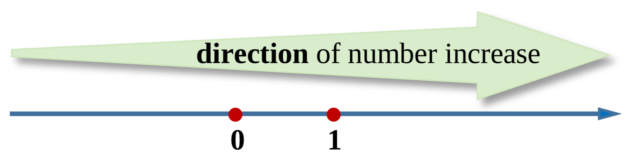
# R.2 Number Line and Interval Notation

As mentioned in the previous section, it is convenient to visualise the set of real numbers by identifying each number with a unique point on a number line.

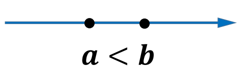
## Order on the Number Line and Inequalities

**Definition 2.1** ► A **number line** is a line with two distinct points chosen on it. One of these points is designated as **0** and the other point is designated as **1**.

The length of the segment from 0 to 1 represents one **unit** and provides the scale that allows to locate the rest of the numbers on the line. The **direction** from 0 to 1, marked by an **arrow** at the end of the line, indicates the **increasing order** on the number line. The numbers corresponding to the points on the line are called the **coordinates** of the points.

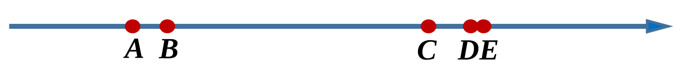

***Note:*** *For simplicity, the coordinates of points on a number line are often identified with the points themselves.*




To compare numbers, we use **inequality signs** such as $<, \leq, >, \geq$, or $\neq$. For example, if $a$ is **smaller than** $b$ we write $a < b$. This tells us that the location of point $a$ on the number line is to the left of point $b$. Equivalently, we could say that $b$ is **larger than** $a$ and write $b > a$. This means that the location of $b$ is to the right of $a$.

**Example 1** ► **Identifying Numbers with Points on a Number Line**

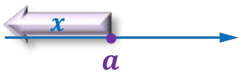
Match the numbers $-2,\ 3.5,\ \pi,\ -1.5,\ \frac{5}{2}$ with the letters on the number line:

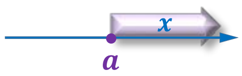



**Solution** ► To match the given numbers with the letters shown on the number line, it is enough to order the numbers from the smallest to the largest. First, observe that negative numbers are smaller than positive numbers and $-2 < -1.5$. Then, observe that $\pi \approx 3.14$ is larger than $\frac{5}{2}$ but smaller than 3.5. Therefore, the numbers are ordered as follows:

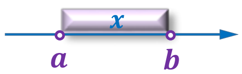
$$-2 < -1.5 < \frac{5}{2} < \pi < 3.5$$

Thus, $\boldsymbol{A = -2}, \boldsymbol{B = -1.5}, \boldsymbol{C = \frac{5}{2}}, \boldsymbol{D = \pi}$, and $\boldsymbol{E = 3.5}$.

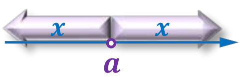



To indicate that a number $x$ is **smaller or equal** $a$, we write $x \leq a$. This tells us that the location of point $x$ on the number line is to the left of point $a$ or exactly at point $a$. Similarly, if $x$ is **larger or equal** $a$, we write $x \geq a$, and we locate $x$ to the right of point $a$ or exactly at point $a$.




To indicate that a number $x$ is **between** $a$ and $b$, we write $a < x < b$. This means that the location of point $x$ on the number line is somewhere on the segment joining points $a$ and $b$, but not at $a$ nor at $b$. Such stream of two inequalities is referred to as a **three-part inequality**.




Finally, to state that a number $x$ is **different than** $a$, we write $x \neq a$. This means that the point $x$ can lie anywhere on the entire number line, except at the point $a$.

Here is a list of some English phrases that indicate the use of particular inequality signs.

| English Phrases | Inequality Sign(s) |
|---|---|
| less than; smaller than | $<$ |
| less or equal; smaller or equal; at most; no more than | $\leq$ |
| more than; larger than; greater than; | $>$ |
| more or equal; larger or equal; greater or equal; at least; no less than | $\geq$ |
| different than | $\neq$ |
| between | $<\quad<$ |

**Example 2** ► **Using Inequality Symbols**

Write each statement as a single or a three-part inequality.

**a.** $-7$ is **less than** $5$
**b.** $2x$ is **greater or equal** $6$
**c.** $3x + 1$ is **between** $-1$ and $7$
**d.** $x$ is **between** $1$ and $8$, **including** $1$ and **excluding** $8$
**e.** $5x - 2$ is **different than** $0$
**f.** $x$ is **negative**

**Solution** ►

**a.** Write $-7 < 5$. *Notice:* The inequality "points" to the smaller number. This is an example of a **strong** inequality. One side is "strongly" smaller than the other side.

**b.** Write $2x \geq 6$. This is an example of a **weak** inequality, as it allows for equation.

**c.** Enclose $3x + 1$ within two strong inequalities to obtain $-1 < 3x + 1 < 7$. *Notice:* The word "between" indicates that the endpoints are not included.

**d.** Since 1 is included, the statement is $1 \leq x < 8$.

**e.** Write $5x - 2 \neq 0$.

**f.** Negative $x$ means that $x$ is smaller than zero, so the statement is $x < 0$.

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**Example 3** ► **Graphing Solutions to Inequalities in One Variable**

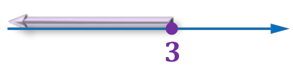
Using a number line, graph all $x$-values that satisfy (are **solutions** of) the given inequality or inequalities:

**a.** $x > -2$ **b.** $x \leq 3$ **c.** $1 \leq x < 4$

**Solution** ►

**a.** The $x$-values that satisfy the inequality $x > -2$ are larger than $-2$, so we shade the part of the number line that corresponds to numbers greater than $-2$. Those are all points to the right of $-2$, but not including $-2$. To indicate that the $-2$ is not a solution to the given inequality, we draw a hollow circle at $-2$.


**b.** The $x$-values that satisfy the inequality $x \leq 3$ are smaller than or equal to 3, so we shade the part of the number line that corresponds to the number 3 or numbers smaller than 3. Those are all points to the right of 3, including the point 3. To indicate that the 3 is a solution to the given inequality, we draw a filled in circle at 3.




**c.** The $x$-values that satisfy the inequalities $1 \leq x < 4$ are larger than or equal to 1 and at the same time smaller than 4. Thus, we shade the part of the number line that corresponds to numbers between 1 and 4, including the 1 but excluding the 4. Those are all the points that lie between 1 and 4, including the point 1 but excluding the point 4. So, we draw a segment connecting 1 with 4, with a filled in circle at 1 and a hollow circle at 4.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## Interval Notation

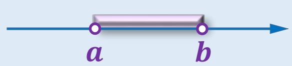
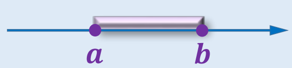
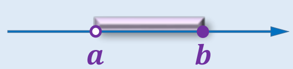
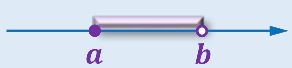
As shown in the solution to *Example 3*, the graphical solutions of inequalities in one variable result in a segment of a number line (if we extend the definition of a segment to include the endpoint at infinity). To record such a solution segment algebraically, it is convenient to write it by stating its left endpoint (corresponding to the lower number) and then the right endpoint (corresponding to the higher number), using appropriate brackets that would indicate the inclusion or exclusion of the endpoint. For example, to record algebraically the segment that starts from 2 and ends on 3, including both endpoints, we write $[2, 3]$. Such notation very closely depicts the graphical representation of the segment, 

, and is called **interval notation**.

**Interval Notation:** A set of numbers satisfying a single inequality of the type $<, \leq, >$, or $\geq$ can be recorded in interval notation, as stated in the table below.

| inequality | set-builder notation | graph | interval notation | comments |
|---|---|---|---|---|
| $x > a$ | $\{x \mid x > a\}$ |   | $(\boldsymbol{a}, \infty)$ | - list the endvalues from **left to right**<br>- to exclude the endpoint use a **round bracket ( or )** |
| $x \geq a$ | $\{x \mid x \geq a\}$ |   | $[\boldsymbol{a}, \infty)$ | - **infinity** sign is used with a **round bracket**, as there is no last point to include<br>- to include the endpoint use a **square bracket [ or ]** |
| $x < a$ | $\{x \mid x < a\}$ |   | $(-\infty, \boldsymbol{a})$ | - to indicate **negative infinity**, use the **negative sign** in front of $\infty$<br>- to indicate **positive infinity**, there is **no need** to write **a positive sign** in front of the infinity sign |
| $x \leq a$ | $\{x \mid x \leq a\}$ |   | $(-\infty, \boldsymbol{a}]$ | - remember to list the endvalues from **left to right**; this also refers to infinity signs |

Similarly, a set of numbers satisfying two inequalities resulting in a segment of solutions can be recorded in interval notation, as stated below.

| inequality | set-builder notation | graph | interval notation | comments |
|---|---|---|---|---|
| $a < x < b$ | $\{x \mid a < x < b\}$ |   | $(\boldsymbol{a}, \boldsymbol{b})$ | - we read:<br>an **open interval** from $a$ to $b$ |
| $a \leq x \leq b$ | $\{x \mid a \leq x \leq b\}$ |   | $[\boldsymbol{a}, \boldsymbol{b}]$ | - we read:<br>a **closed interval** from $a$ to $b$ |
| $a < x \leq b$ | $\{x \mid a < x \leq b\}$ |   | $(\boldsymbol{a}, \boldsymbol{b}]$ | - we read: an interval from $a$ to $b$, without $a$ but with $b$<br>This is called **half-open** or **half-closed** interval. |
| $a \leq x < b$ | $\{x \mid a \leq x < b\}$ |   | $[\boldsymbol{a}, \boldsymbol{b})$ | - we read: an interval from $a$ to $b$, with $a$ but without $b$<br>This is called **half-open** or **half-closed** interval. |

In addition, the set of all real numbers $\mathbb{R}$ is represented in the interval notation as $(-\infty, \infty)$.

**Example 4** ► **Writing Solutions to One Variable Inequalities in Interval Notation**

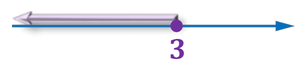
Write solutions to the inequalities from *Example 3* in set-builder and interval notation.

**a.** $x > -2$ **b.** $x \leq 3$ **c.** $1 \leq x < 4$

**Solution** ► **a.** The solutions to the inequality $x > -2$ can be stated in set-builder notation as $\{\boldsymbol{x} | \boldsymbol{x} > -\mathbf{2}\}$. Reading the graph of this set


from **left to right**, we start from $-2$, without $-2$, and go towards infinity. So, the interval of solutions is written as $(-\mathbf{2}, \infty)$. We use the round bracket to indicate that the endpoint is not included. The infinity sign is always written with the round bracket, as infinity is a concept not a number. So, there is no last number to include.

**b.** The solutions to the inequality $x \leq 3$ can be stated in set-builder notation as $\{\boldsymbol{x} | \boldsymbol{x} \leq \mathbf{3}\}$. Again, reading the graph of this set




from **left to right**, we start from $-\infty$ and go up to 3, including 3. So, the interval of solutions is written as $(-\infty, \mathbf{3}]$. We use the square bracket to indicate that the endpoint is included. As before, the infinity sign takes the round bracket. Also, we use "$-$" in front of the infinity sign to indicate negative infinity.

**c.** The solutions to the three-part inequality $1 \leq x < 4$ can be stated in set-builder notation as $\{\boldsymbol{x} | \mathbf{1} \leq \boldsymbol{x} < \mathbf{4}\}$. Reading the graph of this set


from **left to right**, we start from 1, including 1, and go up to 4, excluding 4. So, the interval of solutions is written as $[\mathbf{1}, \mathbf{4})$. We use the square bracket to indicate 1 and the round bracket, to exclude 4.

## Absolute Value, and Distance

The **absolute value** of a number $x$, denoted $|x|$, can be thought of as the distance from $x$ to 0 on a number line. Based on this interpretation, we have $|\boldsymbol{x}| = |-\boldsymbol{x}|$. This is because both numbers $x$ and $-x$ are at the same distance from 0. For example, since both 3 and $-3$ are exactly three units apart from the number 0, then $|3| = |-3| = 3$.

Since distance can not be negative, we have $|\boldsymbol{x}| \geq \mathbf{0}$.

Here is a formal definition of the absolute value operator.

**Definition 2.1** ► For any real number $x$,

$$|\boldsymbol{x}| \stackrel{def}{=} \begin{cases} \boldsymbol{x}, & \boldsymbol{if} \quad \boldsymbol{x} \geq \mathbf{0} \\ -\boldsymbol{x}, & \boldsymbol{if} \quad \boldsymbol{x} < \mathbf{0} \end{cases}.$$

The above definition of absolute value indicates that for $x \geq 0$ we use the equation $|x| = x$, and for $x < 0$ we use the equation $|x| = -x$ (the absolute value of $x$ is the opposite of $x$, which is a positive number).

**Example 5** ► **Evaluating Absolute Value Expressions**

Evaluate.

**a.** $-|-4|$ **b.** $|-5| - |2|$ **c.** $|-5 - (-2)|$

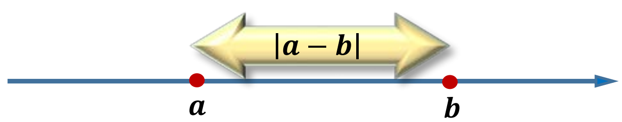
**Solution** ►

**a.** Since $|-4| = 4$ then $-|-4| = \mathbf{-4}$.

**b.** Since $|-5| = 5$ and $|2| = 2$ then $|-5| - |2| = 5 - 2 = \mathbf{3}$.

**c.** Before applying the absolute value operator, first we simplify the expression inside the absolute value sign. So we have $|-5 - (-2)| = |-5 + 2| = |-3| = \mathbf{3}$.

On a number line, the **distance** between two points with coordinates $a$ and $b$ is calculated by taking the difference between the two coordinates. So, if $b > a$, the distance is $b - a$. However, if $a > b$, the distance is $a - b$. What if we don't know which value is larger, $a$ or $b$? Since the distance must be positive, we can choose to calculate any of the differences and apply the absolute value on the result.




**Definition 2.2** ► The **distance** $d(a, b)$ between points $a$ and $b$ on a number line is given by the expression $|\boldsymbol{a - b}|$, or equivalently $|\boldsymbol{b - a}|$.

Notice that $d(x, 0) = |x - 0| = |x|$, which is consistent with the intuitive definition of absolute value of $x$ as the distance from $x$ to 0 on the number line.

**Example 6** ► **Finding Distance Between Two Points on a Number Line**

Find the distance between the two given points on the number line.

**a.** $-3$ and 5 **b.** $x$ and 2

**Solution** ►

**a.** Using the distance formula for two points on a number line, we have $d(-3,5) = |-3 - 5| = |-8| = \mathbf{8}$. Notice that we could also calculate $|5 - (-3)| = |8| = \mathbf{8}$.

**b.** Following the formula, we obtain $d(x, 2) = |x - 2|$. Since $a$ is unknown, the distance between $a$ and 2 is stated as an expression $|\boldsymbol{x - 2}|$ rather than a specific number.

## R.2 Exercises

***Vocabulary Check*** *Fill in each blank with the most appropriate term or phrase from the given list:* ***absolute value, arrow, distance, interval, number.***

**1.** A line with two distinct points designated as 0 and 1 is called a ___________ line.

**2.** An ____________ at the end of a number line indicates the **increasing order** (or **positive direction**) on the number line.

**3.** The solutions to an inequality in one variable can be graphed on a number line or stated algebraically in ______________ notation or set-builder notation.

**4.** The ______________ ___________ of a number is its distance from 0 on the number line.

**5.** $|a - b|$ represents the ______________ between points $a$ and $b$ on a number line.

***Concept Check*** *Use an inequality symbol to write each statement.*

**6.** $-6$ is less than $-3$

**7.** $0$ is more than $-1$

**8.** $17$ is greater or equal to $x$

**9.** $x$ is smaller or equal to $8$

**10.** $2x + 3$ is different than zero

**11.** $2 - 5x$ is negative

**12.** $x$ is between $2$ and $5$

**13.** $3x$ is between $-5$ and $7$

**14.** $2x$ is between $-2$ and $6$, including $-2$ and excluding $6$

**15.** $x + 1$ is between $-5$ and $11$, excluding $-5$ and including $11$

***Concept Check*** *Graph each set of numbers on a number line and write it in* ***interval notation****.*

**16.** $\{x \mid x \geq -4\}$

**17.** $\{x \mid x \leq -3\}$

**18.** $\left\{x \middle| x < \frac{5}{2}\right\}$

**19.** $\left\{x \middle| x > -\frac{2}{5}\right\}$

**20.** $\{x \mid 0 < x < 6\}$

**21.** $\{x \mid -1 \leq x \leq 4\}$

**22.** $\{x \mid -5 \leq x < 16\}$

**23.** $\{x \mid -12 < x \leq 4.5\}$

***Concept Check*** *Evaluate.*

**24.** $-|-7|$

**25.** $|5| - |-13|$

**26.** $|11 - 19|$

**27.** $|-5 - (-9)|$

**28.** $-|9| - |-3|$

**29.** $-|-13 + 7|$

*Replace each* $\square$ *with one of the signs* $<, >, \leq, \geq, =$ *to make the statement true.*

**30.** $-7 \square -5$

**31.** $|-16| \square -|16|$

**32.** $-3 \square -|3|$

**33.** $x^2 \square 0$

**34.** $x \square |x|$

**35.** $|x| \square |-x|$

*Find the distance between the given points on the number line.*

**36.** $-7, -32$ **37.** $46, -13$ **38.** $-\frac{2}{3}, \frac{5}{6}$

**39.** $x, 0$ **40.** $5, y$ **41.** $x, y$

*Find numbers that are at the distance of 5 units from the given point.*

**42.** 0 **43.** 3 **44.** $a$

***Analytic Skills*** *The graph shows sales and profit of certain business for years 2010 through 2015. Use the information given in this graph to answer questions 45 through 51.*


**45.** When were sales greatest? When were profits highest?

**46.** When were sales lowest? When were profits lowest?

**47.** Describe a relationship between profits and sales.

**48.** If the operating cost is the difference between the sales and the profit, in which year the operating cost was the greatest? In which year the operating cost was the lowest?

**49.** What was the decrease in profit between years 2012 and 2013? What was the increase in profit between years 2014 and 2015?

**50.** Between which two consecutive years the sales dropped the most? Between which two consecutive years the sales raised the most?

**51.** During the years 2010 to 2015, what was the average profit per year? What were the average sales per year? *Round the answer to the nearest thousands.*