

35. $4x - 2 > 10$ *and* $8x + 2 \leq -14$

36. $7t - 1 > -1$ *and* $2t - 5 \geq -10$

37. $7z - 6 > 0$ *or* $-\frac{1}{2}z \leq 6$

38. $\frac{5x+4}{2} \geq 7$ *or* $\frac{7-2x}{-3} \leq 2$

39. $\frac{2x-5}{-2} \geq 2$ *and* $\frac{2x+1}{3} \geq 0$

40. $13 - 3x > -8$ *and* $12x + 7 \geq -(1 - 10x)$

41. $1 \leq -\frac{1}{3}(4b - 27) \leq 3$

Discussion Point *Discuss how to solve the three-part inequalities in problem 42 and 43. Then, solve them.*

42. $-4x < 2x - 18 \leq -x$

43. $7x - 5 \leq 4x - 3 \leq 8x - 3$

Analytic Skills *Solve each problem.*

44. On a cross-country move, a couple plans to drive between 450 and 600 miles per day. If they estimate that their average driving speed will be 60 mph, how many hours per day will they be driving?

45. Matter is in a liquid state between its melting point and its boiling point. The melting point of mercury is about $-38.8°C$ and its boiling point is about $356.7°C$. The formula $C = \frac{5}{9}(F - 32)$ is used to convert temperatures in degrees Fahrenheit ($°F$) to temperatures in degrees Celsius ($°C$). To the nearest tenth of a degree, determine the temperatures in $°F$ for which mercury is not in the liquid state.

46. The cost of a wedding reception is \$2500 plus \$50 for each guest. If a couple would like to keep the cost of the reception between \$7500 and \$10,000, how many guests can the couple invite?

47. A factory worker earns \$12 per hour plus an overtime rate of \$16 for every hour over 40 hr she works per week. How many hours of overtime must she work if she wants to make between \$600 and \$800 per week?

48. Average expenses for full-time resident college students at 4-year institutions during the 2007–2008 academic year are shown in the table.

Type of Expense	Public Schools	Private Schools
Tuition and fees	5950	21588
Board rates	3402	3993
Dormitory charges	4072	4812

List the elements of the following sets:

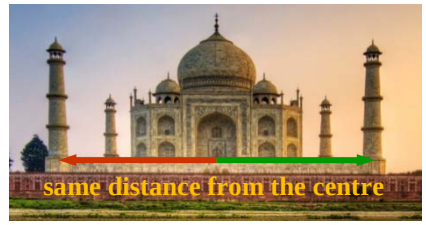
a. the set of expenses that are less than \$6000 for public schools *and* are greater than \$10,000 for private schools

b. the set of expenses that are greater than \$3000 for public schools *and* are less than \$4000 for private schools

c. the set of expenses that are less than \$5000 for public schools *or* are greater than \$10,000 for private schools

d. the set of expenses that are greater than \$4,000 for public schools *or* are between \$4000 and \$6000 for private schools

L.6 Absolute Value Equations and Inequalities



The concept of **absolute value** (also called **numerical value**) was introduced in *Section R2*. Recall that when using geometrical visualisation of real numbers on a number line, the absolute value of a number x, denoted $|x|$, can be interpreted as the distance of the point x from zero. Since distance cannot be negative, the result of absolute value is always nonnegative. In addition, the distance between points x and a can be recorded as $|x-a|$ (see *Definition 2.2* in *Section R2*), which represents the nonnegative difference between the two quantities. In this section, we will take a closer look at absolute value properties, and then apply them to solve absolute value equations and inequalities.

Properties of Absolute Value

The formal definition of absolute value

$$|\boldsymbol{x}| \overset{def}{=} \begin{cases} \boldsymbol{x}, & \boldsymbol{if} \quad \boldsymbol{x} \geq \mathbf{0} \\ -\boldsymbol{x}, & \boldsymbol{if} \quad \boldsymbol{x} < \mathbf{0} \end{cases}$$

tells us that, when $\boldsymbol{x}$ is nonnegative, the absolute value of $\boldsymbol{x}$ is the same as $\boldsymbol{x}$, and when $\boldsymbol{x}$ is negative, the absolute value of it is the **opposite** of $\boldsymbol{x}$.

So, $|2| = 2$ and $|-2| = -(-2) = 2$. Observe that this complies with the notion of a distance from zero on a number line. Both numbers, 2 and -2 are at a distance of 2 units from zero. They are both solutions to the equation $|x| = 2$.

Since $|x|$ represents the distance of the number x from 0, which is never negative, we can claim the first absolute value property:

$|\boldsymbol{x}| \geq \mathbf{0}$, for any real x

Here are several other absolute value properties that allow us to simplify algebraic expressions.

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ are any real numbers. Then

$|\boldsymbol{x}| = \mathbf{0}$ if and only if $\boldsymbol{x} = \mathbf{0}$

Only zero is at the distance zero from zero.

$|-\boldsymbol{x}| = |\boldsymbol{x}|$

The distance of opposite numbers from zero is the same.

$|\boldsymbol{xy}| = |\boldsymbol{x}||\boldsymbol{y}|$

Absolute value of a product is the product of absolute values.

$\left|\frac{\boldsymbol{x}}{\boldsymbol{y}}\right| = \frac{|\boldsymbol{x}|}{|\boldsymbol{y}|}$ for $y \neq 0$

Absolute value of a quotient is the quotient of absolute values.

Attention: *Absolute value doesn't 'split' over addition or subtraction! That means*

$$|\boldsymbol{x} \pm \boldsymbol{y}| \neq |\boldsymbol{x}| \pm |\boldsymbol{y}|$$

For example, $|2 + (-3)| = 1 \neq 5 = |2| + |-3|$.

Example 1 ► **Simplifying Absolute Value Expressions**

Simplify, leaving as little as possible inside each absolute value sign.

a. $|-2x|$ **b.** $|3x^2y|$

c. $\left|\frac{-a^2}{2b}\right|$ **d.** $\left|\frac{-1+x}{4}\right|$

Solution ► **a.** Since absolute value can 'split' over multiplication, we have

$$|-2x| = |-2||x| = \mathbf{2}|\boldsymbol{x}|$$

b. Using the multiplication property of absolute value and the fact that x^2 is never negative, we have

$$|3x^2y| = |3||x^2||y| = \mathbf{3}\boldsymbol{x}^{\mathbf{2}}|\boldsymbol{y}|$$

c. Using properties of absolute value, we have

$$\left|\frac{-a^2}{2b}\right| = \frac{|-1||a^2|}{|2||b|} = \frac{\boldsymbol{a}^{\mathbf{2}}}{\mathbf{2}|\boldsymbol{b}|}$$

c. Since absolute value does not 'split' over addition, the only simplification we can perform here is to take 4 outside of the absolute value sign. So, we have

$$\left|\frac{-1+x}{4}\right| = \frac{|\boldsymbol{x} - \mathbf{1}|}{\mathbf{4}} \text{ or equivalently } \frac{\mathbf{1}}{\mathbf{4}}|\boldsymbol{x} - \mathbf{1}|$$

Remark: *Convince yourself that* $|x - 1|$ *is not equivalent to* $x + 1$ *by evaluating both expressions at, for example,* $x = 1$.

Absolute Value Equations

The formal definition of absolute value (see *Definition 2.1* in *Section R2*) applies not only to a single number or a variable x but also to any algebraic expression. Generally, we have

$$|\boldsymbol{expr.}| \overset{\boldsymbol{def}}{=} \begin{cases} \boldsymbol{expr.}, & \boldsymbol{if} \quad \boldsymbol{expr.} \geq \mathbf{0} \\ -(\boldsymbol{expr.}), & \boldsymbol{if} \quad \boldsymbol{expr.} < \mathbf{0} \end{cases}$$

This tells us that, when an $expression$ is nonnegative, the absolute value of the $expression$ is the **same** as the $\boldsymbol{expression}$, and when the $expression$ is negative, the absolute value of the $expression$ is the **opposite** of the $\boldsymbol{expression}$.

For example, to evaluate $|x-1|$, we consider when the expression $x-1$ is nonnegative and when it is negative. Since $x-1 \geq 0$ for $x \geq 1$, we have

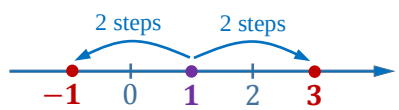
$$|x-1| = \begin{cases} x-1, & for\ \ x \geq 1 \\ -(x-1), & for\ \ x < 1 \end{cases}$$

Notice that both expressions, $x-1\ for\ x \geq 1$ and $-(x-1)\ for\ x < 1$ produce nonnegative values that represent the distance of a number x from 0.

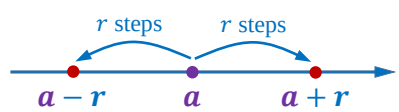
In particular,

if $x = 3$, then $|\boldsymbol{x}-\mathbf{1}| = x-1 = 3-1 = \mathbf{2}$,
and
if $x = -1$, then $|\boldsymbol{x}-\mathbf{1}| = -(x-1) = -(-1-1) = -(-2) = \mathbf{2}$.

As illustrated on the number line below, both numbers, **3** and **−1** are at the distance of **2** units from **1**.



Generally, the equation $|\boldsymbol{x}-\boldsymbol{a}| = \boldsymbol{r}$ tells us that the distance between $\boldsymbol{x}$ and $\boldsymbol{a}$ is equal to $\boldsymbol{r}$. This means that $\boldsymbol{x}$ is $\boldsymbol{r}$ units away from number $\boldsymbol{a}$, in either direction.



Therefore, $\boldsymbol{x} = \boldsymbol{a}-\boldsymbol{r}$ and $\boldsymbol{x} = \boldsymbol{a}+\boldsymbol{r}$ are the solutions of the equation $|\boldsymbol{x}-\boldsymbol{a}| = \boldsymbol{r}$.

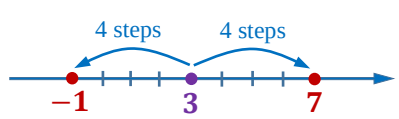
Example 1 ► **Solving Absolute Value Equations Geometrically**

For each equation, state its geometric interpretation, illustrate the situation on a number line, and then find its solution set.

a. $|x-3| = 4$ **b.** $|x+5| = 3$

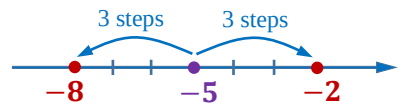
Solution ►

a. Geometrically, $|x-3|$ represents the distance between x and 3. Thus, in $|x-3| = 4$, x is a number whose distance from 3 is 4. So, $x = 3 \pm 4$, which equals either -1 or 7.



Therefore, the solution set is $\{\mathbf{-1}, \mathbf{7}\}$.

b. By rewriting $|x+5|$ as $|x-(-5)|$, we can interpret this expression as the distance between x and -5. Thus, in $|x+5| = 3$, x is a number whose distance from -5 is 3. Thus, $x = -5 \pm 3$, which results in -8 or -2.



Therefore, the solution set is $\{-\mathbf{8}, -\mathbf{2}\}$.

Although the geometric interpretation of absolute value proves to be very useful in solving some of the equations, it can be handy to have an algebraic method that will allow us to solve any type of absolute value equation.

Suppose we wish to solve an equation of the form

$$|\boldsymbol{expr.}| = \boldsymbol{r}, \quad \text{where } r > 0$$

We have two possibilities. Either the ***expression*** inside the absolute value bars is nonnegative, or it is negative. By definition of absolute value, if the ***expression*** is nonnegative, our equation becomes

$$\boldsymbol{expr.} = \boldsymbol{r}$$

If the ***expression*** is negative, then to remove the absolute value bar, we must change the sign of the ***expression***. So, our equation becomes

$$-\boldsymbol{expr.} = \boldsymbol{r},$$

which is equivalent to

$$\boldsymbol{expr.} = -\boldsymbol{r}$$

In summary, for $\boldsymbol{r} > 0$, the equation

$$|\boldsymbol{expr.}| = \boldsymbol{r}$$

is equivalent to the system of equations

$$\boldsymbol{expr.} = \boldsymbol{r} \quad \text{or} \quad \boldsymbol{expr.} = -\boldsymbol{r},$$

with the connecting word *or*.

If $\boldsymbol{r} = 0$, then $|\boldsymbol{expr.}| = \mathbf{0}$ is equivalent to the equation $\boldsymbol{expr.} = \mathbf{0}$ with no absolute value.

If $\boldsymbol{r} < 0$, then $|\boldsymbol{expr.}| = \boldsymbol{r}$ has **NO SOLUTION**, as an absolute value is never negative.

Now, suppose we wish to solve an equation of the form

$$|\boldsymbol{expr.A}| = |\boldsymbol{expr.B}|$$

Since both expressions, $\boldsymbol{A}$ and $\boldsymbol{B}$, can be either nonnegative or negative, when removing absolute value bars, we have four possibilities:

$$\boldsymbol{expr.A} = \boldsymbol{expr.B} \quad \text{or} \quad \boldsymbol{expr.A} = -\boldsymbol{expr.B}$$
$$-\boldsymbol{expr.A} = -\boldsymbol{expr.B} \quad \text{or} \quad -\boldsymbol{expr.A} = \boldsymbol{expr.B}$$

However, observe that the equations in blue are equivalent. Also, the equations in green are equivalent. So, in fact, it is enough to consider just the first two possibilities.

Therefore, the equation

$$|\boldsymbol{expr.A}| = |\boldsymbol{expr.B}|$$

is equivalent to the system of equations

$$\boldsymbol{expr.A} = \boldsymbol{expr.B} \text{ or } \boldsymbol{expr.A} = -(\boldsymbol{expr.B}),$$

with the connecting word *or*.

Example 2 ► **Solving Absolute Value Equations Algebraically**

Solve the following equations.

a. $|2-3x| = 7$

b. $5|x| - 3 = 12$

c. $\left|\frac{1-x}{4}\right| = 0$

d. $|6x+5| = -4$

e. $|2x-3| = |x+5|$

f. $|x-3| = |3-x|$

Solution ►

a. To solve $|2-3x| = 7$, we remove the absolute value bars by changing the equation into the corresponding system of equations with no absolute value anymore. Then, we solve the resulting linear equations. So, we have

$$|2-3x| = 7$$

$$2-3x = 7 \quad \text{or} \quad 2-3x = -7$$

$$2-7 = 3x \quad \text{or} \quad 2+7 = 3x$$

$$x = \frac{-5}{3} \quad \text{or} \quad x = \frac{9}{3} = 3$$

Therefore, the solution set of this equation is $\left\{-\frac{5}{3}, \mathbf{3}\right\}$.

b. To solve $5|x| - 3 = 12$, first, we **isolate the absolute value**, and then replace the equation by the corresponding system of two linear equations.

$$5|x| - 3 = 12$$

$$5|x| = 15$$

$$|x| = 3$$

$$x = 3 \quad \text{or} \quad x = -3$$

So, the solution set of the given equation is $\{\mathbf{-3}, \mathbf{3}\}$.

c. By properties of absolute value, $\left|\frac{1-x}{4}\right| = 0$ if and only if $\frac{1-x}{4} = 0$, which happens when the numerator $1 - x = 0$. So, the only solution to the given equation is $x = \mathbf{1}$.

d. Since an absolute value is never negative, the equation $|6x+5| = -4$ does not have any solution.

e. To solve $|2x-3| = |x+5|$, we remove the absolute value symbols by changing the equation into the corresponding system of linear equations with no absolute value. Then, we solve the resulting equations. So, we have

$$|2x-3| = |x+5|$$

$$2x-3 = x+5 \quad \textit{or} \quad 2x-3 = -(x+5)$$

$$2x-x = 5+3 \quad \textit{or} \quad 2x-3 = -x-5$$
$$x = 8 \quad \textit{or} \quad 3x = -2$$
$$x = \frac{-2}{3}$$

Therefore, the solution set of this equation is $\left\{-\frac{2}{3}, \mathbf{8}\right\}$.

f. We solve $|x-3| = |3-x|$ as in *Example 2e*.

$$|x-3| = |3-x|$$

$$x-3 = 3-x \quad \textit{or} \quad x-3 = -(3-x)$$

$$2x = 6 \quad \textit{or} \quad x-3 = -3+x$$
$$x = 3 \quad \textit{or} \quad 0 = 0$$

Since the equation $0 = 0$ is always true, any real x-value satisfies the original equation $|x-3| = |3-x|$. So, the solution set to the original equation is $\mathbb{R}$.

Remark: *Without solving the equation in Example 2f, one could observe that the expressions $x-3$ and $3-x$ are opposite to each other and as such, they have the same absolute value. Therefore, the equation is always true.*

Summary of Solving Absolute Value Equations

Step 1 **Isolate the absolute value** expression on one side of the equation.

Step 2 **Check for special cases**, such as

$$|\boldsymbol{A}| = \mathbf{0} \leftrightarrow \boldsymbol{A} = \mathbf{0}$$
$$|\boldsymbol{A}| = -\boldsymbol{r} \rightarrow \textbf{No solution}$$

Step 2 **Remove the absolute value symbol** by replacing the equation with the corresponding system of equations with the joining word *or*,

$$|\boldsymbol{A}| = \boldsymbol{r} \quad (\boldsymbol{r} > \mathbf{0}) \qquad\qquad |\boldsymbol{A}| = |\boldsymbol{B}|$$

$$\boldsymbol{A} = \boldsymbol{r} \ \textit{or}\ \boldsymbol{A} = -\boldsymbol{r} \qquad\qquad \boldsymbol{A} = \boldsymbol{B} \ \textit{or}\ \boldsymbol{A} = -\boldsymbol{B}$$

Step 3 **Solve** the resulting equations.

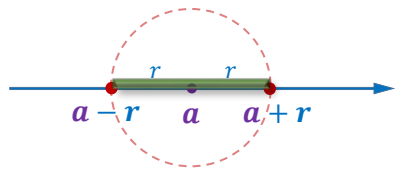
Step 4 **State the solution set** as a union of the solutions of each equation in the system.

Absolute Value Inequalities with One Absolute Value Symbol

GEOMETRIC VISUALIZATION

Suppose we wish to solve inequalities of the form $|x-a|<r$ or $|x-a|>r$, where r is a positive real number. Similarly as in the case of absolute value equations, we can either use a geometric interpretation with the aid of a number line, or we can rely on an algebraic procedure.

Using the geometrical visualization of $|x-a|$ as the distance between x and a on a number line, the inequality $|x-a|<r$ tells us that the number x is less than r units from number a. One could think of drawing a circle centered at a, with radius r. Then, the solutions of the inequality $|x-a|<r$ are all the points on a number line that lie inside such a circle (*see the green segment below*).



Therefore, the solution set is the interval $(a-r, a+r)$.

This result can be achieved algebraically by rewriting the absolute value inequality

$$|x-a|<r$$

in an equivalent three-part inequality form

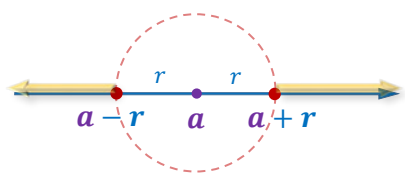
$$-r<x-a<r,$$

and then solving it for x

$$a-r<x<a+r,$$

which confirms that the solution set is indeed $(a-r, a+r)$.

Similarly, the inequality $|x-a|>r$ tells us that the number x is more than r units from number a. As illustrated in the diagram below, the solutions of this inequality are all points on a number line that lie outside of the circle centered at a, with radius r.



Therefore, the solution set is the union $(-\infty, a-r) \cup (a+r, \infty)$.

As before, this result can be achieved algebraically by rewriting the absolute value inequality

$$|x-a|>r$$

in an equivalent system of two inequalities joined by the word *or*

$$x-a<-r \quad \text{or} \quad r<x-a,$$

and then solving it for x

$$x < a - r \quad \text{or} \quad a + r < x,$$

which confirms that the solution set is $(-\infty, a - r) \cup (a + r, \infty)$.

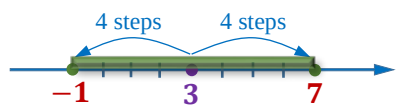
Example 3 ► **Solving Absolute Value Inequalities Geometrically**

For each inequality, state its geometric interpretation, illustrate the situation on a number line, and then find its solution set.

a. $|x - 3| \leq 4$ **b.** $|x + 5| > 3$

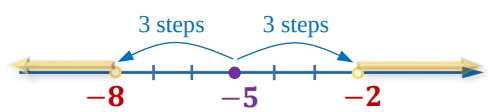
Solution ►

a. Geometrically, $|x - 3|$ represents the distance between x and 3. Thus, in $|x - 3| \leq 4$, x is a number whose distance from 3 is at most 4, in either direction.
So, $3 - 4 \leq x \leq 3 + 4$, which is equivalent to $-1 \leq x \leq 7$.



Therefore, the solution set is $[-1, 7]$.

b. By rewriting $|x + 5|$ as $|x - (-5)|$, we can interpret this expression as the distance between x and -5. Thus, in $|x + 5| > 3$, x is a number whose distance from -5 is more than 3, in either direction. Thus, $x < -5 - 3$ or $-5 + 3 < x$, which results in $x < -8$ or $x > -2$.



Therefore, the solution set equals $(-\infty, -8) \cup (-2, \infty)$.

The algebraic strategy can be applied to any inequality of the form

$$|expr.| < (\leq) r, \quad \text{or} \quad |expr.| > (\geq) r, \quad \text{as long as } r > 0.$$

Depending on the type of inequality, we follow these rules:

$$|expr.| < r \quad \Rightarrow \quad -r < expr. < r$$

or

$$|expr.| > r \quad \Rightarrow \quad expr. < -r \quad \text{or} \quad r < expr.$$

These rules also apply to weak inequalities, such as $\leq$ or $\geq$.

In the above rules, we assume that $r > 0$. **What if** $r = 0$ **?**

Observe that, the inequality $|expr.| < 0$ is never true, so this inequality doesn't have any solution. Since $|expr.| < 0$ is never true, the inequality $|expr.| \leq 0$ is equivalent to the equation $|expr.| = 0$.

On the other hand, $|\boldsymbol{expr.}| \geq \mathbf{0}$ is always true, so the solution set equals to $\mathbb{R}$. However, since $|\boldsymbol{expr.}|$ is either positive or zero, the solution to $|\boldsymbol{expr.}| > \mathbf{0}$ consists of all real numbers except for the solutions of the equation $\boldsymbol{expr.} = \mathbf{0}$.

What if $r < 0$?

Observe that both inequalities $|\boldsymbol{expr.}| > \boldsymbol{negative}$ and $|\boldsymbol{expr.}| \geq \boldsymbol{negative}$ are always true, so the solution set of such inequalities is equal to $\mathbb{R}$.

On the other hand, both inequalities $|\boldsymbol{expr.}| < \boldsymbol{negative}$ and $|\boldsymbol{expr.}| \leq \boldsymbol{negative}$ are never true, so such inequalities result in **NO SOLUTION**.

Example 4 ► **Solving Absolute Value Inequalities with One Absolute Value Symbol**

Solve each inequality. Give the solution set in both interval and graph form.

a. $|5x + 9| \leq 4$ **b.** $|-2x - 5| > 1$

e. $16 \leq |2x - 3| + 9$ **f.** $1 - 2|4x - 7| > -5$

Solution ► **a.** To solve $|5x + 9| \leq 4$, first, we remove the absolute value symbol by rewriting the inequality in the three-part inequality, as below.

$$|5x + 9| \leq 4$$

$$-4 \leq 5x + 9 \leq 4 \qquad / -9$$

$$-13 \leq 5x \leq -5 \qquad / \div 5$$

$$-\frac{13}{5} \leq x \leq -1$$

The solution is shown in the graph below.

The inequality is satisfied by all $x \in \left[-\frac{13}{5}, -\mathbf{1}\right]$.

b. As in the previous example, first, we remove the absolute value symbol by replacing the inequality with the corresponding system of inequalities, joined by the word *or*. So, we have

$$|-2x - 5| > 1$$

$$-2x - 5 < -1 \quad \textit{or} \quad 1 < -2x - 5 \qquad / +5$$

$$-2x < 4 \quad \textit{or} \quad 6 < -2x \qquad / \div (-2)$$

$$x > -2 \quad \textit{or} \quad -3 > x$$

The joining word *or* indicates that we look for the union of the obtained solutions. This union is shown in the graph below.

The inequality is satisfied by all $x \in (-\infty, -\mathbf{3}) \cup (-\mathbf{2}, \infty)$.

c. To solve $16 \le |2x-3|+9$, first, we **isolate the absolute value**, and then replace the inequality with the corresponding system of two linear equations. So, we have

$$16 \le |2x-3|+9 \qquad / -9$$

$$7 \le |2x-3|$$

$$2x-3 \le -7 \quad \text{or} \quad 7 \le 2x-3 \qquad / +3$$

$$2x \le -4 \quad \text{or} \quad 2x \ge 10 \qquad / \div 2$$

$$x \le -2 \quad \text{or} \quad x \ge 5$$

The joining word *or* indicates that we look for the union of the obtained solutions. This union is shown in the graph below.

So, the inequality is satisfied by all $x \in (-\infty, \mathbf{-2}) \cup (\mathbf{5}, \infty)$.

d. As in the previous example, first, we **isolate the absolute value**, and then replace the inequality with the corresponding system of two inequalities.

$$1-2|4x-7| > -5 \qquad / -1$$

$$-2|4x-7| > -6 \qquad / \div (-2)$$

$$|4x-7| < 3$$

$$-3 < 4x-7 < 3 \qquad / +7$$

$$4 < 4x < 10 \qquad / \div 4$$

$$1 < x < \frac{10}{4} = \frac{5}{2}$$

So the solution set is the interval $\left(\mathbf{1}, \frac{\mathbf{5}}{\mathbf{2}}\right)$, visualized in the graph below.

Example 5 ► **Solving Absolute Value Inequalities in Special Cases**

Solve each inequality.

a. $\left|\frac{1}{2}x + \frac{5}{3}\right| \ge -3$

b. $|4x-7| \le 0$

c. $|3-4x| > 0$

d. $1 - 2\left|\frac{3}{2}x - 5\right| > 3$

Solution ►

a. Since an absolute value is always bigger or equal to zero, the inequality $\left|\frac{1}{2}x + \frac{5}{3}\right| \ge -3$ is always true. Thus, it is satisfied by **any real number**. So the solution set is $\mathbb{R}$.

b. Since $|4x-7|$ is never negative, the inequality $|4x-7| \leq 0$ is satisfied only by solutions to the equation $|4x-7| = 0$. So, we solve

$$|4x-7| = 0$$
$$4x-7 = 0 \qquad / +7$$
$$4x = 7 \qquad / \div 4$$
$$x = \frac{7}{4}$$

Therefore, the inequality is satisfied only by $\boldsymbol{x = \frac{7}{4}}$.

c. Inequality $|3-4x| > 0$ is satisfied by all real x-values except for the solution to the equation $3-4x = 0$. Since

$$3-4x = 0 \qquad / +4x$$
$$3 = 4x \qquad / \div 4$$
$$\frac{3}{4} = x,$$

then the solution to the original inequality is $\boldsymbol{\left(-\infty, \frac{3}{4}\right) \cup \left(\frac{3}{4}, \infty\right)}$.

d. To solve $1 - 2\left|\frac{3}{2}x - 5\right| > 3$, first, we **isolate the absolute value**. So, we have

$$1 - 2\left|\frac{3}{2}x - 5\right| > 3 \qquad / +2\left|\frac{3}{2}x - 5\right|, -3$$
$$-2 > 2\left|\frac{3}{2}x - 5\right| \qquad / \div 2$$
$$-1 > \left|\frac{3}{2}x - 5\right|$$

Since $\left|\frac{3}{2}x - 5\right|$ is never negative, it can't be less than -1. So, there is **no solution** to the original inequality.

Summary of Solving Absolute Value Inequalities with One Absolute Value Symbol

Let $\boldsymbol{r}$ be a positive real number. To solve absolute value inequalities with one absolute value symbol, follow the steps:

Step 1 **Isolate the absolute value** expression on one side of the inequality.

Step 2 **Check for special cases**, such as

$$\boldsymbol{|A| < 0 \rightarrow \text{No solution}}$$
$$\boldsymbol{|A| \leq 0 \leftrightarrow A = 0}$$
$$\boldsymbol{|A| \geq 0 \rightarrow \text{All real numbers}}$$
$$\boldsymbol{|A| > 0 \rightarrow \text{All real numbers except for solutions of } A = 0}$$
$$\boldsymbol{|A| > (\geq) - r \rightarrow \text{All real numbers}}$$
$$\boldsymbol{|A| < (\leq) - r \rightarrow \text{No solution}}$$

Step 3 **Remove the absolute value symbol** by replacing the equation with the corresponding system of equations as below:

$$|\boldsymbol{A}| < \boldsymbol{r} \qquad\qquad |\boldsymbol{A}| > \boldsymbol{r}$$

$$-\boldsymbol{r} < \boldsymbol{A} < \boldsymbol{r} \qquad\qquad \boldsymbol{A} < -\boldsymbol{r} \ \ or \ \ \boldsymbol{r} < \boldsymbol{A}$$

This also applies to weak inequalities, such as $\leq$ or $\geq$.

Step 3 **Solve** the resulting equations.

Step 4 **State the solution set** as a union of the solutions of each equation in the system.

Applications of Absolute Value Inequalities

One of the typical applications of absolute value inequalities is in error calculations. When discussing errors in measurements, we refer to the *absolute error* or the *relative error*. For example, if M is the actual measurement of an object and x is the approximated measurement, then the ***absolute error*** is given by the formula $|\boldsymbol{x} - \boldsymbol{M}|$ and the ***relative error*** is calculated according to the rule $\frac{|\boldsymbol{x}-\boldsymbol{M}|}{\boldsymbol{M}}$.

In quality control situations, the relative error often must be less than some predetermined amount. For example, suppose a machine that fills two-litre milk cartons is set for a relative error no greater than 1%. We might be interested in how much milk a two-litre carton can actually contain? What is the absolute error that this machine is allowed to make?

Since $M = 2$ litres and the relative error = $1\% = 0.01$, we are looking for all x-values that would satisfy the inequality

$$\frac{|x-2|}{2} < 0.01.$$

This is the **relative error**.

This is equivalent to

$$|x-2| < 0.02$$

This is the **absolute error**.

$$-0.02 < x - 2 < 0.02$$

$$1.98 < x < 2.02,$$

so, a two-litre carton of milk can contain any amount of milk between 1.98 and 2.02 litres. The absolute error in this situation is $0.02\ l = 20\ ml$.

Example 6 ► **Solving Absolute Value Application Problems**

A technician is testing a scale with a 50 kg block of steel. The scale passes this test if the relative error when weighing this block is less than 0.1%. If x is the reading on the scale, then for what values of x does the scale pass this test?

Solution ► If the relative error must be less than 0.1%=0.001, then x must satisfy the inequality

$$\frac{|x-50|}{50} < 0.001$$

After solving it for x,

$$|x - 50| < 0.05$$

$$-0.05 < x - 50 < 0.05$$

$$49.95 < x < 50.05$$

We conclude that the scale passes the test if it shows a weight between **49.95** and **50.05** kg. This also tells us that the scale may err up to $0.05\ kg = 5\ dkg$ when weighing this block.

L.6 Exercises

Vocabulary Check *Complete each blank with one of the suggested words or the most appropriate term from the given list:* ***absolute value, addition, distance, division, multiplication, opposite, solution set, subtraction, three-part****.*

1. The _____ _____ of a real number represents the distance of this number from zero, on a number line.

2. The expression $|a - b|$ represents the _____ between a and b, on a number line.

3. The absolute value symbol 'splits' over _____ and _____ but not over _____ and _____.

4. The absolute value of any expression is also equal to the absolute value of the _____ expression.

5. If X represents an expression and r represents a positive real number, the _____ _____ of the inequality $|X| > r$ consists of $\underset{one\ /two}{\_\_\_\_\_}$ interval(s) of numbers.

6. To solve $|X| < r$, remove the absolute value symbol by writing the corresponding _____ inequality instead.

Concept Check *Simplify, if possible, leaving as little as possible inside the absolute value symbol.*

7. $|-2x^2|$ **8.** $|3x|$ **9.** $\left|\frac{-5}{y}\right|$ **10.** $\left|\frac{3}{-y}\right|$

11. $|7x^4y^3|$ **12.** $|-3x^5y^4|$ **13.** $\left|\frac{x^2}{y}\right|$ **14.** $\left|\frac{-4x}{y^2}\right|$

15. $\left|\frac{-3x^3}{6x}\right|$ **16.** $\left|\frac{5x^2}{-25x}\right|$ **17.** $|(x-1)^2|$ **18.** $|x^2 - 1|$

19. ***Concept Check*** *How many solutions will $|ax + b| = k$ have for each situation?*

a. $k < 0$ **b.** $k = 0$ **c.** $k > 0$

20. ***Concept Check*** *Match each absolute value equation or inequality in Column I with the graph of its solution set in Column II.*

Column I	Column II
a. $\lvert x\rvert = 3$	**A.** 
b. $\lvert x\rvert > 3$	**B.** 
c. $\lvert x\rvert < 3$	**C.** 
d. $\lvert x\rvert \geq 3$	**D.** 
e. $\lvert x\rvert \leq 3$	**E.** 

Concept Check *Solve each equation.*

21. $|-x| = 4$

22. $|5x| = 20$

23. $|y - 3| = 8$

24. $|2y + 5| = 9$

25. $7|3x - 5| = 35$

26. $-3|2x - 7| = -12$

27. $\left|\frac{1}{2}x + 3\right| = 11$

28. $\left|\frac{2}{3}x - 1\right| = 5$

29. $|2x - 5| = -1$

30. $|7x + 11| = 0$

31. $2 + 3|a| = 8$

32. $10 - |2a - 1| = 4$

33. $\left|\frac{2x-1}{3}\right| = 5$

34. $\left|\frac{3-5x}{6}\right| = 3$

35. $|2p + 4| = |3p - 1|$

36. $|5 - q| = |q + 7|$

37. $\left|\frac{1}{2}x + 3\right| = \left|\frac{1}{5}x - 1\right|$

38. $\left|\frac{2}{3}x - 8\right| = \left|\frac{1}{6}x + 3\right|$

39. $\left|\frac{3x-6}{2}\right| = \left|\frac{5+x}{5}\right|$

40. $\left|\frac{6-5x}{4}\right| = \left|\frac{7+3x}{3}\right|$

Concept Check *Solve each inequality. Give the solution set in both interval and graph form.*

41. $|x + 4| < 3$

42. $|x - 5| > 7$

43. $|x - 12| \geq 5$

44. $|x + 14| \leq 5$

45. $|5x + 3| \leq 8$

46. $|3x - 2| \geq 10$

47. $|7 - 2x| > 5$

48. $|-5x + 4| < 3$

49. $\left|\frac{1}{4}y - 6\right| \leq 24$

50. $\left|\frac{2}{5}x + 3\right| > 5$

51. $\left|\frac{3x-2}{4}\right| \geq 10$

52. $\left|\frac{2x+3}{3}\right| < 10$

53. $|-2x+4|-8 \geq -5$

54. $|6x-2|+3 < 9$

55. $7-2|x+4| \geq 5$

56. $9-3|x-2| < 3$

Concept Check *Solve each inequality.*

57. $\left|\frac{2}{3}x+4\right| \leq 0$

58. $\left|-2x+\frac{4}{5}\right| > 0$

59. $\left|\frac{6x-2}{5}\right| < -3$

60. $|-3x+5| > -3$

61. $|-x+4|+5 \geq 4$

62. $|4x+1|-2 < -5$

Discussion Point

63. Assume that you have solved an inequality of the form $|x-a| < b$, where a is a real number and b is a positive real number. Explain how you can write the solutions of the inequality $|x-a| \geq b$ without solving this inequality. Justify your answer.

Analytic Skills *Solve each problem.*

64. The recommended daily intake (RDI) of calcium for females aged 19–50 is 1000 mg. Actual needs vary from person to person, with a tolerance of up to 100 mg. Write this statement as an absolute value inequality, with x representing the actual needs for calcium intake, and then solve this inequality.

65. A police radar gun is calibrated to have an allowable maximum error of 2 km/h. If a car is detected at 61 km/h, what are the minimum and maximum possible speeds that the car was traveling?

66. A patient placed on a restricted diet is allowed 1300 calories per day with a tolerance of no more than 50 calories.

a. Write an inequality to represent this situation. Use c to represent the number of allowable calories per day.

b. Use the inequality in part (**a**) to find the range of allowable calories per day in the patient's diet.

67. The average annual income of residents in an apartment house is \$39,000. The income of a particular resident is not within \$5000 of the average.

a. Write an absolute value inequality that describes the income I of the resident.

b. Solve the inequality in part (**a**) to find the possible income of this resident.

68. On a highway, the speed limit is 110 km/h, and slower cars are required to travel at least 90 km/h. What absolute value inequality must any legal speed $\boldsymbol{s}$ satisfy?

69. An adult's body temperature T is considered to be normal if it is at least 36.4°C and at most 37.6°C. Express this statement as an absolute value inequality.