Review of Operations on the Set of Real Numbers

Before we start our jurney through algebra, let us review the structure of the real number system, properties of four operations, order of operations, the concept of absolute value, and set-builder and interval notation.

R.1 Structure of the Set of Real Numbers

It is in human nature to group and classify objects with the same properties. For instance, items found in one's home can be classified as furniture, clothing, appliances, dinnerware, books, lighting, art pieces, plants, etc., depending on what each item is used for, what it is made of, how it works, etc. Furthermore, each of these groups could be subdivided into more specific cathegories (groups). For example, furniture includes tables, chairs, bookshelves, desks, etc. Sometimes an item can belong to more than one group. For example, a piece of furniture can also be a piece of art. Sometimes the groups do not have any common items (e.g. plants and appliances). Similarly to everyday life, we like to classify numbers with respect to their properties. For example, even or odd numbers, prime or composite numbers, common fractions, finite or infinite decimals, infinite repeting decimals, negative numbers, etc. In this section, we will take a closer look at commonly used groups (sets) of real numbers and the relations between those groups.

Set Notation and Frequently Used Sets of Numbers

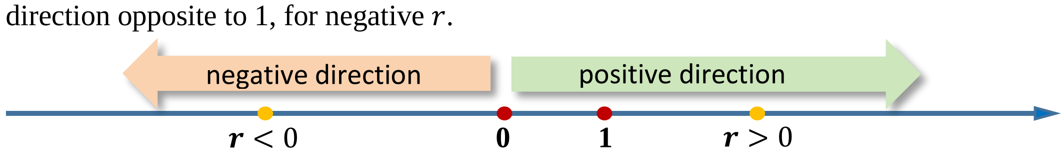
We start with terminology and notation related to sets.

Definition 1.1 ► A **set** is a collection of objects, called **elements** (or **members**) of this set.

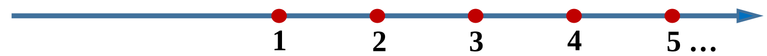
Roster Notation: A set can be given by listing its elements within the **set brackets { }** (braces).
The elements of the set are separeted by commas.
To indicate that a pattern continues, we use three dots … .
Examples:
If set A consists of the numbers 1, 2, and 3, we write $A = \{1,2,3\}$.
If set B consists of all consecutive numbers, starting from 5, we write $B = \{5,6,7,8, \dots\}$.

More on Notation: To indicate that the number 2 **is an element** of set A, we write $\mathbf{2 \in A}$.
To indicate that the number 2 **is <u>not</u> an element** of set B, we write $\mathbf{2 \notin A}$.
A set with no elements, called **empty** set, is denoted by the symbol $\emptyset$ or **{ }**.

In this course we will be working with the set of **real numbers**, denoted by $\mathbb{R}$. To visualise this set, we construct a line and choose two distinct points on it, 0 and 1, to establish direction and scale. This makes it a **number line**. Each real number r can be identified with exactly one point on such a number line by choosing the endpoint of the segment of length $|r|$ that starts from 0 and follows the line in the direction of 1, for positive r, or in the direction opposite to 1, for negative r.

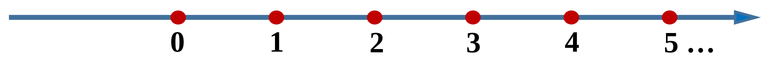


The set of real numbers contains several important subgroups (**subsets**) of numbers. The very first set of numbers that we began our mathematics education with is the set of counting numbers $\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \dots\}$, called **natural numbers** and denoted by $\mathbb{N}$.



Natural numbers $\mathbb{N}$

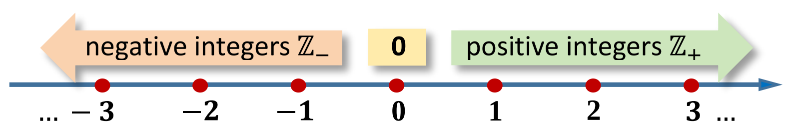
The set of natural numbers together with the number **0** creates the set of **whole numbers** $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \dots\}$, denoted by $\mathbb{W}$.



Whole numbers $\mathbb{W}$

Notice that if we perform addition or multiplication of numbers from either of the above sets, $\mathbb{N}$ and $\mathbb{W}$, the result will still be an element of the same set. We say that the set of **natural numbers** $\mathbb{N}$ and the set of **whole numbers** $\mathbb{W}$ are both **closed** under **addition** and **multiplication**.

However, if we wish to perform subtraction of natural or whole numbers, the result may become a negative number. For example, $2 - 5 = -3 \notin \mathbb{W}$, so neither the set of whole numbers nor natural numbers is not closed under subtraction. To be able to perform subtraction within the same set, it is convenient to extend the set of whole numbers to include negative counting numbers. This creates the set of **integers** $\{\dots, \mathbf{-3}, \mathbf{-2}, \mathbf{-1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \dots\}$, denoted by $\mathbb{Z}$.

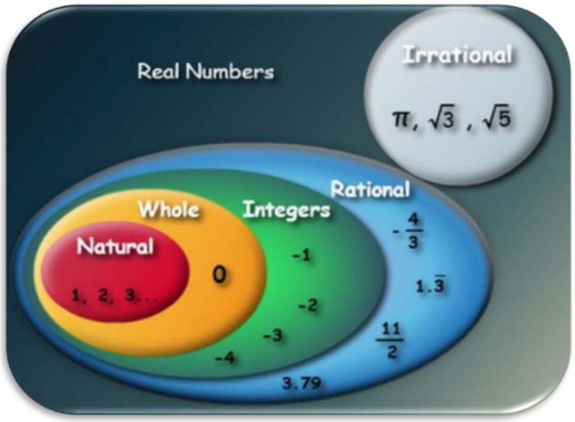


Integers $\mathbb{Z}$

Alternatively, the set of integers can be recorded using the $\pm$ sign: $\{\mathbf{0}, \pm\mathbf{1}, \pm\mathbf{2}, \pm\mathbf{3}, \dots\}$. The $\pm$ sign represents two numbers at once, the positive and the negative.

So the set of **integers** $\mathbb{Z}$ is **closed** under **addition**, **subtraction** and **multiplication**. What about division? To create a set that would be closed under division, we extend the set of integers by including all quotients of integers (all common fractions). This new set is called the set of **rational numbers** and denoted by $\mathbb{Q}$. Here are some examples of rational numbers: $\frac{3}{1} = \mathbf{3}$, $\frac{1}{2} = \mathbf{0.5}$, $-\frac{7}{4}$, or $\frac{4}{3} = \mathbf{1.\overline{3}}$.

Thus, the set of **rational numbers** $\mathbb{Q}$ is **closed** under **all four operations**. It is quite difficult to visualize this set on the number line as its elements are nearly everywhere. Between any two rational numbers, one can always find another rational number, simply by taking an average of the two. However, all the points corresponding to rational numbers still do not fulfill the whole number line. Actually, the number line contains a lot more unassigned points than points that are assigned to rational numbers. The remaining points correspond to numbers called **irrational** and denoted by $\mathbb{IQ}$. Here are some examples of irrational numbers: $\sqrt{\mathbf{2}}, \boldsymbol{\pi}, \boldsymbol{e}$, or $\mathbf{0.1010010001}\dots$.



By the definition, the two sets, $\mathbb{Q}$ and $\mathbb{IQ}$ fulfill the entire number line, which represents the set of **real numbers**, $\mathbb{R}$.

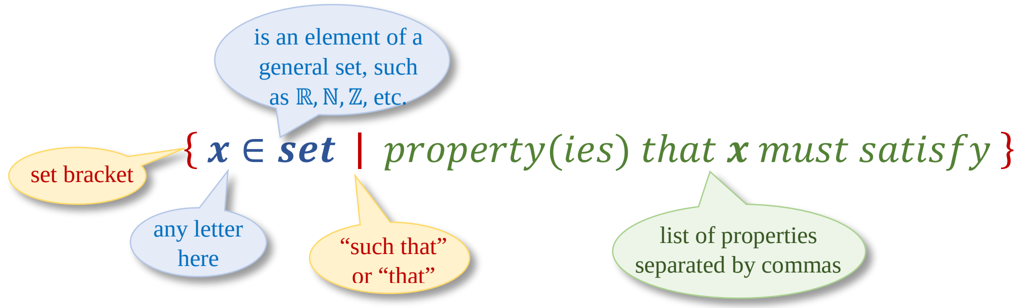
The sets $\mathbb{N}, \mathbb{W}, \mathbb{Z}, \mathbb{Q}, \mathbb{IQ}$, and $\mathbb{R}$ are related to each other as in *Figure 1*. One can make the following observations:

$\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, where $\subset$ (*read* is a **subset**) represents the operator of **inclusion of sets**;

$\mathbb{Q}$ and $\mathbb{IQ}$ are **disjoint** (they have **no common element**);

$\mathbb{Q}$ together with $\mathbb{IQ}$ create $\mathbb{R}$.

So far, we introduced six double-stroke letter signs to denote the main sets of numbers. However, there are many more sets that one might be interested in describing. Sometimes it is enough to use a subindex with the existing letter-name. For instance, the set of all positive real numbers can be denoted as $\mathbb{R}_+$ while the set of negative integers can be denoted by $\mathbb{Z}_-$. But how would one represent, for example, the set of even or odd numbers or the set of numbers divisible by 3, 4, 5, and so on? To describe numbers with a particular property, we use the **set-builder notation**. Here is the structure of set-builder notation:



For example, to describe the set of even numbers, first, we think of a property that distinguishes even numbers from other integers. This is divisibility by 2. So each even number n can be expressed as $2k$, for some integer k. Therefore, the set of even numbers could be stated as $\{n \in \mathbb{Z} \mid n = 2k, k \in \mathbb{Z}\}$ (*read*: The set of all integers n such that each n is of the form $2k$, for some integral k.)

To describe the set of rational numbers, we use the fact that any rational number can be written as a common fraction. Therefore, the set of rational numbers $\mathbb{Q}$ can be described as $\left\{x \,\middle|\, x = \frac{p}{q},\ p, q \in \mathbb{Z},\ q \neq 0\right\}$ (*read*: The set of all real numbers x that can be expressed as a fraction $\frac{p}{q}$, for integral p and q, with $q \neq 0$.)

Convention: *If the description of a set refers to the set of real numbers, there is no need to state $x \in \mathbb{R}$ in the first part of set-builder notation. For example, we can write $\{\boldsymbol{x} \in \mathbb{R} \mid \boldsymbol{x} > \boldsymbol{0}\}$ or $\{\boldsymbol{x} \mid \boldsymbol{x} > \boldsymbol{0}\}$. Both sets represent the set of all positive real numbers, which could also be recorded as simply $\mathbb{R}_+$. However, if we work with any other major set, this set must be stated. For example, to describe all positive integers $\mathbb{Z}_+$ using setbuilder notation, we write $\{\boldsymbol{x} \in \mathbb{Z} \mid \boldsymbol{x} > \boldsymbol{0}\}$ and $\mathbb{Z}$ is essential there.*

Example 1 ► **Listing Elements of Sets Given in Set-builder Notation**

List the elements of each set.

a. $\{n \in \mathbb{Z} \,|\, -2 \leq n < 5\}$ **b.** $\{n \in \mathbb{N} \,|\, n = 5k, k \in \mathbb{N}\}$

Solution ► **a.** This is the set of integers that are at least -2 but smaller than 5. So this is $\{-2, -1, 0, 1, 2, 3, 4\}$.

b. This is the set of natural numbers that are multiples of 5. Therefore, this is the infinite set $\{5, 10, 15, 20, \ldots\}$.

Example 2 ► **Writing Sets with the Aid of Set-builder Notation**

Use set-builder notation to describe each set.

a. $\{1, 4, 9, 25, \ldots\}$ **b.** $\{-2, 0, 2, 4, 6\}$

Solution ► **a.** First, we observe that the given set is composed of consecutive perfect square numbers, starting from 1. Since all the elements are natural numbers, we can describe this set using the set-builder notation as follows: $\{n \in \mathbb{N} \,|\, n = k^2, \text{for } k \in \mathbb{N}\}$.

b. This time, the given set is finite and lists all even numbers starting from -2 up to 6. Since the general set we work with is the set of integers, the corresponding set in set-builder notation can be writen as $\{n \in \mathbb{Z} \,|\, n \text{ is even}, -2 \leq n \leq 6\}$, or $\{n \in \mathbb{Z} \,|\, -2 \leq n \leq 6, n = 2k, \text{for } k \in \mathbb{Z}\}$.

Observations:

* *There are many equivalent ways to describe a set using the set-builder notation.*

* *The commas used between the conditions (properties) stated after the "such that" bar play the same role as the connecting word "and".*

Rational Decimals

How can we recognize if a number in decimal notation is rational or irrational?

A **terminating** decimal (with a **finite** number of nonzero digits after the decimal dot, like 1.25 or 0.1206) can be converted to a common fraction by replacing the decimal dot with the division by the corresponding power of 10 and then simplifying the resulting fraction. For example,

$$1.25 = \frac{125}{100} = \frac{5}{4}, \quad \text{or} \quad 0.1206 = \frac{1206}{10000} = \frac{603}{5000}.$$

2 places → 2 zeros; simplify by 25

4 places → 4 zeros; simplify by 2

Therefore, any **terminating decimal is a rational number**.

One can also convert a **nonterminating** (**infinite**) decimal to a common fraction, as long as there is a **recuring** (**repeating**) sequence of digits in the decimal expantion. This can be done using the method shown in *Example 3a*. Hence, any **infinite repeating decimal is a rational number**.

Also, notice that any fraction $\frac{m}{n}$ can be converted to either a finite or infinite repeated decimal. This is because since there are only finitely many numbers occurring as remainders in the long division process when dividing by n, eventually, either a remainder becomes zero, or the sequence of remainders starts repeating.

So **a number is rational if and only if it can be represented by a finite or infinite repeating decimal**. Since the irrational numbers are defined as those that are not rational, we can conclude that **a number is irrational if and only if it can be represented as an infinite non-repeating decimal**.

Example 3 ► **Proving that an Infinite Repeated Decimal is a Rational Number**

Show that the given decimal is a rational number.

a. $0.333\ldots$ **b.** $4.3\overline{25}$

Solution ► **a.** Let $a = 0.333\ldots$. After multiplying this equation by 10, we obtain $10a = 3.333\ldots$. Since in both equations, the number after the decimal dot is exactly the same, after subtracting the equations side by side, we obtain

$$-\begin{cases} 10a = 3.333\ldots \\ \ \ a = 0.333\ldots \end{cases}$$
$$9a = 3$$

which solves to $a = \frac{3}{9} = \frac{1}{3}$. So $0.333\ldots = \frac{1}{3}$ is a rational number.

b. Let $a = 4.6\overline{25}$. The bar above 25 tells us that the sequence 25 reapeats forever. To use the subtraction method as in solution to *Example 3a*, we need to create two equations involving the given number with the decimal dot moved after the repeating sequence and before the repeating sequence. This can be obtained by multiplying the equation $a = 2.6\overline{25}$ first by 1000 and then by 10, as below.

$$-\begin{cases} 1000a = 4325.\overline{25} \\ \ \ 10a = 43.\overline{25} \end{cases}$$
$$990a = 4302$$

Therefore, $a = \frac{4302}{990} = \frac{239}{55} = 4\frac{19}{55}$, which proves that $4.6\overline{25}$ is rational.

Example 4 ► **Identifying the Main Types of Numbers**

List all numbers of the set

$\left\{-10,\ -5.34,\ 0,\ 1,\ \frac{12}{3},\ 3.\overline{16},\ \frac{4}{7},\ \sqrt{2},\ -\sqrt{36},\ \sqrt{-4},\ \pi,\ 9.010010001\ldots\right\}$ that are

a. natural **b.** whole **c.** integral **d.** rational **e.** irrational

Solution ►

a. The only natural numbers in the given set are 1 and $\frac{12}{3} = 4$.

b. The whole numbers include the natural numbers and the number 0, so we list $0, 1$ and $\frac{12}{3}$.

c. The integral numbers in the given set include the previously listed $0, 1, \frac{12}{3}$, and the negative integers -10 and $-\sqrt{36} = -6$.

d. The rational numbers in the given set include the previously listed integers $0, 1, \frac{12}{3}, -10, -\sqrt{36}$, the common fraction $\frac{4}{7}$, and the decimals -5.34 and $3.\overline{16}$.

e. The only irrational numbers in the given set are the constant π and the infinite decimal $9.010010001 \ldots$.

Note: $\sqrt{-4}$ is not a real number.

R.1 Exercises

Vocabulary Check *Fill in each blank.*

1. A set with no _______________ is called an empty set and it is denoted ______ .

2. Two sets are _______________ if they contain no common elements.

3. The statement $\mathbb{Z} \subset \mathbb{Q}$ tells us that the set of integers is a _______________ of the set of rational numbers.

4. The set of all digits can be recorded, for example, as $\{0,1,2,3,4,5,6,7,8,9\}$ or $\{n \in \mathbb{W} | \ 0 \leq n \leq 9\}$. The first form is calld the _______________ notation and the second form is called the ________________________ notation.

5. The set of all numbers of the form $\frac{a}{b}$ such that a and b are integers with b not equal to zero is the set of _________________ numbers.

6. The decimal representations of _______________ numbers neither terminate nor repeat.

Concept Check *True or False? If it is false, explain why.*

7. Every natural number is an integer.

8. Some rational numbers are irrational.

9. Some real numbers are integers.

10. Every integer is a rational number.

11. Every infinite decimal is irrational.

12. Every square root of an odd number is irrational.

Concept Check *Use **roster notation** to list all elements of each set.*

13. The set of all positive integers less than 9

14. The set of all odd whole numbers less than 11

15. The set of all even natural numbers

16. The set of all negative integers greater than -5

17. The set of natural numbers between 3 and 9

18. The set of whole numbers divisible by 4

Concept Check *Use set-builder notation to describe each set.*

19. $\{0, 1, 2, 3, 4, 5\}$

20. $\{4, 6, 8, 10, 12, 14\}$

21. The set of all real numbers greater than -3

22. The set of all real numbers less than 21

23. The set of all multiples of 3

24. The set of perfect square numbers up to 100

Concept Check *Fill in each box with one of the signs* $\in, \notin, \subset, \not\subset$ *or* $=$ *to make the statement true.*

25. $-3 \,\square\, \mathbb{Z}$

26. $\{0\} \,\square\, \mathbb{W}$

27. $\mathbb{Q} \,\square\, \mathbb{Z}$

28. $0.3555\ldots \,\square\, \mathbb{IQ}$

29. $\sqrt{3} \,\square\, \mathbb{Q}$

30. $\mathbb{Z}_- \,\square\, \mathbb{Z}$

31. $\pi \,\square\, \mathbb{R}$

32. $\mathbb{N} \,\square\, \mathbb{Q}$

33. $\mathbb{Z}_+ \,\square\, \mathbb{N}$

Which elements of each set are ***(a)*** *natural numbers,* ***(b)*** *whole numbers,* ***(c)*** *integers,* ***(d)*** *rational numbers,* ***(e)*** *irrational numbers,* ***(f)*** *real numbers?*

34. $\left\{-1,\ 2.16,\ -\sqrt{25},\ \frac{12}{2}, -\frac{12}{5},\ 3.\overline{25},\ \sqrt{5},\ \pi,\ 3.565665666\ldots\right\}$

35. $\left\{0.999\ldots,\ -5.001,\ 0,\ 5\frac{3}{4},\ 1.4\overline{05},\ \frac{7}{8},\ \sqrt{2},\ \sqrt{16},\ \sqrt{-9},\ 9.010010001\ldots\right\}$

Analytic Skills *Show that the given decimal is a rational number.*

36. $0.555\ldots$

37. $1.\overline{02}$

38. $0.1\overline{34}$

39. $2.0\overline{125}$

40. $0.25\overline{7}$

41. $5.22\overline{54}$