R.3 Properties and Order of Operations on Real Numbers

In algebra, we are often in need of changing an expression to a different but equivalent form. This can be observed when simplifying expressions or solving equations. To change an expression equivalently from one form to another, we use appropriate properties of operations and follow the order of operations.

Properties of Operations on Real Numbers

The four basic operations performed on real numbers are addition $(+)$, subtraction $(-)$, multiplication $(\cdot)$, and division $(\div)$. Here are the main properties of these operations:

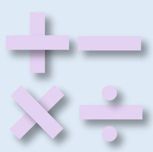
Closure: The result of an operation on real numbers is also a real number. We can say that the **set of real numbers** is **closed** under **addition**, **subtraction** and **multiplication**.

We cannot say this about division, as **division by zero is not allowed**.

Neutral Element: A real number that leaves other real numbers unchanged under a particular operation.

For example, **zero** is the **neutral element** (also called **additive identity**) of **addition**, since $\boldsymbol{a+0=a}$, and $\boldsymbol{0+a=a}$, for any real number a.
Similarly, **one** is the **neutral element** (also called **multiplicative identity**) of **multiplication**, since $\boldsymbol{a\cdot 1=a}$, and $\boldsymbol{1\cdot a=a}$, for any real number a.

Inverse Operations: Operations that reverse the effect of each other. For example, **addition and subtraction** are **inverse operations**, as $\boldsymbol{a+b-b=a}$, and $\boldsymbol{a-b+b=a}$, for any real a and b.



Similarly, **multiplication and division** are **inverse operations**, as $\boldsymbol{a\cdot b\div b=a}$, and $\boldsymbol{a\cdot b\div b=a}$ for any real a and $b\neq 0$.

Opposites: Two quantities are **opposite** to each other if they **add to zero**. Particularly, $\boldsymbol{a}$ and $\boldsymbol{-a}$ are **opposites** (also referred to as **additive inverses**), as $\boldsymbol{a+(-a)=0}$. For example, the opposite of 3 is -3, the opposite of $-\frac{3}{4}$ is $\frac{3}{4}$, the opposite of $x+1$ is $-(x+1)=-x-1$.

Reciprocals: Two quantities are **reciprocals** of each other if they **multiply to one**. Particularly, $\boldsymbol{a}$ and $\boldsymbol{\frac{1}{a}}$ are **reciprocals** (also referred to as **multiplicative inverses**), since $\boldsymbol{a\cdot\frac{1}{a}=1}$. For example, the reciprocal of 3 is $\frac{1}{3}$, the reciprocal of $-\frac{3}{4}$ is $-\frac{4}{3}$, the reciprocal of $x+1$ is $\frac{1}{x+1}$.

Multiplication by 0: Any real quantity **multiplied by zero** becomes **zero**. Particularly, $\boldsymbol{a\cdot 0=0}$, for any real number a.

Zero Product: If a product of two real numbers is zero, then at least one of these numers must be zero. Particularly, for any real $\boldsymbol{a}$ and $\boldsymbol{b}$, if $\boldsymbol{a\cdot b=0}$, then $\boldsymbol{a=0}$ or $\boldsymbol{b=0}$.

For example, if $x(x-1)=0$, then either $x=0$ or $x-1=0$.

Commutativity:

The order of numbers does not change the value of a particular operation. In particular, **addition** and **multiplication** is **commutative**, since

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a} \text{ and } \boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a},$$

for any real a and b. For example, $5 + 3 = 3 + 5$ and $5 \cdot 3 = 3 \cdot 5$.

Note: Neither subtraction nor division is commutative. See a counterexample: $5 - 3 = 2$ but $3 - 5 = -2$, so $5 - 3 \neq 3 - 5$. Similarly, $5 \div 3 \neq 3 \div 5$.

Associativity:

Association (grouping) of numbers does not change the value of an expression involving only one type of operation. In particular, **addition** and **multiplication** is **associative**, since

$$(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) \text{ and } (\boldsymbol{a} \cdot \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a} \cdot (\boldsymbol{b} \cdot \boldsymbol{c}),$$

for any real a and b. For example, $(5 + 3) + 2 = 5 + (3 + 2)$ and $(5 \cdot 3) \cdot 2 = 5 \cdot (3 \cdot 2)$.

Note: Neither subtraction nor division is associative. See a counterexample:
$(8 - 4) - 2 = 2$ but $8 - (4 - 2) = 6$, so $(8 - 4) - 2 \neq 8 - (4 - 2)$.
Similarly, $(8 \div 4) \div 2 = 1$ but $8 \div (4 \div 2) = 4$, so $(8 \div 4) \div 2 \neq 8 \div (4 \div 2)$.

Distributivity:

Multiplication can be **distributed** over addition or subtraction by following the rule:

$$\boldsymbol{a}(\boldsymbol{b} \pm \boldsymbol{c}) = \boldsymbol{ab} \pm \boldsymbol{ac},$$

for any real a, b and c. For example, $2(3 \pm 5) = 2 \cdot 3 \pm 2 \cdot 5$, or $2(x \pm y) = 2x \pm 2y$.

Note: The reverse process of distribution is known as **factoring a common factor** out. For example, $2ax + 2ay = 2a(x + y)$.

Example 1 ► **Showing Properties of Operations on Real Numbers**

Complete each statement so that the indicated property is illustrated.

a. $mn =$ _______ (commutativity of multiplication)
b. $5x + (7x + 8) =$ ___________________ (associativity of addition)
c. $5x(2 - x) =$ ___________________ (distributivity of multiplication)
d. $-y +$ _____ $= 0$ (additive inverse)
e. $-6 \cdot$ _____ $= 1$ (multiplicative inverse)
f. If $7x = 0$, then _____ $= 0$ (zero product)

Solution ►

a. To show that multiplication is commutative, we change the order of letters, so $mn = nm$.

b. To show that addition is associative, we change the position of the bracket, so $5x + (7x + 8) = (5x + 7x) + 8$.

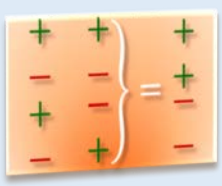
c. To show the distribution of multiplication over subtraction, we multiply $5x$ by each term of the bracket. So we have $5x(2 - x) = 5x \cdot 2 - 5x \cdot x$.

d. Additive inverse to $-y$ is its opposite, which equals to $-(-y) = y$.
So we write $-y + y = 0$.

e. Multiplicative inverse of -6 is its reciprocal, which equals to $-\frac{1}{6}$.
So we write $-6 \cdot \left(-\frac{1}{6}\right) = 1$.

f. By the zero product property, one of the factors, 7 or x, must equal to zero.
Since $7 \neq 0$, then x must equal to zero. So, we write: If $7x = 0$, then $x = 0$.

Sign Rule:



When multiplying or dividing two numbers of the **same sign**, the result is **positive**.

When multiplying or dividing two numbers of **different signs**, the result is **negative**.

This rule also applies to double signs. If the two **signs** are the **same**, they can be replaced by a **positive** sign. For example $+(+3) = 3$ and $-(-3) = 3$.
If the two **signs** are **different**, they can be replaced by a **negative** sign. For example $-(+3) = -3$ and $+(-3) = -3$.

Observation:

*Since a double negative $(--)$ can be replaced by a positive sign $(+)$, the **opposite of an opposite** leaves the original quantity unchanged. For example, $-(-2) = 2$, and generally $-(-a) = a$.*
*Similarly, taking the **reciprocal of a reciprocal** leaves the original quantity unchanged. For example, $\frac{1}{\frac{1}{2}} = 1 \cdot \frac{2}{1} = 2$, and generally $\frac{1}{\frac{1}{a}} = 1 \cdot \frac{a}{1} = a$.*

Example 2 ► **Using Properties of Operations on Real Numbers**

Use properties of real numbers to simplify each expression.

a. $-\frac{-2}{-3}$

b. $3 + (-2) - (-7) - 11$

c. $2x(-3y)$

d. $3a - 2 - 5a + 4$

e. $-(2x - 5)$

f. $2(x^2 + 1) - 2(x - 3x^2)$

g. $-\frac{100ab}{25a}$

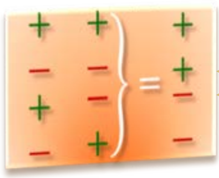
h. $\frac{2x-6}{2}$

Solution ►

a. The quotient of two negative numbers is positive, so $-\frac{-2}{-3} = -\frac{2}{3}$.

Note: *To determine the overall sign of an expression involving only multiplication and division of signed numbers, it is enough to count how many of the negative signs appear in the expression. An **even number of negatives** results in a **positive** value; an **odd number of negatives** leaves the answer **negative**.*

b. First, according to the sign rule, replace each double sign by a single sign. Therefore,



$$3 + (-2) - (-7) - 11 = 3 - 2 + 7 - 11.$$

It is convenient to treat this expression as a **sum** of signed numbers. So, it really means

$$3 + -2 + 7 + -11$$

but, for shorter notation, we tend not to write the plus signs.

Then, using the commutative property of addition, we collect all positive numbers, and all negative numbers to obtain

$$3 \underbrace{-2 + 7}_{\substack{switch\\addends}} - 11 = \underbrace{3 + 7}_{\substack{collect\\positive}} \underbrace{-2 - 11}_{\substack{collect\\negative}} = \underbrace{10 - 13}_{subtract} = \mathbf{-3}.$$

c. Since associativity of multiplication tells us that the order of performing multiplication does not change the outcome, there is no need to use any brackets in expressions involving only multiplication. So, the expression $2x(-3y)$ can be written as $2 \cdot x \cdot (-3) \cdot y$. Here, the bracket is used only to isolate the negative number, not to prioritize any of the multiplications. Then, applying commutativity of multiplication to the middle two factors, we have

$$2 \cdot \underbrace{x \cdot (-3)}_{\substack{switch\\factors}} \cdot y = \underbrace{2 \cdot (-3)}_{\substack{perform\\multiplication}} \cdot x \cdot y = \boldsymbol{-6xy}$$

d. First, use commutativity of addition to switch the two middle addends, then factor out the a, and finally perform additions where possible.

$$3a \underbrace{-2 - 5a}_{\substack{switch\\addends}} + 4 = \underbrace{3a - 5a}_{factor\ \boldsymbol{a}\ out} - 2 + 4 = \underbrace{(3-5)}_{combine} a \underbrace{-2 + 4}_{combine} = \boldsymbol{-2a + 2}$$

Note: *In practice, to combine terms with the same variable, add their coefficients.*

e. The expression $-(2x - 5)$ represents the **opposite** to $2x - 5$, which is $\boldsymbol{-2x + 5}$. This expression is indeed the opposite because

$$-2x + 5 + 2x - 5 = -2x \underbrace{+2x + 5}_{\substack{commutativity\\of\ addition}} - 5 = \underbrace{2x - 2x}_{opposites} \underbrace{-5 + 5}_{opposites} = 0 + 0 = 0.$$

Notice that the negative sign in front of the bracket in the expression $-(2x - 5)$ can be treated as multiplication by -1. Indeed, using the distributive property of multiplication over subtraction and the sign rule, we achieve the same result

$$-1(2x - 5) = -1 \cdot 2x + (-1)(-5) = \boldsymbol{-2x + 5}.$$

Note: *In practice, to release a bracket with a negative sign (or a negative factor) in front of it, change all the addends into opposites. For example*

$$-(2x-y+1)=-2x+y-1$$
$$\text{and} \quad -3(2x-y+1)=-6x+3y-3$$

f. To simplify $2(x^2+1)-2(x-3x^2)$, first, we apply the distributive property of multiplication and the sign rule.

$$2(x^2+1)-2(x-3x^2)=2x^2+2-2x+6x^2$$

Then, using the commutative property of addition, we group the terms with the same powers of x. So, the equivalent expression is

$$2x^2+6x^2-2x+2$$

Finally, by factoring x^2 out of the first two terms, we can add them to obtain

$$(2+6)x^2-2x+2=\mathbf{8x^2-2x+2}.$$

Note: *In practice, to combine terms with the same powers of a variable (or variables), add their coefficients. For example*

$$\underline{\underline{2x^2}}\ \underline{\underline{-5x^2}}\ \underline{+3xy}\ \underline{-xy}\ -3+2=\underline{\underline{-3x^2}}\ \underline{+2xy}-1.$$

g. To simplify $-\frac{100ab}{25a}$, we reduce the common factors of the numerator and denominator by following the property of the neutral element of multiplication, which is one. So,

$$-\frac{100ab}{25a}=-\frac{25\cdot 4ab}{25a}=-\frac{25a\cdot 4b}{25a\cdot 1}=-\frac{25a}{25a}\cdot\frac{4b}{1}=-1\cdot\frac{4b}{1}=-\mathbf{4b}.$$

This process is called **canceling** and can be recorded in short as

$$-\frac{\overset{4}{\cancel{100}}\cancel{a}b}{\cancel{25}\cancel{a}}=-\mathbf{4b}.$$

h. To simplify $\frac{2x-6}{2}$, factor the numerator and then remove from the fraction the factor of one by canceling the common factor of 2 in the numerator and the denominator. So, we have

$$\frac{2x-6}{2}=\frac{\cancel{2}\cdot(2x-6)}{\cancel{2}}=\mathbf{2x-6}.$$

In the solution to *Example 2d* and *2f*, we used an intuitive understanding of what a "term" is. We have also shown how to combine terms with a common variable part (like terms). Here is a more formal definition of a term and of like terms.

Definition 3.1 ► A **term** is a **product** of numbers, letters, or expressions. Here are examples of single terms:

$$1,\ x,\ \frac{1}{2}x^2,\ -3xy^2,\ 2(x+1),\ \frac{x+2}{x(x+1)},\ \pi\sqrt{x}.$$

Observe that the expression $2x+2$ consists of two terms connected by addition, while the equivalent expression $2(x+1)$ represents just one term, as it is a product of the number 2 and the expression $(x+1)$.

Like terms are the terms that have exactly the same variable part (the same letters or expressions raised to the same exponents). Like terms can be **combined** by adding their **coefficients** (numerical part of the term).

For example, $5x^2$ and $-2x^2$ are like, so they can be combined (added) to $3x^2$,
$(x+1)$ and $3(x+1)$ are like, so they can be combined to $4(x+1)$,
but $5x$ and $2y$ are unlike, so they cannot be combined.

Example 3 ► **Combining Like Terms**

Simplify each expression by combining like terms.

a. $-x^2+3y^2+x-6+2y^2-x+1$

b. $\frac{2}{x+1}-\frac{5}{x+1}+\sqrt{x}-\frac{\sqrt{x}}{2}$

Solution ► **a.** Before adding like terms, it is convenient to underline the groups of like terms by the same type of underlining. So, we have

add to zero

$$\underline{-x^2}\,\underline{+3y^2}\,\underline{\underline{+x}}\,\underline{-6}\,\underline{+2y^2}\,\underline{\underline{-x}}\,\underline{+1} = \boldsymbol{-x^2\underline{+5y^2}-5}$$

b. Notice that the numerical coefficients of the first two like terms in the expression

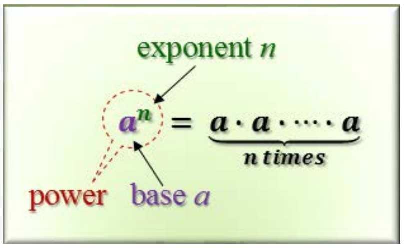
$$\frac{2}{x+1}-\frac{5}{x+1}+\sqrt{x}-\frac{\sqrt{x}}{2}$$

are 2 and -5, and of the last two like terms are 1 and $-\frac{1}{2}$. So, by adding these coefficients, we obtain

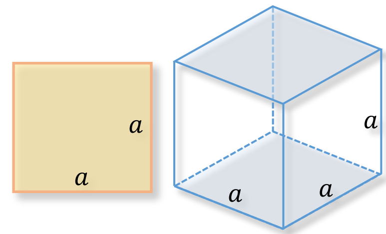
$$\boldsymbol{-\frac{3}{x+1}+\frac{1}{2}\sqrt{x}}$$

Observe that $\frac{1}{2}\sqrt{x}$ can also be written as $\frac{\sqrt{x}}{2}$. Similarly, $-\frac{3}{x+1}$, $\frac{-3}{x+1}$, or $-3\cdot\frac{1}{x+1}$ are equivalent forms of the same expression.

Exponents and Roots



Exponents are used as a shorter way of recording repeated multiplication by the same quantity. For example, to record the product $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, we write 2^5. The **exponent** 5 tells us how many times to multiply the **base** 2 by itself to evaluate the product, which is 32. The expression 2^5 is referred to as the 5^{th} **power** of 2, or "2 to the 5^{th}". In the case of exponents 2 or 3, terms "squared" or "cubed" are often used. This is because of the connection to geometric figures, a square and a cube.



The area of a square with sides of length a is expressed by a^2 (*read:* "a squared" or "the square of a") while the volume of a cube with sides of length a is expressed by a^3 (*read:* "a cubed" or "the cube of a").

If a negative number is raised to a certain exponent, a bracket must be used around the base number. For example, if we wish to multiply -3 by itself two times, we write $(-3)^2$, which equals $(-3)(-3) = 9$. The notation -3^2 would indicate that only 3 is squared, so $-3^2 = -3 \cdot 3 = -9$. This is because an **exponent refers only to the number immediately below the exponent**. Unless we use a bracket, a negative sign in front of a number is not under the influence of the exponent.

Example 4 ► **Evaluating Exponential Expressions**

Evaluate each exponential expression.

a. -3^4 **b.** $(-2)^6$

c. $(-2)^5$ **d.** $-(-2)^3$

e. $\left(-\frac{2}{3}\right)^2$ **f.** $-\left(-\frac{2}{3}\right)^5$

Solution ► **a.** $-3^4 = (-1) \cdot 3 \cdot 3 \cdot 3 \cdot 3 = \mathbf{-81}$

b. $(-2)^6 = (-2)(-2)(-2)(-2)(-2)(-2) = \mathbf{64}$

c. $(-2)^5 = (-2)(-2)(-2)(-2)(-2) = \mathbf{-32}$

Observe: *Negative sign in front of a power works like multiplication by* -1.

*A **negative** base raised to an **even exponent** results in a **positive** value.*
*A **negative** base raised to an **odd exponent** results in a **negative** value.*

d. $-(-2x)^3 = -(-2x)(-2x)(-2x)$

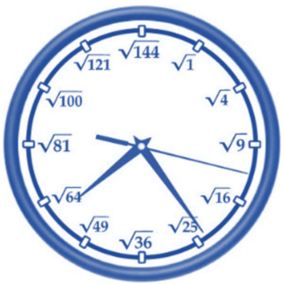
$= -(-2)(-2)(-2)xxx = -(-2)^3x^3 = -(-8)x^3 = \mathbf{8x^3}$

e. $\left(-\frac{2}{3}\right)^2 = \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) = \frac{(-2)^2}{3^2} = \mathbf{\frac{4}{9}}$

f. $-\left(-\frac{2}{3}\right)^5 = -\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) = -\frac{(-2)^5}{3^5} = -\frac{-32}{243} = \mathbf{\frac{32}{243}}$

Observe: *Exponents apply to every factor of the numerator and denominator of the base. This exponential property can be stated as*

$$(ab)^n = a^n b^n \quad \textit{and} \quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$



To reverse the process of squaring, we apply a **square root**, denoted by the **radical sign** $\sqrt{\ }$. For example, since $5 \cdot 5 = 25$, then $\sqrt{25} = 5$. Notice that $(-5)(-5) = 25$ as well, so we could also claim that $\sqrt{25} = -5$. However, we wish to define the operation of taking square root in a unique way. We choose to take the **positive** number (called **principal square root**) as the value of the square root. Therefore $\sqrt{25} = 5$, and generally

$$\sqrt{x^2} = |x|.$$

Since the square of any nonzero real number is positive, the square root of a negative number is not a real number. For example, we can say that $\sqrt{-16}$ **does not exist** (in the set of real numbers), as there is no real number a that would satisfy the equation $a^2 = -16$.

Example 5 ► **Evaluating Radical Expressions**

Evaluate each radical expression.

a. $\sqrt{0}$ **b.** $\sqrt{64}$

c. $-\sqrt{121}$ **d.** $\sqrt{-100}$

e. $\sqrt{\frac{1}{9}}$ **f.** $\sqrt{0.49}$

Solution ►

a. $\sqrt{0} = \mathbf{0}$, as $0 \cdot 0 = 0$

b. $\sqrt{64} = \mathbf{8}$, as $8 \cdot 8 = 64$

c. $-\sqrt{121} = \mathbf{-11}$, as we copy the negative sign and $11 \cdot 11 = 121$

d. $\sqrt{-100} = \boldsymbol{DNE}$ (*read:* doesn't exist), as no real number squared equals -100

e. $\sqrt{\frac{1}{9}} = \mathbf{\frac{1}{3}}$, as $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$.

Notice that $\frac{\sqrt{1}}{\sqrt{9}}$ also results in $\frac{1}{3}$. So, $\sqrt{\frac{1}{9}} = \frac{\sqrt{1}}{\sqrt{9}}$ and generally $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ for any nonnegative real numbers a and $b \neq 0$.

f. $\sqrt{0.49} = \mathbf{0.7}$, as $0.7 \cdot 0.7 = 0.49$

Order of Operations

In algebra, similarly as in arithmetic, we like to perform various operations on numbers or on variables. To record in what order these operations should be performed, we use grouping signs, mostly brackets, but also division bars, absolute value symbols, radical symbols, etc. In an expression with many grouping signs, we perform operations in the **innermost grouping sign first**. For example, the innermost grouping sign in the expression

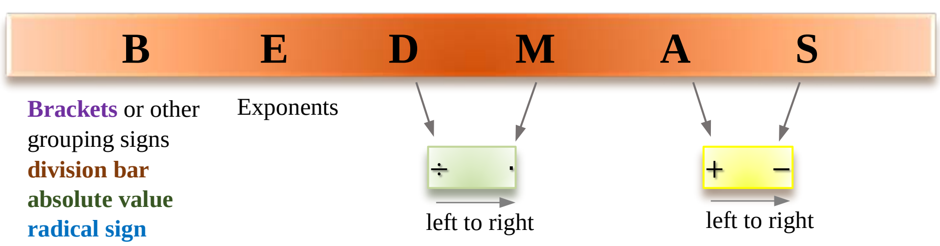
$$[4+(3\cdot|2-4|)]\div 2$$

is the absolute value sign, then the round bracket, and finally, the square bracket. So first, perform subtraction, then apply the absolute value, then multiplication, addition, and finally the division. Here are the calculations:

$$\begin{aligned}&[4+(3\cdot|2-4|)]\div 2\\ &=[4+(3\cdot|-2|)]\div 2\\ &=[4+(3\cdot 2)]\div 2\\ &=[4+6]\div 2\\ &=10\div 2\\ &=\mathbf{5}\end{aligned}$$

Observe that the more operations there are to perform, the more grouping signs would need to be used. To simplify the notation, additional rules of order of operations have been created. These rules, known as BEDMAS, allow for omitting some of the grouping signs, especially brackets. For example, knowing that multiplication is performed before addition, the expression $[4+(3\cdot|2-4|)]\div 2$ can be written as $[4+3\cdot|2-4|]\div 2$ or $\frac{4+3\cdot|2-4|}{2}$.

Let's review the BEDMAS rule.



BEDMAS Rule:

1. Perform operations in the innermost **B**racket (or other grouping sign) first.
2. Then work out **E**xponents.
3. Then perform **D**ivision and **M**ultiplication in order of their occurrence (left to right). *Notice that there is no priority between division and multiplication. However, both division and multiplication have priority before any addition or subtraction.*
4. Finally, perform **A**ddition and **S**ubtraction in order of their occurrence (left to right). *Again, there is no priority between addition and subtraction.*

Example 5 ► **Simplifying Arithmetic Expressions According to the Order of Operations**

Use the order of operations to simplify each expression.

a. $3 + 2 \cdot 6$

b. $4 \cdot 6 \div 3 - 2$

c. $12 \div 4 + 2|3 - 4|$

d. $2 \cdot 3^2 - 3(-2 + 6)$

e. $\sqrt{30 - 5} - 2\big(3 + 4 \cdot (-2)\big)^2$

f. $\frac{3+(-3^2)(-2)}{3\cdot\sqrt{4}-6\cdot 2}$

Solution ► **a.** Out of the two operations, + and · , multiplication is performed first. So, we have

$3 + 2 \cdot 6$
$= 3 + 12$
$= \mathbf{15}$

b. There is no priority between multiplication and division, so we perform these operations in the order in which they appear, from left to right. Then we subtract. Therefore,

$4 \cdot 6 \div 3 - 2$
$= 24 \div 3 - 2$
$= 8 - 2$
$= \mathbf{6}$

c. In this expression, we have a grouping sign (the absolute value bars), so we perform the subtraction inside the absolute value first. Then, we apply the absolute value and work out the division and multiplication before the final addition. So, we obtain

$12 \div 4 + 2|3 - 4|$
$= 12 \div 4 + 2|-1|$
$= 12 \div 4 + 2 \cdot 1$
$= 3 + 2$
$= \mathbf{5}$

d. In this expression, work out the bracket first, then perform the exponent, then both multiplications, and finally the subtraction. So, we have

$2 \cdot 3^2 - 3(-2 + 6)$
$= 2 \cdot 3^2 - 3(4)$
$= 2 \cdot 9 - 3(4)$
$= 18 - 12$
$= \mathbf{6}$

e. The expression $\sqrt{30 - 5} - 2\big(3 + 4 \cdot (-2)\big)^2$ contains two grouping signs, the bracket and the radical sign. Since these grouping signs are located at separate places (they are not nested), they can be worked out simultaneously. As usual, out of the operations inside the bracket, multiplication is done before subtruction. Here are the steps:

$\sqrt{30 - 5} - 2\big(3 + 4 \cdot (-2)\big)^2$
$= \sqrt{25} - 2\big(3 + (-8)\big)^2$
$= 5 - 2(-5)^2$

work out the power first, then multiply, and finally subtract.

$= 5 - 2 \cdot 25$
$= 5 - 50$
$= \mathbf{-45}$

f. To simplify the expression $\frac{3+(-3^2)(-2)}{3\cdot\sqrt{4}-6\cdot 2}$, work on the numerator and the denominator before performing the division. Follow the steps:

$$\frac{3+(-3^2)(-2)}{3\cdot\sqrt{4}-6\cdot 2}$$ ($-3^2 = -9$)

$$= \frac{3+(-9)(-2)}{3\cdot 2-6\cdot 2}$$

$$= \frac{3+(-18)}{6-12}$$

$$= \frac{-15}{-6}$$ (reduce the common factor of -3)

$$= \mathbf{\frac{5}{2}}$$

Evaluation of Algebraic Expressions

An **algebraic expression** consists of letters, numbers, operation signs, and grouping symbols. Here are some examples of algebraic expressions:

$$6ab,\quad x^2 - y^2,\quad 3(2a+5b),\quad \frac{x-3}{3-x},\quad 2\pi r,\quad \frac{d}{t},\quad Prt,\quad \sqrt{x^2+y^2}$$

When a letter is used to stand for various numerical values, it is called a **variable**. For example, if t represents the number of hours needed to drive between particular towns, then t is a variable, since t changes depending on the average speed used during the trip. Notice, however, that the distance d between the two towns represents a constant number. So, even though letters in algebraic expressions usually represent variables, sometimes they may represent a **constant** value. One such constant is the letter π, which represents approximately 3.14.

Notice that algebraic expressions do not contain any comparison signs (equality or inequality, such as $=$, $\neq$, $<$, $\leq$, $>$, $\geq$), therefore, they are **not to be solved** for any variable. Algebraic expressions can only be **simplified** by implementing properties of operations (see *Example 2* and *3*) or **evaluated** for particular values of the variables. The evaluation process involves substituting given values for the variables and evaluating the resulting arithmetic expression by following the order of operations.

Advice: *To evaluate an algebraic expression for given variables, first rewrite the expression replacing each variable with* ***empty brackets*** *and then write appropriate values inside these brackets. This will help to avoid possible errors of using incorrect signs or operations.*

Example 6 ► **Evaluating Algebraic Expressions**

Evaluate each expression if $a = -2$, $b = 3$, and $c = 6$.

a. $b^2 - 4ac$ **b.** $2c \div 3a$ **c.** $\frac{|a^2-b^2|}{-a^2+\sqrt{a+c}}$

Solution ►

a. First, we replace each letter in the expression $b^2 - 4ac$ with an empty bracket. So, we write

$$(\ \ )^2 - 4(\ \ )(\ \ ).$$

Now, we fill in the brackets with the corresponding values and evaluate the resulting expression. So, we have

$$(3)^2 - 4(-2)(6) = 9 - (-48) = 9 + 48 = \mathbf{57}.$$

b. As above, we replace the letters with their corresponding values to obtain

$$2c \div 3a = 2(6) \div 3(-2).$$

Since we work only with multiplication and division here, they are to be performed in order from left to right. Therefore,

$$2(6) \div 3(-2) = 12 \div 3(-2) = 4(-2) = \mathbf{-8}.$$

c. As above, we replace the letters with their corresponding values to obtain

$$\frac{|a^2 - b^2|}{-a^2 + \sqrt{b + c}} = \frac{|(-2)^2 - (3)^2|}{-(-2)^2 + \sqrt{(3) + (6)}} = \frac{|4 - 9|}{-4 + \sqrt{9}} = \frac{|-5|}{-4 + 3} = \frac{5}{-1} = \mathbf{-5}.$$

Equivalent Expressions

Algebraic expressions that produce the same value for all allowable values of the variables are reffered to as **equivalent expressions**. Notice that properties of operations allow us to rewrite algebraic expressions in a simpler but equivalent form. For example,

$$\frac{\boldsymbol{x-3}}{\boldsymbol{3-x}} = \frac{\cancel{x-3}}{-(\cancel{x-3})} = \mathbf{-1}$$

or

$$(\boldsymbol{x}+\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y}) = (x+y)x - (x+y)y = x^2 + \cancel{yx} - \cancel{xy} - y^2 = \boldsymbol{x^2 - y^2}.$$

To show that two expressions are **<u>not</u> equivalent**, it is enough to find a particular set of variable values for which the two expressions evaluate to a different value. For example,

$$\sqrt{x^2 + y^2} \neq x + y$$

because if $x = 1$ and $y = 1$ then $\sqrt{x^2 + y^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$ while $x + y = 1 + 1 = 2$. Since $\sqrt{2} \neq 2$ the two expressions $\sqrt{x^2 + y^2}$ and $x + y$ are not equivalent.

Example 7 ► **Determining Whether a Pair of Expressions is Equivalent**

Determine whether the given expressions are equivalent.

a. $(a+b)^2$ and a^2+b^2 **b.** $\frac{x^8}{x^4}$ and x^4

Solution ► **a.** Suppose $a=1$ and $b=1$. Then

$$(a+b)^2=(1+1)^2=2^2=4$$

but

$$a^2+b^2=1^2+1^2=2.$$

So the expressions $(a+b)^2$ and a^2+b^2 are not equivalent.

Using the distributive propery and commutativity of multiplication, check on your own that

$$(a+b)^2=a^2+2ab+b^2.$$

b. Using properties of exponents and then removing a factor of one, we show that

$$\frac{x^8}{x^4}=\frac{x^4\cdot x^4}{x^4}=x^4.$$

So the two expressions are indeed equivalent.

Review of Operations on Fractions

A large part of algebra deals with performing operations on algebraic expressions by generalising the ways that these operations are performed on real numbers, particularly, on common fractions. Since operations on fractions are considered to be one of the most challenging topics in arithmetic, it is a good idea to review the rules to follow when performing these operations before we move on to other topics of algebra.

Operations on Fractions:

Simplifying To simplify a fraction to its lowest terms, **remove** the **greatest common factor** (**GCF**) of the numerator and denominator. For example, $\frac{48}{64}=\frac{3\cdot 16}{4\cdot 16}=\frac{3}{4}$, and generally $\frac{ak}{bk}=\frac{a}{b}$.

This process is called ***reducing*** or ***canceling***.

Note that the reduction can be performed several times, if needed. In the above example, if we didn't notice that 16 is the greatest common factor for 48 and 64, we could reduce the fraction by dividing the numerator and denominator by any common factor (2, or 4, or 8) first, and then repeat the reduction process until there is no common factors (other than 1) anymore. For example,

$$\frac{48}{64}=\frac{24}{32}=\frac{6}{8}=\frac{\mathbf{3}}{\mathbf{4}}.$$

÷ by 2 ÷ by 4 ÷ by 2

Multiplying

To multiply fractions, we multiply their numerators and denominators. So generally,

$$\frac{\boldsymbol{a}}{\boldsymbol{b}} \cdot \frac{\boldsymbol{c}}{\boldsymbol{d}} = \frac{\boldsymbol{ac}}{\boldsymbol{bd}}.$$

However, before performing multiplication of numerators and denominators, it is a good idea to reduce first. This way, we work with smaller numbers, which makes the calculations easier. For example,

$$\frac{18}{15} \cdot \frac{25}{14} = \frac{18 \cdot 25}{15 \cdot 14} = \frac{18 \cdot 5}{3 \cdot 14} = \frac{6 \cdot 5}{1 \cdot 14} = \frac{3 \cdot 5}{1 \cdot 7} = \frac{\mathbf{15}}{\mathbf{7}}.$$

÷ by 5 ÷ by 3 ÷ by 2

Dividing

To divide fractions, we **multiply** the dividend (the first fraction) **by the reciprocal** of the **divisor** (the second fraction). So generally,

$$\frac{\boldsymbol{a}}{\boldsymbol{b}} \div \frac{\boldsymbol{c}}{\boldsymbol{d}} = \frac{\boldsymbol{a}}{\boldsymbol{b}} \cdot \frac{\boldsymbol{d}}{\boldsymbol{c}} = \frac{\boldsymbol{ad}}{\boldsymbol{bc}}.$$

· by reciprocal

For example,

÷ by 4

$$\frac{8}{15} \div \frac{4}{5} = \frac{8}{15} \cdot \frac{5}{4} = \frac{2 \cdot 1}{3 \cdot 1} = \frac{\mathbf{2}}{\mathbf{3}}.$$

÷ by 5

Adding or Subtracting

To add or subtract fractions, follow the steps:

1. Find the **Lowest Common Denominator** (**LCD**).
2. Extend each fraction to higher terms to obtain the desired common denominator.
3. Add or subtract the numerators, keeping the common denominator.
4. Simplify the resulting fraction, if possible.

For example, to evaluate $\frac{5}{6} + \frac{3}{4} - \frac{4}{15}$, first we find the LCD for denominators 6, 4, and 15. We can either guess that 60 is the least common multiple of 6, 4, and 15, or we can use the following method of finding LCD:

2	**6**	**4**	**15**	
· 3	3	2	15	
·	**1**	**· 2**	**· 5**	**= 60**

- divide by a common factor of at least two numbers; for example, by 2
- write the quotients in the line below; 15 is not divisible by 2, so just copy it down
- keep dividing by common factors till all numbers become relatively prime
- the LCD is the product of all numbers listed in the letter L, so it is 60

Then, we extend the fractions so that they share the same denominator of 60, and finally perform the operations in the numerator. Therefore,

$$\frac{5}{6} + \frac{3}{4} - \frac{4}{5} = \underbrace{\frac{5 \cdot 10}{6 \cdot 10} + \frac{3 \cdot 15}{4 \cdot 15} - \frac{4 \cdot 12}{5 \cdot 12}}_{\text{in practice, this step doesn't have to be written}} = \frac{5 \cdot 10 + 3 \cdot 15 - 4 \cdot 12}{60} = \frac{50 + 45 - 48}{60} = \frac{\mathbf{47}}{\mathbf{60}}.$$

Example 8 ► **Evaluating Fractional Expressions**

Simplify each expression.

a. $-\frac{2}{3}-\left(-\frac{5}{12}\right)$ **b.** $-3\left[\frac{3}{2}+\frac{5}{6}\div\left(-\frac{3}{8}\right)\right]$

Solution ► **a.** After replacing the double negative by a positive sign, we add the two fractions, using 12 as the lowest common denominator. So, we obtain

$$-\frac{2}{3}-\left(-\frac{5}{12}\right)=-\frac{2}{3}+\frac{5}{12}=\frac{-2\cdot 4+5}{12}=\frac{-\cancel{3}}{\cancel{12}}=-\mathbf{\frac{1}{4}}.$$

b. Following the order of operations, we calculate

$$-3\left[\frac{3}{2}+\frac{5}{6}\div\left(-\frac{3}{8}\right)\right]$$

First, perform the division in the bracket by converting it to a multiplication by the reciprocal. The quotient becomes negative.

$$=-3\left[\frac{3}{2}-\frac{5}{\cancel{6}_3}\cdot\frac{\cancel{8}^4}{3}\right]$$

Reduce, before multiplying.

$$=-3\left(\frac{3}{2}-\frac{20}{9}\right)$$

Extend both fractions to higher terms using the common denominator of 18.

$$=-3\left(\frac{27-40}{18}\right)$$

Perform subtraction.

$$=-\cancel{3}\left(\frac{-13}{\cancel{18}}\right)$$

Reduce before multiplying. The product becomes positive.

$$=\mathbf{\frac{13}{6}} \text{ or equivalently } \mathbf{2\frac{1}{6}}.$$

R.3 Exercises

Vocabulary Check *Fill in each blank with the most appropriate term or phrase from the given list:* ***like, coefficients, reciprocal, least common, term, common factor.***

1. Only ________ terms can be combined by adding their ______________.

2. To divide two real numbers, multiply the first number by the ______________ of the second number.

3. The first step in adding or subtracting fractions is finding their ___________ ______________ denominator.

4. A product of numbers, letters, or expressions is called a _________ .

5. To simplify a fraction completely, cancel out the greatest ____________ _________ of the numerator and denominator.

Concept Check *True or False?*

6. The set of integers is closed under multiplication.

7. The set of natural numbers is closed under subtraction.

8. The set of real numbers different than zero is closed under division.

9. According to the BEDMAS rule, division should be performed before multiplication.

10. For any real number $\sqrt{x^2} = x$.

11. Square root of a negative number is not a real number.

12. If the value of a square root exists, it is positive.

13. $-x^3 = (-x)^3$

14. $-x^2 = (-x)^2$

Discussion Point

15. What is the value of the expression $\frac{2x-2x}{x-x}$?

Concept Check *Complete each statement so that the indicated property is illustrated.*

16. $x + (-y) =$ _________ , commutative property of addition

17. $(7 \cdot 5) \cdot 2 =$ ___________ , associative property of multiplication

18. $(3 + 8x) \cdot 2 =$ ______________ , distributive property of multiplication over addition

19. $a +$ _____ $= 0$, additive inverse

20. $-\frac{a}{b} \cdot$ _____ $= 1$, multiplicative inverse

21. $\frac{3x}{4y} \cdot$ _____ $= \frac{3x}{4y}$, multiplicative identity

22. _____ $+ (-a) = -a$, additive identity

23. $(2x - 7) \cdot$ ____ $= 0$, multiplication by zero

24. If $(x + 5)(x - 1) = 0$, then ________$= 0$ or ________$= 0$, zero product property

Concept Check *Perform operations.*

25. $-\frac{2}{5} + \frac{3}{4}$

26. $\frac{5}{6} - \frac{2}{9}$

27. $\frac{5}{8} \cdot \left(-\frac{2}{3}\right) \cdot \frac{18}{15}$

28. $-3\left(-\frac{5}{9}\right)$

29. $-\frac{3}{4}(8x)$

30. $\frac{15}{16} \div \left(-\frac{9}{12}\right)$

Concept Check *Use order of operations to evaluate each expression.*

31. $64 \div (-4) \div 2$

32. $3 + 3 \cdot 5$

33. $8 - 6(5 - 2)$

34. $20 + 4^3 \div (-8)$

35. $6\left(9 - 3\sqrt{9-5}\right)$

36. $-2^5 - 8 \div 4 - (-2)$

37. $-\frac{5}{6} + \left(-\frac{7}{4}\right) \div 2$

38. $\left(-\frac{3}{2}\right) \cdot \frac{1}{6} - \frac{2}{5}$

39. $-\frac{3}{2} \div \left(-\frac{4}{9}\right) - \frac{5}{4} \cdot \frac{2}{3}$

40. $-3\left(-\frac{4}{9}\right) - \frac{1}{4} \div \frac{3}{5}$

41. $2 - 3|3 - 4 \cdot 6|$

42. $\frac{3|5-7|-6\cdot 4}{5\cdot 6-2|4-1|}$

Concept Check *Simplify each expression.*

43. $-(x - y)$

44. $-2(3a - 5b)$

45. $\frac{2}{3}(24x + 12y - 15)$

46. $\frac{3}{4}(16a - 28b + 12)$

47. $5x - 8x + 2x$

48. $3a + 4b - 5a + 7b$

49. $5x - 4x^2 + 7x - 9x^2$

50. $8\sqrt{2} - 5\sqrt{2} + \frac{1}{x} + \frac{3}{x}$

51. $2 + 3\sqrt{x} - 6 - \sqrt{x}$

52. $\frac{a-b}{b-a}$

53. $\frac{2(x-3)}{3-x}$

54. $-\frac{100ab}{75a}$

55. $-(5x)^2$

56. $\left(-\frac{2}{3}a\right)^2$

57. $5a - (4a - 7)$

58. $6x + 4 - 3(9 - 2x)$

59. $5x - 4(2x - 3) - 7$

60. $8x - (-4x + 7) + (9x - 1)$

61. $6a - [4 - 3(9a - 2)]$

62. $5\{x + 3[4 - 5(2x - 3) - 7]\}$

63. $-2\{2 + 3[4x - 3(5x + 1)]\}$

64. $4\{[5(x - 3) + 5^2] - 3[2(x + 5) - 7^2]\}$

65. $3\{[6(x + 4) - 3^3] - 2[5(x - 8) - 8^2]\}$

Concept Check *Evaluate each algebraic expression if $a = -2$, $b = 3$, and $c = 2$.*

66. $b^2 - a^2$

67. $6c \div 3a$

68. $\frac{c-a}{c-b}$

69. $b^2 - 3(a - b)$

70. $\frac{-b+\sqrt{b^2-4ac}}{2a}$

71. $c\left(\frac{a}{b}\right)^{|a|}$

Determine whether each pair of expressions is equivalent.

72. $x^3 \cdot x^2$ and x^5

73. $a^2 - b^2$ and $(a - b)^2$

74. $\sqrt{x^2}$ and x

75. $(x^3)^2$ and x^5

Use the distributive property to calculate each value mentally.

76. $96 \cdot 18 + 4 \cdot 18$

77. $29 \cdot 70 + 29 \cdot 30$

78. $57 \cdot \frac{3}{5} - 7 \cdot \frac{3}{5}$

79. $\frac{8}{5} \cdot 17 + \frac{8}{5} \cdot 13$

Analytic Skills *Insert one pair of parentheses to make the statement true.*

80. $2 \cdot 3 + 6 \div 5 - 3 = 9$

81. $9 \cdot 5 + 2 - 8 \cdot 3 + 1 = 22$