

L.2

Formulas and Applications

In the previous section, we studied how to solve linear equations. Those skills are often helpful in problem solving. However, the process of solving an application problem has many components. One of them is the ability to construct a mathematical model of the problem. This is usually done by observing the relationship between the variable quantities in the problem and writing an equation that describes this relationship.

Definition 2.1 ▶ An equation that represents or models a relationship between two or more quantities is called a **formula**.

To model real situations, we often use well-known formulas, such as $R \cdot T = D$, or $a^2 + b^2 = c^2$. However, sometimes we need to construct our own models.

Data Modelling

Example 1 ▶ **Constructing a Formula to Model a Set of Data Following a Linear Pattern**

The table below shows the cost C of driving n miles in a taxi cab in Santa Barbara, CA.

number of miles driven, n	1	2	3	4	5
cost C in dollars	5.50	8.50	11.50	14.50	17.50

- Construct a formula relating the cost (C) and the number of miles driven (n) for this taxi cab.
- Use the constructed formula to find the cost of a 16-mile trip by this taxi.
- If a customer paid a fare of \$29.50, what was the distance driven by the taxi?

Solution ▶

- Observe that the increase in cost when driving each additional mile after the first is constantly \$3.00. This is because

$$17.5 - 14.5 = 14.5 - 11.5 = 11.5 - 8.5 = 8.5 - 5.5 = 3$$

If n represents the number of miles driven, then the number of miles after the first can be represented by $(n - 1)$. The cost of driving n miles is the cost of driving the first mile plus the cost of driving the additional miles, after the first one. So, we can write

$$\text{total cost } C = \left(\begin{array}{l} \text{cost of the} \\ \text{first mile} \end{array} \right) + \left(\begin{array}{l} \text{cost increase} \\ \text{per mile} \end{array} \right) \cdot \left(\begin{array}{l} \text{number of} \\ \text{additional miles} \end{array} \right)$$

or symbolically,

$$C = 5.5 + 3(n - 1)$$

The above equation can be simplified to

$$C = 5.5 + 3n - 3 = 3n + 2.5.$$

Therefore, $C = 3n + 2.5$ is the formula that models the given data.

- Since the number of driven miles is $n = 16$, we evaluate

$$C = 3 \cdot 16 + 2.5 = 50.5$$

Therefore, the cost of a 16-mile trip is **\$50.50**.

- c. This time, we are given the cost $C = 29.50$ and we are looking for the corresponding number of miles n . To find n , we substitute 31.7 for C in our formula $C = 3n + 2.5$ and then solve the resulting equation for n . We obtain

$$\begin{array}{rcl} 29.5 = 3n + 2.5 & & /-2.5 \\ 27 = 3n & & /\div 3 \\ n = 9 & & \end{array}$$

So, the taxi drove **9 miles**.

Notice that in the solution to *Example 1c*, we could first solve the equation $C = 3n + 2.5$ for n :

$$\begin{array}{rcl} C = 3n + 2.5 & & /-2.5 \\ C - 2.5 = 3n & & /\div 3 \\ n = \frac{C - 2.5}{3}, & & \end{array}$$

and then use the resulting formula to evaluate n at $C = 29.50$.

$$n = \frac{29.5 - 2.5}{3} = \frac{27}{3} = 9.$$

The advantage of solving the formula $C = 3n + 2.5$ for the variable n first is such that the resulting formula $n = \frac{C-2.5}{3}$ makes evaluations of n for various values of C easier. For example, to find the number of miles n driven for the fare of \$35.5, we could evaluate directly using $n = \frac{35.5-2.5}{3} = \frac{33}{3} = 11$ rather than solving the equation $35.5 = 3n + 2.5$ again.

Solving Formulas for a Variable

If a formula is going to be used for repeated evaluation of a specific variable, it is convenient to rearrange this formula in such a way that the desired variable is **isolated on one side** of the equation and it does not appear on the other side. Such a formula may also be called a function.

Definition 2.2 ▶ A **function** is a rule for determining the value of one variable from the values of one or more other variables, in a unique way. We say that the first variable **is a function** of the other variable(s).

For example, consider the uniform motion relation between distance, rate, and time. To evaluate rate when distance and time is given, we use the formula

$$R = \frac{D}{T}.$$

This formula describes **rate as a function of distance and time**, as rate can be uniquely calculated for any possible input of distance and time.

To evaluate time when distance and rate is given, we use the formula

$$T = \frac{D}{R}$$

This formula describes **time as a function of distance and rate**, as time can be uniquely calculated for any possible input of distance and rate.

Finally, to evaluate distance when rate and time is given, we use the formula

$$D = R \cdot T$$

Here, the **distance is** presented as **a function of rate and time**, as it can be uniquely calculated for any possible input of rate and time.

To **solve a formula for a given variable** means to rearrange the formula so that the desired **variable equals to an expression that contains only other variables** but not the one that we solve for. This can be done the same way as when solving equations.

Here are some hints and guidelines to keep in mind when solving formulas:

- **Highlight** the variable of interest and solve the equation as if the other variables were just numbers (think of easy numbers), without actually performing the given operations.

Example: To solve $mx + b = c$ for m ,

we pretend to solve, for example:

$$m \cdot 2 + 3 = 1 \quad /-3$$

$$m \cdot 2 = 1 - 3 \quad /\div 2$$

$$m = \frac{1-3}{2}$$

so we write:

$$mx + b = c \quad /-b$$

$$mx = c - b \quad /\div x$$

$$m = \frac{c-b}{x}$$

- **Reverse (undo) operations** to isolate the desired variable.

Example: To solve $2L + 2W = P$ for W , first, observe the operations applied to W :

$$W \xrightarrow{\cdot 2} 2W \xrightarrow{+2L} 2L + 2W$$

Then, reverse these operations, starting from the last one first.

$$W \xleftarrow{\div 2} 2W \xleftarrow{-2L} 2L + 2W$$

So, we solve the formula as follows:

$$2L + 2W = P \quad /-2L$$

$$2W = P - 2L \quad /\div 2$$

$$W = \frac{P - 2L}{2}$$

Notice that the last equation can also be written in the equivalent form $W = \frac{P}{2} - L$.



- **Keep the desired variable in the numerator.**

Example: To solve $R = \frac{D}{T}$ for T , we could take the reciprocal of each side of the equation to keep T in the numerator,

$$\frac{T}{D} = \frac{1}{R}$$

and then multiply by D to “undo” the division. Therefore, $T = \frac{D}{R}$.

Observation: Another way of solving $R = \frac{D}{T}$ for T is by multiplying both sides by T and dividing by R .

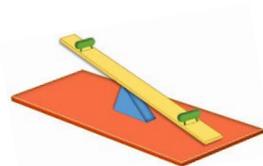
$$\frac{T}{R} \cdot R = \frac{D}{T} \cdot \frac{T}{R}$$

This would also result in $T = \frac{D}{R}$. Observe, that no matter how we solve this formula for T , the result differs from the original formula by interchanging (swapping) the variables T and R .

Note: When working only with multiplication and division, by applying inverse operations, any factor of the numerator can be moved to the other side into the denominator, and likewise, any factor of the denominator can be moved to the other side into the numerator. Sometimes it helps to think of this movement of variables as the movement of a “teeter-totter”.

For example, the formula $\frac{bh}{2} = A$ can be solved for h by dividing by b and multiplying by 2 . So, we can write directly $h = \frac{2A}{b}$.

*2 was down so now goes up
and
b was up so now goes down*



- **Keep the desired variable in one place.**

Example: To solve $A = P + Prt$ for P , we can factor P out,

$$A = P(1 + rt)$$

and then divide by the bracket. Thus,

$$P = \frac{A}{1 + rt}$$

Example 2 ▶ Solving Formulas for a Variable

Solve each formula for the indicated variable.

a. $a_n = a_1 + (n - 1)d$ for n

b. $\frac{PV}{T} = \frac{P_0V_0}{T_0}$ for T

Solution ▶

- a. To solve $a_n = a_1 + (n - 1)d$ for n we use the reverse operations strategy, starting with reversing the addition, then multiplication, and finally the subtraction.

$$a_n = a_1 + (n - 1)d \quad /-a_1$$

$$a_n - a_1 = (n - 1)d \quad / \div d$$

$$\frac{a_n - a_1}{d} = n - 1 \quad / +1$$

$$n = \frac{a_n - a_1}{d} + 1$$

The last equation can also be written in the equivalent form $n = \frac{a_n - a_1 + d}{d}$.

- b. To solve $\frac{PV}{T} = \frac{P_0V_0}{T_0}$ for T , first, we can take the reciprocal of each side of the equation to keep T in the numerator,

$$\frac{T}{PV} = \frac{T_0}{P_0V_0},$$

and then multiply by PV to “undo” the division. So,

$$T = \frac{T_0PV}{P_0V_0}.$$

Attention: Taking reciprocal of each side of an equation is a good strategy only if both sides are in the form of a **single fraction**. For example, to use the reciprocal property when solving $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$ for a , first, we perform the addition to create a single fraction, $\frac{1}{a} = \frac{c+b}{bc}$. Then, taking reciprocals of both sides will give us an instant result of $a = \frac{bc}{c+b}$.

Warning! The reciprocal of $\frac{1}{b} + \frac{1}{c}$ is **not** equal to $b + c$.

Example 3 ▶ Using Formulas in Application Problems

Body mass index I can be used to determine whether an individual has a healthy weight for his or her height. An index in the range 18.5–24.9 indicates a normal weight. Body mass index is given by the formula $I = \frac{W}{H^2}$, where W represents weight, in kilograms, and H represents height, in meters.

- What is the body index of 182 cm tall Allen, who weighs 89 kg ?
- Katia has a body mass index of 24.5 and a height of 1.7 meters. What is her weight?

Solution ▶ a. Since the formula calls for height H is in meters, first, we convert 182 cm to 1.82 meters, and then we substitute $H = 1.82$ and $W = 89$ into the formula. So,

$$I = \frac{89}{1.82^2} \approx 26.9$$

When rounded to one decimal place. Thus Allen is overweighted.

- To find Katia’s weight, we may want to solve the given formula for W first, and then plug in the given data. So, Katia’s weight is

$$W = I H^2 = 24.5 \cdot 1.7^2 \approx 70.8 \text{ kg}$$

L.2 Exercises

1. **Concept Check** When a formula is solved for a particular variable, several different equivalent forms may be possible. When solving formula $A = \frac{a+b}{2}h$ for b , one possible correct answer is $b = \frac{2A}{h} - a$. Which of the following are equivalent to this?

A. $b = \frac{A}{\frac{1}{2}h} - a$

B. $b = \frac{2A-a}{h}$

C. $b = \frac{2A-ah}{h}$

D. $b = \frac{A-ah}{\frac{1}{2}h}$

2. **Concept Check** When a group of students solved the formula $A = P + Prt$ for P , they obtained several different answers. Which of the following answers are **not** correct and why?

A. $P = \frac{A}{rt}$

B. $P = \frac{A}{1+rt}$

C. $P = A - Prt$

D. $P = \frac{A-P}{rt}$

Solve each formula for the specified variable.

3. $I = Prt$ for r (simple interest)

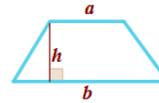
4. $C = 2\pi r$ for r (circumference of a circle)

5. $E = mc^2$ for m (mass-energy relation)

6. $F = \frac{mv^2}{r}$ for m (force in a circular motion)

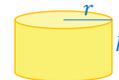
7. $A = \frac{(a+b)}{2}$ for b (average)

8. $Ax + By = c$ for y (equation of a line)



9. $P = 2l + 2w$ for l (perimeter of a rectangle)

10. $A = \frac{h}{2}(a + b)$ for a (area of a trapezoid)



11. $S = \pi rs + \pi r^2$ for π (surface area of a cone)

12. $S = 2\pi rh + 2\pi r^2$ for h (surface area of a cylinder)

13. $F = \frac{9}{5}C + 32$ for C (Celsius to Fahrenheit)

14. $C = \frac{9}{5}(F - 32)$ for F (Fahrenheit to Celsius)

15. $Q = \frac{p-q}{2}$ for p

16. $Q = \frac{p-q}{2}$ for q

17. $T = B + Bqt$ for q

18. $d = R - Rst$ for t

19. $d = R - Rst$ for R

20. $T = B + Bqt$ for B

Analytic Skills Solve each problem.

21. The table below shows the cost C of driving n kilometers in a taxi cab in Abbotsford, BC.

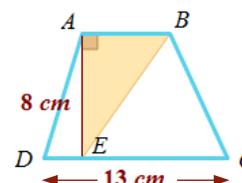
number of kilometers driven, n	1	2	3	4	5
cost C in dollars	5.10	7.00	8.90	10.80	12.70

- Construct a formula relating the cost (C) and the number of kilometers driven (n) for this taxi cab.
- Use the constructed formula to find the cost of a 10-km trip by this taxi.
- If a customer paid the fare of \$29.80, what was the distance driven by the taxi?

22. The table below shows the cost C of driving n kilometers in a taxi cab in Vancouver, BC.

number of kilometers driven, n	1	2	3	4	5
cost C in dollars	5.35	7.20	9.05	10.90	12.75

- a. Construct a formula relating the cost (C) and the number of kilometers driven (n) for this taxi cab.
 b. Use the constructed formula to find the cost of a 20-km trip by this taxi.
 c. If a customer paid the fare of \$25.70, what was the distance driven by the taxi?
23.  Assume that the rule for determining the amount of a medicine dosage for a child is given by the formula
- $$c = \frac{ad}{a + 12},$$
- where a represents the child's age, in years, and d represents the usual adult dosage, in milligrams.
- a. If the usual adult dosage of a particular medication is 250 mg. Find the dosage for a child of age 3.
 b. Solve the formula for d .
 c. Find the corresponding adult dosage, if an eight year old child uses 30 ml of a certain medication.
24. Colleges accommodate students who need to take different total-credit-hour loads. They determine the number of "full-time-equivalent" students, F , using the formula $F = \frac{n}{15}$, where n is the total number of credits students enroll in for a given semester.
- a. Determine the number of full-time-equivalent students on a campus in which students register for 42,690 credits.
 b. Solve the formula for n .
 c. Find the total number of credits students enroll in a particular semester if the number of full-time-equivalent students in this semester is 2854.
25. During a strenuous skating workout, an athlete can burn 530 calories in 40 minutes.
- a. Write a formula that calculates the calories C burned from skating 40 minutes a day for x days.
 b. How many calories could be burned in 30 days?
26. Refer to information given in problem 25.
- a. If a person loses 1 pound for every 3500 calories burned, write a formula that gives the number of pounds P lost in x days from skating 40 minutes per day.
 b. How many pounds could be lost in 100 days?
27. Write a formula that expresses the width W of a rectangle as a function of its area A and length L .
28. Write a formula that expresses the area A of a circle as a function of its diameter d .
29. a. Solve the formula $I = Prt$ for t .
 b. Using this formula, determine how long it will take a deposit of \$75 to earn \$6 interest when invested at 4% simple interest.
30. Refer to information given in the accompanying figure.



If the area of the shaded triangle ABE is 20 cm^2 , what is the area of the trapezoid $ABCD$?