Linear Equations and Inequalities

One of the main concepts in Algebra is solving equations or inequalities. This is because solutions to most application problems involve setting up and solving equations or inequalities that describe the situation presented in the problem. In this unit, we will study techniques of solving linear equations and inequalities in one variable, linear forms of absolute value equations and inequalities, and applications of these techniques in word problems.

L.1 Linear Equations in One Variable

When two algebraic expressions are compared by an equal sign $(=)$, an **equation** is formed. An equation can be interpreted as a scale that balances two quantities. It can also be seen as a mathematical sentence with the verb "equals" or the verb phrase "is equal to". For example, the equation $3x - 1 = 5$ corresponds to the sentence:

One less than three times an unknown number equals five.

Unless we know the value of the unknown number (the variable x), we are unable to determine whether or not the above sentence is a true or false statement. For example, if $x = 1$, the equation $3x - 1 = 5$ becomes a false statement, as $3 \cdot 1 - 1 \neq 5$ (the "scale" is not in balance); while, if $x = 2$, the equation $3x - 1 = 5$ becomes a true statement, as $3 \cdot 2 - 1 = 5$ (the "scale" is in balance). For this reason, such sentences (equations) are called **open sentences**. Each variable value that satisfies an equation (i.e., makes it a true statement) is a **solution** (i.e., a ***root***, or a ***zero***) of the equation. An equation is **solved** by finding its **solution set**, the set of all solutions.

Attention:

* ***Equations** can be **solved** by finding the variable value(s) satisfying the equation.*

Example: $\overbrace{\mathbf{2x + x - 1}}^{\textit{left side}} \underset{\textit{equal sign}}{=} \overbrace{\mathbf{5}}^{\textit{right side}}$ *can be solved for x*

* ***Expressions** can only be **simplified** or **evaluated***

Example: $\mathbf{2x + x - 1}$ *can be simplified to* $3x - 1$
or evaluated for a particular x-value

Example 1 ► **Distinguishing Between Expressions and Equations**

Decide whether each of the following is an expression or an equation.

a. $4x - 16$ **b.** $4x - 16 = 0$

Solution ► **a.** $4x - 16$ is an **expression** as it does not contain any symbol of equality.

*This expression can be **evaluated** (for instance, if* $x = 4$*, the expression assumes the value 0), or it can be written in a different form. For example, we could **factor** it. So, we could write*

$$4x - 16 = 4(x - 4).$$

Notice that the equal symbol (=) in the above line does not indicate an equation, but rather an equivalency between the two expressions, $4x - 16$ *and* $4(x - 4)$.

b. $4x - 16 = 0$ is an **equation** as it contains an equal symbol $(=)$ that connects two sides of the equation.

To solve this equation we could factor the left-hand side expression,

$$4(x - 4) = 0,$$

and then from the zero product property, we have

$$x - 4 = 0,$$

which leads us to the solution

$$x = \mathbf{4}.$$

Attention: *Even though the two algebraic forms, $4x - 16$ and $4x - 16 = 0$ are related to each other, it is important that we neither **voluntarily add** the "$= 0$" part when we want to change the form of the expression, nor **voluntarily drop** the "$= 0$" part when we solve the equation.*

Example 2 ► **Determining if a Given Number is a Solution to an Equation**

Determine whether the number -2 is a solution to the given equation.

a. $4x = 10 + x$ **b.** $x^2 - 4 = 0$

Solution ► **a.** To determine whether -2 is a solution to the equation $4x = 10 + x$, we substitute -2 in place of the variable x and find out whether the resulting equation is a true statement. This gives us

$$4(-2) \stackrel{?}{=} 10 + (-2)$$
$$-8 \stackrel{?}{=} 8$$

Since the resulting equation is not a true statement, the number $\mathbf{-2}$ **is not a solution** to the given equation.

b. After substituting -2 for x in the equation $x^2 - 4 = 0$, we obtain

$$(-2)^2 - 4 \stackrel{?}{=} 0$$
$$4 - 4 \stackrel{?}{=} 0,$$

which becomes $0 = 0$, a true statement. Therefore, the number $\mathbf{-2}$ **is a solution** to the given equation.

Equations can be classified with respect to the number of solutions. There are **identities**, **conditional** equations, and **contradictions** (or ***inconsistent*** equations).

Definition 1.1 ► An **identity** is an equation that is satisfied by every real number for which the expressions on both sides of the equation are defined. Some examples of identities are

$$2x + 5x = 7x, \quad x^2 - 4 = (x+2)(x-2), \quad \text{or} \quad \frac{x}{x} = 1.$$

The solution set of the first two identities is the set of all real numbers, $\mathbb{R}$. *However, since the expression* $\frac{x}{x}$ *is undefined for* $x = 0$, *the solution set of the equation* $\frac{x}{x} = 1$ *is the set of all nonzero real numbers,* $\{\boldsymbol{x} | \boldsymbol{x} \neq \mathbf{0}\}$.

A **conditional** equation is an equation that is satisfied by at least one real number, but is not an identity. This is the most commonly encountered type of equation. Here are some examples of conditional equations:

$$3x - 1 = 5, \quad x^2 - 4 = 0, \quad \text{or} \quad \sqrt{x} = 2.$$

The solution set of the first equation is $\{2\}$; *of the second equation is* $\{-2,2\}$; *and of the last equation is* $\{4\}$.

A **contradiction** (an ***inconsistent*** equation) is an equation that has **no solution**. Here are some examples of contradictions:

$$5 = 1, \quad 3x - 3x = 8, \quad \text{or} \quad 0x = 1.$$

The solution set of any contradiction is the empty set, $\emptyset$.

Example 3 ► **Recognizing Conditional Equations, Identities, and Contradictions**

Determine whether the given equation is *conditional*, an *identity*, or a *contradiction*.

a. $x = x$ **b.** $x^2 = 0$ **c.** $\frac{1}{x} = 0$

Solution ►

a. This equation is satisfied by any real number. Therefore, it is an **identity**.

b. This equation is satisfied by $x = 0$, as $0^2 = 0$. However, any nonzero real number when squared becomes a positive number. So, the left side of the equation $x^2 = 0$ does not equal to zero for a nonzero x. That means that a nonzero number does not satisfy the equation. Therefore, the equation $x^2 = 0$ has exactly one solution, $x = 0$. So, the equation is **conditional**.

c. A fraction equals zero only when its numerator equals to zero. Since the numerator of $\frac{1}{x}$ does not equal to zero, then no matter what the value of x would be, the left side of the equation will never equal zero. This means that there is no x-value that would satisfy the equation $\frac{1}{x} = 0$. Therefore, the equation has no solution, which means it is a **contradiction**.

Attention: *Do not confuse the solution* $x = 0$ *to the equation in Example 3b with an empty set* $\emptyset$. *An empty set means that there is no solution.* $x = 0$ *means that there is one solution equal to zero.*

Solving Linear Equations in One Variable

In this section, we will focus on solving linear (up to the first degree) equations in one variable. Before introducing a formal definition of a linear equation, let us recall the definition of a term, a constant term, and a linear term.

Definition 1.2 ► A **term** is a **product** of numbers, letters, and possibly other algebraic expressions.

Examples of terms: $2,\ -3x,\ \frac{2}{3}(x+1),\ 5x^2y,\ -5\sqrt{x}$

A **constant term** is a number or a product of numbers.

Examples of constant terms: $2,\ -3,\ \frac{2}{3},\ 0,\ -5\pi$

A **linear term** is a product of numbers and the first power of a single variable.

Examples of linear terms: $-3x,\ \frac{2}{3}x,\ x,\ -5\pi x$

Definition 1.3 ► A **linear** equation is an equation with only **constant** or **linear terms**. A linear equation in one variable can be written in the form $\boldsymbol{Ax+B=0}$, for some real numbers A and B, and a variable x.

Here are some examples of *linear* equations: $2x+1=0,\quad 2=5,\quad 3x-7=6+2x$

Here are some examples of *nonlinear* equations: $x^2=16,\quad x+\sqrt{x}=-1,\quad 1+\frac{1}{x}=\frac{1}{x+1}$

So far, we have been finding solutions to equations mostly by guessing a value that would make the equation true. To find a methodical way of solving equations, observe the relations between equations with the same solution set. For example, equations

$$3x-1=5,\ \ 3x=6,\ \ and\ \ x=2$$

all have the same solution set $\{2\}$. While the solution to the last equation, $x=2$, is easily "seen" to be 2, the solution to the first equation, $3x-1=5$, is not readily apparent. Notice that the second equation is obtained by adding 1 to both sides of the first equation. Similarly, the last equation is obtained by dividing the second equation by 3. This suggests that to solve a linear equation, it is enough to write a sequence of simpler and simpler equations that preserve the solution set, and eventually result in an equation of the form:

$$x=constant \quad \text{or} \quad 0=constant.$$

If the resulting equation is of the form $x=constant$, the solution is this constant.
If the resulting equation is $\mathbf{0}=\mathbf{0}$, then the original equation is an **identity**, as it is true for **all real values** $\boldsymbol{x}$.
If the resulting equation is $\mathbf{0}=\boldsymbol{constant\ other\ than\ zero}$, then the original equation is a **contradiction**, as there is **no real values** $\boldsymbol{x}$ that would make it true.

Definition 1.4 ► **Equivalent equations** are equations with the same solution set.

How can we transform an equation to obtain a simpler but equivalent one?

We can certainly simplify expressions on both sides of the equation, following properties of operations listed in *section R3*. Also, recall that an equation works like a scale in balance.

Therefore, adding (or subtracting) the same quantity to (from) both sides of the equation will preserve this balance. Similarly, multiplying (or dividing) both sides of the equation by a nonzero quantity will preserve the balance.

Suppose we work with an equation $A = B$, where A and B represent some algebraic expressions. In addition, suppose that C is a real number (or another expression).
Here is a summary of the basic equality operations that can be performed to produce equivalent equations:

Equality Operation	**General Rule**	**Example**
Simplification	Write each expression in a simpler but equivalent form	$2(x-3) = 1 + 3$ can be written as $2x - 6 = 4$
Addition	$A + C = B + C$	$2x - 6 + 6 = 4 + 6$
Subtraction	$A - C = B - C$	$2x - 6 - 4 = 4 - 4$
Multiplication	$CA = CB, \quad if\ C \neq 0$	$\frac{1}{2} \cdot 2x = \frac{1}{2} \cdot 10$
Division	$\frac{A}{C} = \frac{B}{C}, \quad if\ C \neq 0$	$\frac{2x}{2} = \frac{10}{2}$

Example 4 ► **Using Equality Operations to Solve Linear Equations in One Variable**

Solve each equation.

a. $4x - 12 + 3x = 3 + 5x - 2x$ **b.** $2[3(x-6) - x] = 3x - 2(5 - x)$

Solution ► **a.** First, simplify each side of the equation and then isolate the linear terms (terms containing x) on one side of the equation. Here is a sequence of equivalent equations that leads to the solution:

$$4x - 12 + 3x = 3 + 6x - 2x \qquad \text{collect like terms} \qquad (1)$$

$$7x - 12 = 3 + 4x \qquad (2)$$

move the x-terms to the left side by subtracting $4x$ and the constant terms to the right side by adding 12

$$7x - \cancel{12} - 4x + \cancel{12} = 3 + \cancel{4x} - \cancel{4x} + 12 \qquad (3)$$

$$7x - 4x = 3 + 12 \qquad \text{collect like terms} \qquad (4)$$

$$3x = 15 \qquad (5)$$

isolate x by dividing both sides by 3

$$\frac{\cancel{3}x}{\cancel{3}} = \frac{15}{3} \qquad \text{Simplify each side} \qquad (6)$$

$$x = \mathbf{5} \qquad (7)$$

Each equation is written underneath the previous one, with the **“=” symbol aligned in a column**

There is only **one “=” symbol** in each line

Let us analyze the relation between line (2) and (4).

$$7x \boxed{\mathbf{-\,12}} = 3 \boxed{\mathbf{+\,4x}} \qquad (2)$$

$$7x \boxed{\mathbf{-\,4x}} = 3 \boxed{\mathbf{+\,12}} \qquad (4)$$

By subtracting $4x$ from both sides of the equation (2), we actually 'moved' the term $+4x$ to the left side of equation (4) as $-4x$. Similarly, the addition of 12 to both sides of equation (2) caused the term -12 to 'move' to the other side as $+12$, in equation (4). This shows that the addition and subtraction property of equality allows us to change the position of a term from one side of an equation to another, by simply changing its sign. Although line (3) is helpful when explaining why we can move particular terms to another side by changing their signs, it is often cumbersome, especially when working with longer equations. So, in practice, we will avoid writing lines such as (3). Since it is important to indicate what operation is applied to the equation, we will record the operations performed in the right margin, after a slash symbol ($/$). Here is how we could record the solution to equation (1) in a concise way.

$$4x - 12 + 3x = 3 + 6x - 2x$$

$$7x - 12 = 3 + 4x$$

$$7x - 4x = 3 + 12$$

$$3x = 15$$

$$x = \mathbf{5}$$

b. First, release all the brackets, starting from the inner-most brackets. If applicable, remember to collect like terms after releasing each bracket. Finally, isolate x by applying appropriate equality operations. Here is our solution:

$$2[3(x-6) - x] = 3x - 2(5 - x) \qquad \text{release red brackets}$$

$$2[3x - 18 - x] = 3x - 10 + 2x \qquad \text{collect like terms}$$

$$2[2x - 18] = 5x - 10 \qquad \text{release the blue bracket}$$

$$4x - 36 = 5x - 10 \qquad / \; -5x, +36$$

$$-x = 26 \qquad / \; \div(-1)$$

$$x = \mathbf{-26}$$

multiplication by -1 works as well

Note: *Notice that we could choose to collect x-terms on the right side of the equation as well. This would shorten the solution by one line and save us the division by -1. Here is the alternative ending of the above solution.*

$$4x - 36 = 5x - 10 \qquad / \; -4x, +10$$

$$-26 = x$$

Example 5 ► **Solving Linear Equations Involving Fractions**

Solve

$$\frac{x-4}{4}+\frac{2x+1}{6}=5.$$

Solution ► First, clear the fractions and then solve the resulting equation as in *Example 4*. To clear fractions, multiply both sides of the equation by the LCD of 4 and 6, which is 12.

$$\frac{x-4}{4}+\frac{2x+1}{6}=5 \qquad / \cdot 12 \qquad (8)$$

$$\overset{3}{\cancel{12}}\left(\frac{x-4}{\cancel{4}}\right)+\overset{2}{\cancel{12}}\left(\frac{2x+1}{\cancel{6}}\right)=12\cdot 5 \qquad (9)$$

$$3(x-4)+2(2x+1)=60 \qquad (10)$$

$$3x-12+4x+2=60 \qquad (11)$$

$$7x-10=60 \qquad (12)$$

$$7x=70 \qquad (13)$$

$$x=\frac{70}{7}=\mathbf{10} \qquad (14)$$

When multiplying each term by the $LCD = 12$, **simplify** it with the **denominator** before **multiplying** the result by the **numerator**.

So the solution to the given equation is $x=\mathbf{10}$.

Note: *Notice, that if the division of 12 by 4 and then by 6 can be performed fluently in our minds, writing equation* (9) *is not necessary. One could write equation* (10) *directly after the original equation* (8). *One could think: 12 divided by 4 is 3 so I multiply the resulting 3 by the numerator* $(x-4)$. *Similarly, 12 divided by 6 is 2 so I multiply the resulting 2 by the numerator* $(2x+1)$. *It is important though that each term, including the free term 5, gets multiplied by 12.*

Also, notice that the reason we multiply equations involving fractions by LCD's is to clear the denominators of those fractions. That means that if the multiplication by an appropriate LCD is performed correctly, the resulting equation should not involve any denominators!

Example 6 ► **Solving Linear Equations Involving Decimals**

Solve $0.07x-0.03(15-x)=0.05(14)$.

Solution ► To solve this equation, it is convenient (although not necessary) to clear the decimals first. This is done by multiplying the given equation by 100.

Each **term** (product of numbers and variable expressions) needs to be multiplied by 100.

$$0.07x-0.03(15-x)=0.05(14) \qquad / \cdot 100$$

$$7x-3(15-x)=5(14)$$

$$7x - 45 + 3x = 70$$

$$10x = 70 + 45$$

$$x = \frac{115}{10} = \mathbf{11.5}$$

So the solution to the given equation is $x = \mathbf{11.5}$.

Note: *In general, if n is the highest number of decimal places to clear in an equation, we multiply it by 10^n.*

Attention: *To multiply a product AB by a number C, we multiply just one factor of this product, either A or B, but not both! For example,*

$$10 \cdot 0.3(0.5 - x) = (10 \cdot 0.3)(0.5 - x) = 3(0.5 - x) \quad ✓$$

or

$$10 \cdot 0.3(0.5 - x) = 0.3 \cdot [10(0.5 - x)] = 0.3(5 - 10x) \quad ✓$$

but

$$10 \cdot 0.3(0.5 - x) \neq (10 \cdot 0.3)[10(0.5 - x)] = 3(5 - 10x) \quad ✗$$

Summary of Solving a Linear Equation in One Variable

1. **Clear fractions or decimals**. Eliminate fractions by multiplying each side by the least common denominator (LCD). Eliminate decimals by multiplying by a power of 10.

2. **Clear brackets (**starting from the inner-most ones) by applying the distributive property of multiplication. **Simplify** each side of the equation by **combining like terms**, as needed.

3. **Collect** and **combine variable terms** on one side and free terms on the other side of the equation. Use the addition property of equality to collect all variable terms on one side of the equation and all free terms (numbers) on the other side.

4. **Isolate the variable** by dividing the equation by the linear coefficient (coefficient of the variable term).

L.1 Exercises

Vocabulary Check *Fill in each blank with the most appropriate term or phrase from the given list:* ***conditional, empty set, equations, equivalent, evaluated, identity, linear, solution.***

1. A _______________ equation can be written in the form $Ax + B = 0$, for some $A, B \in \mathbb{R}$.

2. A ________________ of an equation is a value of the variable that makes the equation a true statement.

3. Two equations are ________________ if they have exactly the same solution sets.

4. An _____________ is an equation that is satisfied by all real numbers for which both sides are defined.

5. The solution set of any contradiction is the ____________ _______ .

6. A ________________ equation has at least one solution but is not an identity.

7. ________________ can be solved while expressions can only be simplified or __________________.

Concept Check *True or False? Explain.*

8. The equation $5x - 1 = 9$ is equivalent to $5x - 5 = 5$.

9. The equation $x + \sqrt{x} = -1 + \sqrt{x}$ is equivalent to $x = -1$.

10. The solution set to $12x = 0$ is $\emptyset$.

11. The equation $x - 0.3x = 0.97x$ is an identity.

12. To solve $-\frac{2}{3}x = \frac{3}{5}$, we could multiply each side by the reciprocal of $-\frac{2}{3}$.

13. If a and b are real numbers, then $ax + b = 0$ has a solution.

Concept Check *Decide whether each of the following is an equation to solve or an expression to simplify.*

14. $3x + 2(x - 6) - 1$

15. $3x + 2(x - 6) = 1$

16. $-5x + 19 = 3x - 5$

17. $-5x + 19 - 3x + 5$

Concept Check *Determine whether or not the given equation is linear.*

18. $4x + 2 = x - 3$

19. $12 = x^2 + x$

20. $x + \frac{1}{x} = 1$

21. $2 = 5$

22. $\sqrt{16} = x$

23. $\sqrt{x} = 9$

Discussion Point

24. If both sides of the equation $2 = \frac{1}{x}$ are multiplied by x, we obtain $2x = 1$. Since the last equation is linear, does this mean that the original equation $2 = \frac{1}{x}$ is linear as well? Are these two equations equivalent?

Concept Check *Determine whether the given value is a solution of the equation.*

25. $2,\ 3x - 4 = 2$

26. $-2,\ \frac{1}{x} - \frac{1}{2} = -1$

27. $6,\ \sqrt{2x+4} = -4$

28. $-4,\ (x-1)^2 = 25$

Solve each equation. If applicable, tell whether the equation is an ***identity*** *or a* ***contradiction****.*

29. $6x - 5 = 0$

30. $-2x + 5 = 0$

31. $-3x + 6 = 12$

32. $5x - 3 = -13$

33. $3y - 5 = 4 + 12y$

34. $9y - 4 = 14 + 15y$

35. $2(2a - 3) - 7 = 4a - 13$

36. $3(4 - 2b) = 4 - (6b - 8)$

37. $-3t + 5 = 4 - 3t$

38. $5p - 3 = 11 + 4p + p$

39. $13 - 9(2n + 3) = 4(6n + 1) - 15n$

40. $5(5n - 7) + 40 = 2n - 3(8n + 5)$

41. $3[1 - (4x - 5)] = 5(6 - 2x)$

42. $-4(3x + 7) = 2[9 - (7x + 10)]$

43. $3[5 - 3(4 - t)] - 2 = 5[3(5t - 4) + 8] - 16$

44. $6[7 - 4(8 - t)] - 13 = -5[3(5t - 4) + 8]$

45. $\frac{2}{3}(9n - 6) - 5 = \frac{2}{5}(30n - 25) - 7n$

46. $\frac{1}{2}(18 - 6n) + 5n = 10 - \frac{1}{4}(16n + 20)$

47. $\frac{8x}{3} - \frac{5x}{4} = -17$

48. $\frac{7x}{2} - \frac{5x}{10} = 5$

49. $\frac{3x-1}{4} + \frac{x+3}{6} = 3$

50. $\frac{3x+2}{7} - \frac{x+4}{5} = 2$

51. $\frac{2}{3}\left(\frac{7}{8} + 4x\right) - \frac{5}{8} = \frac{3}{8}$

52. $\frac{3}{4}\left(3x - \frac{1}{2}\right) - \frac{2}{3} = \frac{1}{3}$

53. $x - 2.3 = 0.08x + 3.5$

54. $x + 1.6 = 0.02x - 3.6$

55. $0.05x + 0.03(5000 - x) = 0.04 \cdot 5000$

56. $0.02x + 0.04 \cdot 3000 = 0.03(x + 3000)$