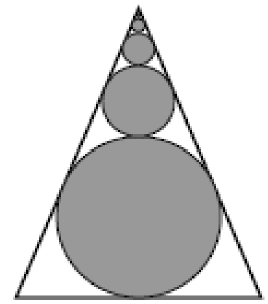
Math Match 2011



1. The base of an isosceles triangle is 20, and the equal sides are both 26. There are infinitely many shaded circles. The largest is inscribed in the triangle, and each of the others is tangent to two sides of the triangle and to the circle below it. What is the sum of the perimeters of all the circles?

2. Let $0 < x < 1$, and $F(n) = (1+x)(1+x^2)(1+x^4)...(1+x^{2^n})$.

   A) Show that $F(n) < \dfrac{1}{1-x}$

   B) Find the smallest number ***a*** such that $\left(1+\dfrac{2}{3}\right)\left(1+\dfrac{4}{9}\right)\left(1+\dfrac{16}{81}\right)...\left(1+\dfrac{2^{2^n}}{3^{2^n}}\right) < a$ for all ***n***.

3. Let ***a*** be the number of digits in the decimal expansion of 2^{2010} and let ***b*** be the number of digits in the decimal expansion of 5^{2010}. What is the value of ***a*** + ***b***?

4. Find numbers ***a*** and ***b*** such that $\lim\limits_{x\to 0} \dfrac{\sqrt{ax+b}-2}{x} = 1$.

5. If ***f*** is differentiable at *a* > 0, evaluate $\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{\sqrt{x}-\sqrt{a}}$ in terms of $f'(a)$.

6. A) Show that $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$ is an odd function.

   B) Find the inverse function of ***f***.

7. Prove that $1+3+5+...+(2n-1) = n^2$.

8. Show that 7 divides $6^{2n} - 1$ for every natural *n*.

9. How many positive integers not exceeding 1000 are divisible by 7 or 11?

10. There are **20** people in the room, and everyone shakes hands with everyone else. Haw many handshakes are there in all?
    Attempt to derive a formula for the number of handshakes if **n** people are in a room.