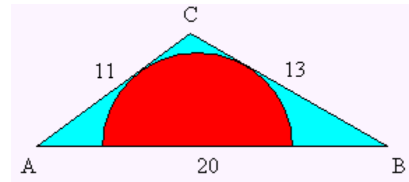
4th Annual Math Match – February 13, 2013

1. Can an irrational number raised to an irrational power be rational?



2. In ΔABC, side $AB = 20, AC = 11$, and $BC = 13$. Find the diameter of the semicircle inscribed in *ABC*, whose diameter lies on *AB*, and that is tangent to *AC* and *BC*.

3. Evaluate the sum

$$\sum_{n=1}^{2012} (-1)^n \frac{n^2+n+1}{n!}.$$

4. Find the exact solution of $\sqrt{4+\sqrt{4-\sqrt{4+\sqrt{4-x}}}} = x$.

5. Show that among any twelve composite numbers selected from the first 1200 natural numbers, there will be always two which have a common factor greater than 1.

6. There are M gold fish and K silver fish in a lake. They are caught and eaten one at a time at random until only one color of fish remains in the lake. One of the silver fish is named George. Find the probability George is not eaten.

7. Let C be a smooth closed curve (no corners) in the plane with a convex interior, and P a given point on C. Show that there are points Q and R on C such that ΔPQR is equilateral.

8. Let f be a function from the Euclidean plane R^2 to R with the property: if A, B, C are the vertices of any triangle in R^2, with circumcentre O, then $\frac{1}{3}[f(A) + f(B) + f(C)] = f(O)$. Show that f is constant.

Solutions or hints:

1. Assume that $\sqrt{2} \notin \mathbb{Q}$. $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational.

If $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$ then we are done.

If $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$, then $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = 2$, and we are done.

2. Let D and E be the tangent points of the circle (with centre at O and radius r) to the sides AC and BC, correspondingly.

Applying Heron's Formula, we have $[ABC] = \sqrt{22(22-11)(22-13)(22-20)} = 66$.
Since $[AOC] = \frac{1}{2}AC\,r = \frac{11r}{2}$ and $[BOC] = \frac{1}{2}BC\,r = \frac{13r}{2}$, then $[ABC] = \frac{11r}{2} + \frac{13r}{2} = 66$, so $r = \frac{11}{2}$ and finally diameter $= 11$ units.

3. $\sum_{n=1}^{2012}(-1)^n \frac{n^2+n+1}{n!} = \sum_{n=1}^{2012}(-1)^n \left(\frac{n}{(n-1)!} + \frac{n+1}{n!}\right) = -1 + \frac{2013}{2012!}$

4. Consider $f(x) = \sqrt{4+\sqrt{4-x}}$. Then $f(f(x)) = x$, so the solution to $f(x) = x$, if it exists, will also be a solution to $f(f(x)) = x$.

Let $y = \sqrt{4-x}$. Then $y^2 = 4 - x$. Also, from $x = \sqrt{4+\sqrt{4-x}} = \sqrt{4+y}$, we have $x^2 = 4 + y$. Therefore $x^2 - y^2 = x + y$. Hence $(x+y)(x-y-1) = 0$.
If $x + y = 0$, then $x = 0$, as $x, y \geq 0$. However it's easy to check that 0 is not a root of $f(x) = x$.
If $x - y - 1 = 0$, then substituting $y = x^2 - 4$, we obtain $x^2 - x - 3 = 0$, which produces the root $\frac{1+\sqrt{13}}{2}$.

5. A composite number less than 1200 must contain a prime factor less than $\sqrt{1200} \cong 34.6$. These primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31. Since there are only 11 such primes, one of them must divide two of the 12 composite numbers.

6. The probability is $1/(M+1)$. The easiest way to see this is to realize that the $K-1$ fish not named George are irrelevant to the problem. George survives if and only if all the M gold fish are eaten first. There are $M+1$ ways that we can permute M gold fish and George, giving the solution above.

7. Let an angle of 60° revolve counter-clockwise about P, with initial position of one of the arms tangent to C at P. The intercepts of the two arms are initially 0 and some $q > 0$. Turn the angle until the other arm becomes tangent to C, and the intercepts are now some $r > 0$ and 0. Hence the difference of the intercepts changes from $0 - q < 0$ to $r - 0 > 0$. By continuity there is a position of the two arms PQ, PR where $|PQ| = |PR|$, hence ΔPQR is equilateral.

8. Consider any two points in the plane P and Q. Draw any circle thru P and Q. Let O be its center and A and B two other points on the circle. Then by the given $f(O) = \frac{1}{2}(f(A) + f(B) + f(P)) = \frac{1}{3}(f(A) + f(B) + f(Q))$. Hence $f(P) = f(Q)$.