Sequences and Series

In everyday life, we can observe sequences or series of events in many contexts. For instance, we line up to enter a store in a sequence, we make a sequence of mortgage payments, or we observe a series of events that lead to a particular outcome. In this section, we will consider mathematical definitions for sequences and series, and explore some applications of these concepts.

S.1 Sequences and Series

Think of a **sequence of numbers** as an **ordered list of numbers**. For example, the waiting time, in minutes, of each person standing in line to Tim Horton's to be served

$$0, 1, 2, 2, 3, 5, 5, 5, 7, 9, 10, 11$$

or the number of bacteria in a colony after each hour, if the colony starts with one bacteria and each bacteria divides into two every hour

$$1, 2, 4, 8, 16, 32, \ldots, 2^{n-1}, \ldots$$

The first example illustrates a finite sequence, while the second example shows an infinite sequence. Notice that numbers listed in a sequence, called **terms**, can repeat, like in the first example, or they can follow a certain pattern, like in the second example. If we can recognize the pattern of the listed terms, it is convenient to state it as a general rule by listing the n-th term. The sequence of numbers in our second example shows consecutive powers of two, starting with 2^0, so the n-th term of this sequence is 2^{n-1}.

Sequences as Functions

Formally, the definition of sequence can be stated by using the terminology of functions.

Definition 1.1 ► An **infinite sequence** is a function whose **domain** is the set of all **natural numbers**.
A **finite sequence** is a function whose **domain** is the first n natural numbers $\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots, \boldsymbol{n}\}$.
The **terms** (or elements) of a sequence are the function values, the entries of the ordered list of numbers.
The **general term** of a sequence is its n-th term.

Notation: Customarily, sequence functions assume names such as a, b, c, rather than f, g, h.
If the name of a sequence function is a, than the function values (the **terms** of the sequence) are denoted $\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}, \ldots$ rather than $a(1), a(2), a(3), \ldots$.
The **index** $\boldsymbol{k}$ in the notation $\boldsymbol{a_k}$ indicates the position of the term in the sequence.
$\boldsymbol{a_n}$ denotes the **general term** of the sequence and $\{\boldsymbol{a_n}\}$ represents the entire sequence.

Example 1 ► **Finding Terms of a Sequence When Given the General Term**

Given the sequence $a_n = \frac{n-1}{n+1}$, find the following

a. the first four terms of $\{a_n\}$ **b.** the 12-th term a_{12}

Solution ► **a.** To find the first four terms of the given sequence, we evaluate a_n for $n = 1,2,3,4$.

We have
$$a_1 = \frac{1-1}{1+1} = 0$$
$$a_2 = \frac{2-1}{2+1} = \frac{1}{3}$$
$$a_3 = \frac{3-1}{3+1} = \frac{2}{4} = \frac{1}{2}$$
$$a_4 = \frac{4-1}{4+1} = \frac{3}{5}$$

so the first four terms are $0, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}$.

b. The twelfth term is $a_{12} = \frac{12-1}{12+1} = \frac{11}{13}$.

Example 2 ► **Finding the General Term of a Sequence**

Determine the expression for the general term a_n of the sequence

a. $3, 9, 27, 81, \ldots$ **b.** $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots$

Solution ► **a.** Observe that all terms of the given sequence are powers of 3.

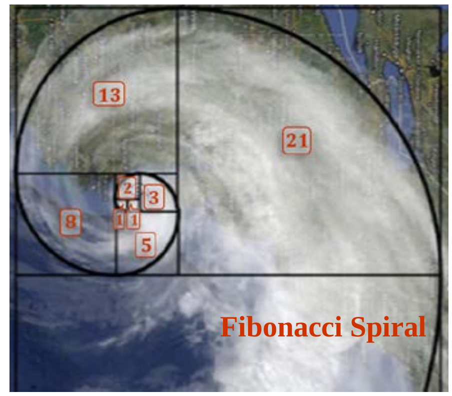
We have
$$a_1 = 3 = 3^1$$
$$a_2 = 9 = 3^2$$
$$a_3 = 27 = 3^3$$
$$a_4 = 81 = 3^4, \text{ and so on.}$$

Notice that in each term, the exponent of 3 is the same as the index of the term. The above pattern suggests the candidate $a_n = 3^n$ for the general term of this sequence. To convince oneself that this is indeed the general term, one may want to generate the given terms with the aid of the developed formula $a_n = 3^n$. If the generated terms match the given ones, the formula is correct.

b. Here, observe that the signs of the given terms alter, starting with the negative sign. While building a formula for the general term, to accommodate for this change in signs, we may want to use the factor of $(-1)^n$. This is because
$$(-1)^n = \begin{cases} -1, & \text{for } n = 1, 3, 5, \ldots \\ 1, & \text{for } n = 2, 4, 6, \ldots \end{cases}$$
Then we observe that all terms may be seen as fractions with the numerator equal to 1 and denominator matching the index of the term,
$$a_1 = -1 = (-1)^1 \frac{1}{1}$$
$$a_2 = \frac{1}{2} = (-1)^2 \frac{1}{2}$$
$$a_3 = -\frac{1}{3} = (-1)^3 \frac{1}{3}$$
$$a_4 = \frac{1}{4} = (-1)^4 \frac{1}{4}$$
$$a_5 = -\frac{1}{5} = (-1)^5 \frac{1}{5}, \text{ and so on.}$$

The above pattern suggests that the formula $a_n = (-1)^n \frac{1}{n}$ would work for the general term of this sequence. As before, please convince yourself that this is indeed the general term of the sequence by generating the given terms with the aid of the suggested formula.

Sometimes it is difficult to describe a sequence by stating the explicit formula for its general term. For example, in the case of the **Fibonacci sequence** $1, 1, 2, 3, 5, 8, 13, 21, \ldots$, one can observe the rule of obtaining the next term by adding the previous two terms (for terms after the second term), but it would be very difficult to come up with an explicit formula for the general term a_n. Yet the Fibonacci sequence can be defined through the following equations $a_1 = a_2 = 1$ and $a_n = a_{n-2} + a_{n-1}$, for $n \geq 3$. Notice that the n-th term is not given explicitly but it can be found as long as the previous terms are known. In such a case we say that the sequence is **defined recursively**.



Definition 1.2 ► A sequence is defined **recursively** if

- the initial term or terms are given, and
- the n-th term is defined by a formula that refers to the *preceding* terms.

Example 3 ► **Finding Terms of a Sequence Given Recursively**

Find the first 5 terms of the sequence given by the conditions
$a_1 = 1$, $a_2 = 2$, and $a_n = 2a_{n-1} + a_{n-2}$, for $n \geq 3$.

Solution ► The first two terms are given, $a_1 = \mathbf{1}$, $a_2 = \mathbf{2}$. To find the third term, we substitute $n = 3$ into the recursive formula, to obtain

$$a_3 = 2a_{3-1} + a_{3-2} = 2a_2 + a_1 = 2 \cdot \mathbf{2} + \mathbf{1} = \mathbf{5}$$

Similarly
$$a_4 = 2a_{4-1} - a_{4-2} = 2a_3 - a_2 = 2 \cdot \mathbf{5} + \mathbf{2} = \mathbf{12}$$

and
$$a_5 = 2a_{5-1} - a_{5-2} = 2a_4 - a_3 = 2 \cdot \mathbf{12} + \mathbf{5} = \mathbf{29}$$

So the first five terms of this sequence are: **1**, **2**, **5**, **12**, and **29**.

Example 4 ► **Using Sequences in Application Problems**

Peter borrows \$5000 and agrees to pay \$500 monthly, plus interest of 1% per month on the unpaid balance. If his first payment is due one month from the date of borrowing, find

a. the total number of payments needed to pay off the debt,
b. the sequence of his first four payments,
c. the general term of the sequence of payments,
d. the last payment.

Solution ► **a.** Since Peter pays \$500 off his \$5000 principal each time, the total number of payments is $\frac{5000}{500} = 10$.

b. Let $a_1, a_2, \dots, a_{10}$ be the sequence of Peter's payments.
After the first month, Peter pays $a_1 = \$500 + 0.01 \cdot \$5000 = \mathbf{\$550}$ and the remaining balance becomes $\$5000 - \$500 = \$4500$.
Then, Peter's second payment is $a_2 = \$500 + 0.01 \cdot \$4500 = \mathbf{\$545}$ and the remaining balance becomes $\$4500 - \$500 = \$4000$.
The third payment is equal to $a_3 = \$500 + 0.01 \cdot \$4000 = \mathbf{\$540}$ and the remaining balance becomes $\$4000 - \$500 = \$3500$.
Finally, the fourth payment is $a_4 = \$500 + 0.01 \cdot \$3500 = \mathbf{\$535}$ with the remaining balance of $\$3500 - \$500 = \$3000$.

So the sequence of Peter's first four payments is $\mathbf{\$550, \$545, \$540, \$535.}$

c. Notice that the terms of the above sequence diminish by 5.
We have

$$\begin{aligned} a_1 &= 550 = 550 - \mathbf{0} \cdot 5 \\ a_2 &= 545 = 550 - \mathbf{1} \cdot 5 \\ a_3 &= 540 = 550 - \mathbf{2} \cdot 5 \\ a_4 &= 535 = 550 - \mathbf{3} \cdot 5, \text{ and so on.} \end{aligned}$$

Since the blue coefficient by "5" is one lower than the index of the term, we can write the general term as $a_n = 550 - (\boldsymbol{n-1}) \cdot 5$, which after simplifying can take the form $a_n = 550 - 5n + 5 = \mathbf{555 - 5n}$.

d. Since there are 10 payments, the last one equals to $a_{10} = 555 - 5 \cdot 10 = \mathbf{\$505}$.

Series and Summation Notation

Often, we take interest in finding sums of terms of a sequence. For instance, in *Example 4*, we might be interested in finding the total amount paid in the first four months $\$550 + \$545 + \$540 + \$535$, or the total cost of borrowing $\$550 + \$545 + \cdots + \$505$. The terms of a sequence connected by the operation of addition create an expression called a **series**.

Note: *The word "**series**" is both singular and plural.*

Definition 1.3 ► A **series** is the sum of terms of a finite or infinite sequence, before evaluation.
The value of a **finite series** can always be determined because addition of a finite number of values can always be performed.
The value of an **infinite series** may not exist. For example, $\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots = 1$ but $1 + 2 + \cdots + n + \cdots = DNE$ (doesn't exist).

Series involve writing sums of many terms, which is often cumbersome. To write such sums in compact form, we use summation notation referred to as **sigma notation**, where the Greek letter $\boldsymbol{\Sigma}$ (sigma) is used to represent the operation of adding all the terms of a sequence. For example, the finite series $1^2 + 2^2 + 3^2 + \cdots + 10^2$ can be recorded in sigma notation as

$$\sum_{i=1}^{10} i^2 \quad \text{or} \quad \sum\nolimits_{i=1}^{10} i^2$$

Here, the letter i is called the **index of summation** and takes integral values from 1 to 10. The expression i^2 (the general term of the corresponding sequence) generates the terms being added. The number 1 is the lower limit of the summation, and the number 10 is the upper limit of the summation. We read "the sum from $i = 1$ to 10 of i^2." To find this sum, we replace the letter i in i^2 with 1, 2, 3, …, 10, and add the resulting terms.

Note: *Any letter can be used for the index of summation; however, the most commonly used letters are i,j,k,m,n.*

A finite series with an unknown number of terms, such as $1 + 2 + \cdots + n$, can be recorded as

$$\sum\nolimits_{i=1}^{n} i$$

Here, since the last term equals to n, the value of the overall sum is an expression in terms of n, rather than a specific number.

An infinite series, such as $0.3 + 0.03 + 0.003 + \cdots$ can be recorded as

$$\sum\nolimits_{i=1}^{\infty} \frac{3}{10^i}$$

In this case, the series can be evaluated and its sum equals to $0.333 \ldots = \frac{1}{3}$.

Example 5 ► **Evaluating Finite Series Given in Sigma Notation**

Evaluate the sum.

a. $\sum_{i=1}^{5}(2i + 1)$ **b.** $\sum_{k=1}^{6}(-1)^k \frac{1}{k}$

Solution ►

a. $\sum_{i=0}^{5}(2i + 1) = (2 \cdot 0 + 1) + (2 \cdot 1 + 1) + (2 \cdot 2 + 1) + (2 \cdot 3 + 1) + (2 \cdot 4 + 1)$

$= 1 + 3 + 5 + 7 + 9 = \mathbf{25}$

b. $\sum_{k=1}^{6}(-1)^k \frac{1}{k} = (-1)^1 \frac{1}{1} + (-1)^2 \frac{1}{2} + (-1)^3 \frac{1}{3} + (-1)^4 \frac{1}{4} + (-1)^5 \frac{1}{5} + (-1)^6 \frac{1}{6}$

$= -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} = \frac{-60+30-20+15-12+10}{60} = -\mathbf{\frac{37}{60}}$

Example 6 ► **Writing Series in Sigma Notation**

Write the given series using sigma notation.

a. $5 + 7 + 9 + \cdots + 47 + 49$ **b.** $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots$

Solution ►

a. Observe that the series consists of a sequence of odd integers, from 5 to 49. An odd integer can be represented by the expression $2n + 1$ or $2n - 1$. The first expression assumes a value of 5 for $n = 2$, and a value of 49 for $n = 24$. Therefore, the general term of the series could be written as $a_n = 2n + 1$, for $n = 2,3, \ldots ,24$. Hence, the series might be written in the form

$$\sum_{i=2}^{24}(2i+1)$$

Using the second expression, $2n-1$, the general term $a_n = 2n-1$ would work for $n = 3,4,\ldots,25$. Hence, the series might also be written in the form

$$\sum_{i=3}^{25}(2i-1)$$

The sequence can also be written with the index of summation set to start with 1. Then we would have

$$\sum_{i=1}^{23}(2i+3)$$

Check that all of the above sigma expressions produce the same series.

b. In this infinite series $\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots$ the signs of consecutive terms alter. To accommodate for the change of signs, we may want to use a factor of $(-1)^n$ or $(-1)^{n+1}$, depending on the sign of the first term. Since the first term is positive, we use the factor of $(-1)^{n+1}$ that equals to 1 for $n = 1$. In addition, the terms consist of fractions with constant numerators equal to 1 and denominators equal to consecutive even numbers that could be represented by $2n$. Hence, the series might be written in the form

$$\sum_{i=1}^{\infty}(-1)^{i+1}\frac{1}{2i}$$

Notice that by renaming the index of summation to, for example, $k = i-1$, the series takes the form

$$\sum_{k=0}^{\infty}(-1)^{k}\frac{1}{2(k+1)}$$

Check on your own that both of the above sigma expressions produce the same series.

Observation: *Series in sigma notation can be written in many different yet equivalent forms. This is because the starting value of the index of summation is arbitrary. Commonly, we start at 1, or 0, unless other values make the general term formula simpler.*

Example 7 ► **Adjusting the Index of Summation**

In each series, change index j to index k that starts at 1.

a. $\sum_{j=2}^{7}(-1)^{j-1}j^3$ **b.** $\sum_{j=0}^{\infty}3^{2j-1}$

Solution ► **a.** If $k = 1$ when $j = 2$, then $j-k = 1$, or equivalently $j = k+1$. In this relation, the upper limit $j = 7$ corresponds to $k = 6$. So by substitution, we obtain

$$\sum_{j=2}^{7}(-1)^{j-1}j^3 = \sum_{k=1}^{6}(-1)^{k+1-1}(k+1)^3 = \sum_{k=1}^{6}(-1)^{k}(k+1)^3$$

b. If $k = 1$ when $j = 0$, then $k - j = 1$, or equivalently $j = k - 1$. By substitution, we have

$$\sum_{j=0}^{\infty} 3^{2j-1} = \sum_{k=1}^{\infty} 3^{2(k-1)-1} = \sum_{k=1}^{\infty} 3^{2k-3}$$

Example 8 ► **Using Series in Application Problems**

In reference to Example 4 of this section:

Peter borrows \$5000 and agrees to pay \$500 monthly, plus interest of 1% per month on the unpaid balance. If his first payment is due one month from the date of borrowing, find

a. the sequence $\{b_n\}$, where b_n represents the remaining balance before the n-th payment,

b. the total interest paid by Peter.

Solution ►

a. As indicated in the solution to *Example 4b*, the sequence of monthly balances before the n-th payment is $5000, 4500, 4000, \dots, 1000, 500$. Since the balance decreases each month by 500, the general term of this sequence is

$$\boldsymbol{b_n} = 5000 - (n-1)500 = \mathbf{5500 - 500}\boldsymbol{n}.$$

b. Since Peter pays 1% on the unpaid balance b_n each month and the number of payments is 10, the total interest paid can be represented by the series

$$\sum_{k=1}^{10}(0.01 \cdot b_k) = \sum_{k=1}^{10}[0.01 \cdot (5500 - 500k)] = \sum_{k=1}^{10}(55 - 5k)$$

$$= 50 + 45 + 40 + 35 + 30 + 25 + 20 + 15 + 10 + 5 = 275$$

Therefore, Peter paid the total interest of **\$275**.

Arithmetic Mean

When calculating the final mark in a course, we often take an average of a sequence of marks we received on assignments, quizzes, or tests. We do this by adding all the marks and dividing the sum into the number of marks used. This average gives us some information about the overall performance on the particular task.

We are often interested in finding averages in many other life situations. For example, we might like to know the average number of calories consumed per day, the average lifetime of particular appliances, the average salary of a particular profession, the average yield from a crop, the average height of a Christmas tree, etc.
In mathematics, the commonly used term "**average**" is called the **arithmetic mean** or just **mean**. If a mean is calculated for a large set of numbers, it is convenient to write it using summation notation.

Definition 1.4 ► The **arithmetic mean (average)** of the numbers $\boldsymbol{x_1, x_2, x_3, \ldots, x_n}$, often denoted $\boldsymbol{\bar{x}}$, is given by the formula

$$\boldsymbol{\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}}$$

Example 9 ► **Calculating the Arithmetic Mean**

a. Find the arithmetic mean of $-12, 3, 0, 4, -2, 10$.

b. Evaluate the mean $\bar{x} = \frac{\sum_{i=1}^{5} i^2}{5}$.

Solution ► **a.** The arithmetic mean equals

$$\frac{-12 + 3 + 0 + 4 + (-2) + 10}{6} = \frac{3}{6} = \boldsymbol{\frac{1}{2}}$$

b. To evaluate this mean by may want to write out all the terms first. We have

$$\bar{x} = \frac{\sum_{i=1}^{5} i^2}{5} = \frac{1^2 + 2^2 + 3^2 + 4^2 + 5^2}{5} = \frac{1 + 4 + 9 + 16 + 25}{5} = \frac{55}{5} = \boldsymbol{11}$$

S.1 Exercises

Vocabulary Check *Fill in each blank with the most appropriate term or phrase from the given list:* ***alternating, arithmetic mean, finite, function, general, infinite, inputs, limit, recursively, sequence, series, sigma, summation, terms, upper.***

1. A ________________ of numbers is an ordered list of numbers.

2. A ____________ whose domain is the set of all natural numbers is an ______________ sequence. The values of such a function are called ___________ of the sequence. The ___________ of the function are the indexes of the terms.

3. A sequence $\{a_n\}$ can be defined explicitly, by stating the ______________ term a_n, or _______________ , by stating the initial term(s) and the relation of the n-th term to the preceding terms.

4. The sum of terms of a sequence before evaluation is called a ___________. The sum of a ___________ series always exists while the sum of an _____________ series may or may not exist.

5. ____________ notation provides a way of writing a sum without writing out all of the terms.

6. The index of _______________ in sigma notation assumes integral values between the lower ____________ and the ____________ limit.

7. In an __________________ series the signs of consecutive terms alternate.

8. “Average” is a popular name of an ________________ ____________ .

Concept Check *Find the **first four terms** and the **10-th** term of each infinite sequence whose n-th term is given.*

9. $a_n = 2n - 3$

10. $a_n = \frac{n+2}{n}$

11. $a_n = (-1)^{n-1}3n$

12. $a_n = 1 - \frac{1}{n}$

13. $a_n = \frac{(-1)^n}{n^2}$

14. $a_n = (n+1)(2n-3)$

15. $a_n = (-1)^n(n-2)^2$

16. $a_n = \frac{1}{n(n+1)}$

17. $a_n = \frac{(-1)^{n+1}}{2n-1}$

Analytic Skills *Write a formula for the n-th term of each sequence.*

18. $2, 5, 8, 11, \dots$

19. $1, -1, 1, -1, \dots$

20. $\frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \dots$

21. $3, 9, 27, 81, \dots$

22. $15, 10, 5, 0, \dots$

23. $6, 9, 12, 15, \dots$

24. $-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{25}, \dots$

25. $1, -\frac{1}{8}, \frac{1}{27}, -\frac{1}{64}, \dots$

26. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$

Concept Check *Find the first five terms of each infinite sequence given by a recursion formula.*

27. $a_n = 2a_{n-1} + 5,\ a_1 = -3$

28. $a_n = 1 - \frac{1}{a_{n-1}},\ a_1 = 2$

29. $a_n = 2a_{n-1} + a_{n-2},\ a_1 = 1, a_2 = 2$

30. $a_n = (a_{n-1})^2 - 1,\ a_1 = 2$

Analytic Skills *Solve each applied problem by writing the first few terms of a sequence.*

31. Lucy borrows \$1600 and agrees to pay \$200 plus interest of 2% on the unpaid balance each month. Find the payments for the first six months and the remaining debt at the end of that period.

32. Maria is offered a new modeling job with a salary of $20{,}000 + 1500n$ dollars per year at the end of the n-th year. Write a sequence showing her salary at the end of each of the first 5 years. What would her salary be at the end of the tenth year?

33. If a contractor does not complete a multimillion-dollar construction project on time, he must pay a penalty of \$500 for the first day that he is late, \$700 for the second day, \$900 for the third day, and so on. Each day the penalty is \$200 larger than the previous day. Write a formula for the penalty on the n-th day. What is the penalty for the 10-th day?

34. The MSRP (manufacturer's suggested retail price) for a 2008 Jeep Grand Cherokee was \$43,440. Analysts estimated that prices increase 6% per year for the next five years. To the nearest dollar, find the estimated price of this model in the years 2009 through 2013. Write a formula for this sequence.

35. Chlorine is often added to swimming pools to control microorganisms. If the level of chlorine rises above 3 ppm (parts per million), swimmers will experience burning eyes and skin discomfort. If the level drops below 1 ppm, there is a possibility that the water will turn green because of a large algae count. Chlorine must be added to pool water at regular intervals. If no chlorine is added to a pool during a 24-hour period, approximately 20% of the chlorine will dissipate into the atmosphere and 80% will remain in the water.

a. Determine a sequence a_n that expresses the amount of chlorine present after n days if the pool has a_0 ppm of chlorine initially and no chlorine is added.

b. If a pool has 7 ppm of chlorine initially, construct a table to determine the first day on which the chlorine level is expected to drop below 3 ppm.

Concept Check *Evaluate each sum.*

36. $\sum_{i=1}^{5}(i+2)$ **37.** $\sum_{i=1}^{10} 5$ **38.** $\sum_{i=1}^{8}(-1)^i i$

39. $\sum_{i=1}^{4}(-1)^i(2i-1)$ **40.** $\sum_{i=1}^{6}(i^2-1)$ **41.** $\sum_{i=3}^{7} 2^i$

42. $\sum_{i=0}^{5}(i-1)\,i$ **43.** $\sum_{i=2}^{5}\frac{(-1)^i}{i}$ **44.** $\sum_{i=0}^{7}(i-2)\,(i-3)$

Analytic Skills *Write each series using sigma notation.*

45. $2+4+6+8+10+12$

46. $3-6+9-12+15-18$

47. $\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\cdots+\frac{1}{50}$

48. $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\cdots-\frac{1}{100}$

49. $1+8+27+64+\cdots$

50. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\cdots$

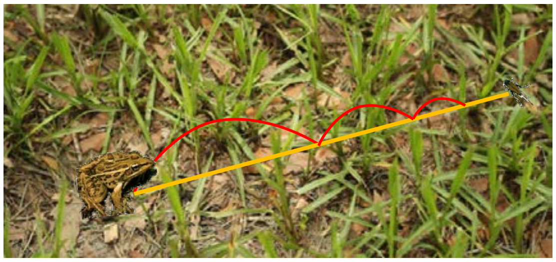
Concept Check *In each series change index **n** to index **m** that starts at **1**.*

51. $\sum_{n=0}^{9}(3n-1)$ **52.** $\sum_{n=3}^{7} 2^{n-2}$ **53.** $\sum_{n=2}^{6}\frac{n}{n+2}$

54. $\sum_{n=0}^{\infty}(-1)^{n+1}n$ **55.** $\sum_{n=2}^{\infty}(-1)^n(n-1)$ **56.** $\sum_{n=2}^{\infty}\frac{1}{n^2}$

Analytic Skills *Use a series to model the situation in each of the following problems.*

57. A frog with a vision problem is 1 yard away from a dead cricket. He spots the cricket and jumps halfway to the cricket. After the frog realizes that he has not reached the cricket, he again jumps halfway to the cricket. Write a series in summation notation to describe how far the frog has moved after nine such jumps.



58. Cleo deposited \$1000 at the beginning of each year for 5 years into an account paying 10% interest compounded annually. Write a series using summation notation to describe how much she has in the account at the end of the fifth year. Note that the first \$1000 will receive interest for 5 years, the second \$1000 will receive interest for 4 years, and so on.

Concept Check *Find the arithmetic mean of each sequence, or evaluate the mean written in sigma notation.*

59. $10, 12, 8, 0, 2, 19, 23, 6$

60. $5, -9, 8, 2, -4, 7, 5$

61. $\bar{x} = \frac{\sum_{i=-5}^{5} i}{11}$

62. $\bar{x} = \frac{\sum_{i=1}^{8} 2^i}{8}$