T.3 Evaluation of Trigonometric Functions

In the previous section, we defined sine, cosine, and tangent as functions of real angles. In this section, we will take interest in finding values of these functions for angles $\theta \in [0°,\ 360°)$. As shown before, one can find exact values of trigonometric functions of an angle θ with the aid of a right triangle with the acute angle θ and given side lengths, or by using coordinates of a given point on the terminal side of the angle θ in standard position. What if such data is not given? Then, one could consider approximating trigonometric function values by measuring sides of a right triangle with the desired angle θ and calculating corresponding ratios. However, this could easily prove to be a cumbersome process, with inaccurate results. Luckily, we can rely on calculators, which are programmed to return approximated values of the three primary trigonometric functions for any angle.

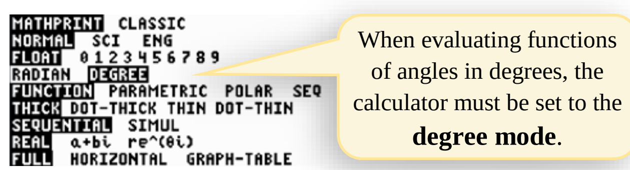
Attention: *In this section, any calculator instruction will refer to graphing calculators such as* ***TI-83*** *or* ***TI-84****.*

Example 1 ► **Evaluating Trigonometric Functions Using a Calculator**

Find each function value up to four decimal places.

a. $\sin 39°12'10''$ **b.** $\tan 102.6°$

Solution ► **a.** Before entering the expression into the calculator, we need to check if the calculator is in degree mode by pressing the MODE key and highlighting the "degree" option, with the aid of arrows and the ENTER key.



To go back to the main screen, we press 2nd MODE, and now we can enter $\sin 39°12'10''$. The degree (°) and minute (′) signs can be found under the "ANGLE" list. To get the second (″) sign, we press ALPHA + .

Thus $\sin 39°12'10'' \approx \mathbf{0.6321}$ when rounded to four decimal places.

b. When evaluating trigonometric functions of angles in decimal degrees, it is not necessary to write the degree (°) sign when in degree mode. We simply key in TAN 102.6 ENTER to obtain $\tan 102.6° \approx \mathbf{-4.4737}$ when rounded to four decimal places.

Special Angles

It has already been discussed how to find the **exact values** of trigonometric functions of **quadrantal angles** using the definitions in terms of x, y, and r. *See section T.2, Example 3.b, and table 2.1.*

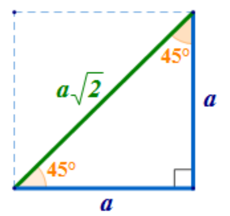


Figure 3.1

Are there any other angles for which the trigonometric functions can be evaluated exactly? Yes, we can find the exact values of trigonometric functions of any angle that can be modelled by a right triangle with known sides. For example, angles such as 30°, 45°, or 60° can be modeled by half of a square or half of an equilateral triangle. In each triangle, the relations between the lengths of sides are easy to establish.

In the case of half a square *(see Figure 3.1)*, we obtain a right triangle with two acute angles of **45°**, and two equal sides of certain length $\boldsymbol{a}$.
Hence, by The Pythagorean Theorem, the diagonal $d = \sqrt{a^2 + a^2} = \sqrt{2a^2} = \boldsymbol{a\sqrt{2}}$.

Summary: The sides of any $\mathbf{45° - 45° - 90°}$ triangle are in the relation $\boldsymbol{a - a - a\sqrt{2}}$.

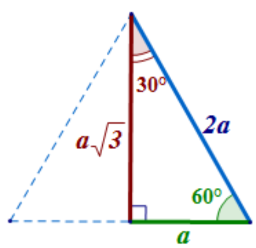


Figure 3.2

By dividing an equilateral triangle *(see Figure 3.1)* along its height, we obtain a right triangle with acute angles of **30°** and **60°**. If the length of the side of the original triangle is denoted by $\boldsymbol{2a}$, then the length of half a side is $\boldsymbol{a}$, and the length of the height can be calculated by applying The Pythagorean Theorem, $h = \sqrt{(2a)^2 - a^2} = \sqrt{3a^2} = \boldsymbol{a\sqrt{3}}$.

Summary: The sides of any $\mathbf{30° - 60° - 90°}$ triangle are in the relation $\boldsymbol{a - 2a - a\sqrt{3}}$.

Since the trigonometric ratios do not depend on the size of a triangle, for simplicity, we can assume that $a = 1$ and work with the following **special triangles**:

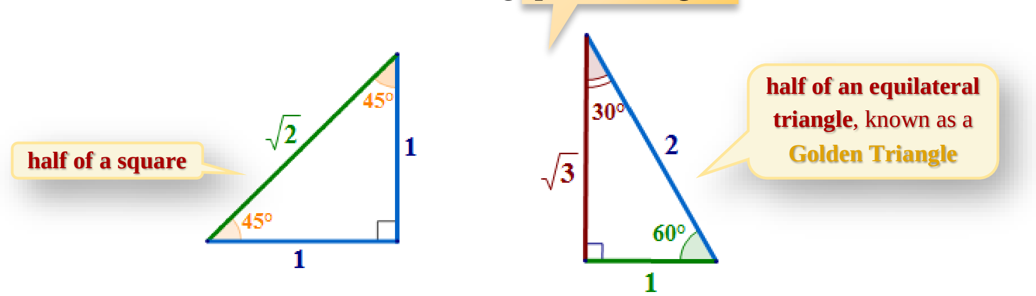


Figure 3.3

Special angles such as **30°**, **45°**, and **60°** are frequently seen in applications. We will often refer to the exact values of trigonometric functions of these angles. Special triangles give us a tool for finding those values.

Advice: *Make sure that you can **recreate the special triangles** by taking half of a square or half of an equilateral triangle, anytime you wish to **recall the relations** between their sides.*

Example 2 ► **Finding Exact Values of Trigonometric Functions of Special Angles**

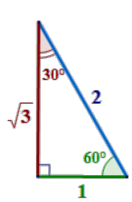
Find the <u>*exact*</u> value of each expression.

a. $\cos 60°$ **b.** $\tan 30°$ **c.** $\sin 45°$ **d.** $\tan 45°$

Solution ►

a. Refer to the $30° - 60° - 90°$ triangle and follow the SOH-CAH-TOA definition of sine:

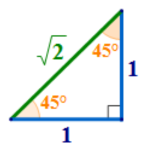
$$\cos 60° = \frac{adj.}{hyp.} = \frac{1}{2}$$



b. Refer to the same triangle as above:

$$\tan 30° = \frac{opp.}{adj.} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

c. Refer to the $45° - 45° - 90°$ triangle:



$$\sin 45° = \frac{opp.}{hyp.} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

d. Refer to the $45° - 45° - 90°$ triangle:

$$\tan 45° = \frac{opp.}{adj.} = \frac{1}{1} = 1$$

The exact values of trigonometric functions of special angles are summarized in the table below.

Table 3.1 Function Values of Special Angles

function \ $\theta =$	$30°$	$45°$	$60°$
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Observations:

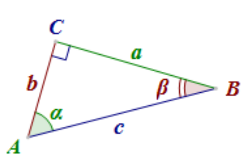
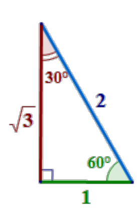


Figure 3.4

- Notice that $\sin 30° = \cos 60°$, $\sin 60° = \cos 30°$, and $\sin 45° = \cos 45°$. Is there any general rule to explain this fact? Lets look at a right triangle with acute angles $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ (*see Figure 3.4*). Since the sum of angles in any triangle is $180°$ and $\angle C = 90°$, then $\boldsymbol{\alpha} + \boldsymbol{\beta} = 90°$, therefore they are **complementary angles**. From the definition, we have $\sin\boldsymbol{\alpha} = \frac{\boldsymbol{a}}{\boldsymbol{b}} = \cos\boldsymbol{\beta}$. Since angle $\boldsymbol{\alpha}$ was chosen arbitrarily, this rule applies to any pair of acute complementary angles. It happens that this rule actually applies to all complementary angles. So we have the following claim:

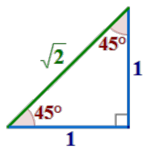
$\sin\boldsymbol{\alpha} = \cos(90° - \boldsymbol{\alpha})$

The **cofunctions** (like sine and cosine) of **complementary** angles are equal.



- Notice that $\tan 30° = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}} = \frac{1}{\tan 60°}$, or equivalently, $\tan 30° \cdot \tan 60° = 1$. This is because of the previously observed rules:

$$\tan\theta \cdot \tan(90° - \theta) = \frac{\sin\theta}{\cos\theta} \cdot \frac{\sin(90° - \theta)}{\cos(90° - \theta)} = \frac{\sin\theta}{\cos\theta} \cdot \frac{\cos\theta}{\sin\theta} = 1$$



In general, we have:

$$\tan\theta = \frac{1}{\tan(90° - \theta)}$$

- Observe that $\tan 45° = 1$. This is an easy, but very useful value to memorize.

Example 3 ► **Using the Cofunction Relationship**

Rewrite $\cos 75°$ in terms of the cofunction of the complementary angle.

Solution ► Since the complement of $75°$ is $90° - 75° = 15°$, then $\cos 75° = \mathbf{\sin 15°}$.

Reference Angles

Can we determine exact values of trigonometric functions of nonquadrantal angles that are larger than 90°?

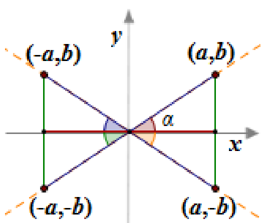
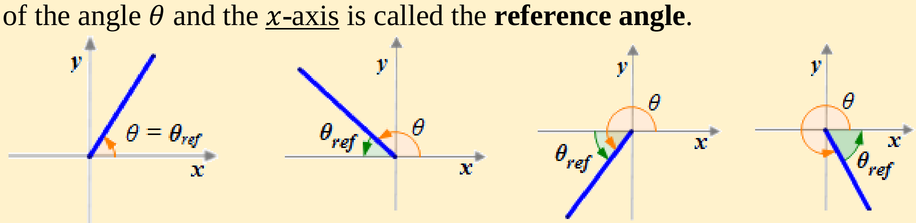


Figure 3.5

Assume that point (a, b) lies on the terminal side of acute angle α. By definition 2.2, the values of trigonometric functions of angles with terminals containing points $(-a, b), (-a, -b)$, and $(a, -b)$ are the same as the values of corresponding functions of the angle α, except for their signs.

Therefore, to find the value of a trigonometric function of any angle θ, it is enough to evaluate this function at the corresponding acute angle θ_{ref}, called the **reference angle**, and apply the sign appropriate to the quadrant of the terminal side of θ.

Definition 3.1 ► Let θ be an angle in standard position. The acute angle $\boldsymbol{\theta_{ref}}$ formed by the terminal side of the angle θ and the x-axis is called the **reference angle**.



Attention: *Think of a **reference angle** as the smallest rotation of the terminal arm required to line it up with the **x-axis**.*

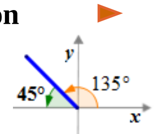
Example 4 ► **Finding the Reference Angle**

Find the **reference angle** for each of the given angles.

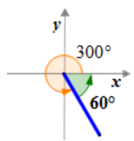
a. 40° **b.** 135° **c.** 210° **d.** 300°

Solution ►

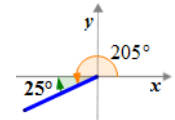
a. Since $\mathbf{40°} \in QI$, this is already the reference angle.



b. Since $135° \in QII$, the reference angle equals $180° - 135° = \mathbf{45°}$.



c. Since $205° \in QIII$, the reference angle equals $205° - 180° = \mathbf{25°}$.



d. Since $300° \in QIV$, the reference angle equals $360° - 300° = \mathbf{60°}$.

CAST Rule

Using the x, y, r definition of trigonometric functions, we can determine and summarize the signs of those functions in each of the quadrants.

Since $\sin\theta = \frac{y}{r}$ and r is positive, then the sign of the sine ratio is the same as the sign of the y-value. This means that the values of sine are positive only in quadrants where y is positive, thus in QI and QII.

Since $\cos\theta = \frac{x}{r}$ and r is positive, then the sign of the cosine ratio is the same as the sign of the x-value. This means that the values of cosine are positive only in quadrants where x is positive, thus in QI and QIV.

Since $\tan\theta = \frac{y}{x}$, then the values of the tangent ratio are positive only in quadrants where both x and y have the same signs, thus in QI and $QIII$.

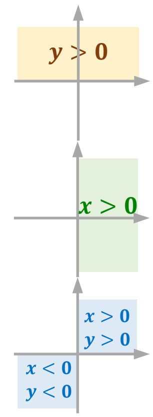


Table 3.2 Signs of Trigonometric Functions in Quadrants

function \ $\theta \in$	QI	QII	$QIII$	QIV
$\sin\theta$	$+$	$+$	$-$	$-$
$\cos\theta$	$+$	$-$	$-$	$+$
$\tan\theta$	$+$	$-$	$+$	$-$

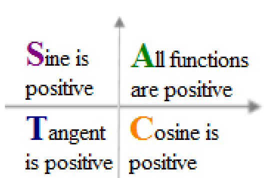


Figure 3.6

Since we will be making frequent decisions about signs of trigonometric function values, it is convenient to have an acronym helping us memorizing these signs in different quadrants. The first letters of the names of functions that are positive in particular quadrants, starting from the fourth quadrant and going counterclockwise, spells **CAST**, which is very helpful when working with trigonometric functions of any angles.

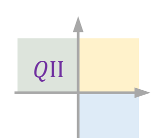
Example 5 ► **Identifying the Quadrant of an Angle**

Identify the quadrant or quadrants for each angle satisfying the given conditions.

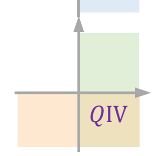
a. $\sin\theta > 0$; $\tan\theta < 0$ **b.** $\cos\theta > 0$; $\sin\theta < 0$

Solution ►

a. Using **CAST**, we have $\sin\theta > 0$ in QI(**A**ll) and QII(**S**ine) and $\tan\theta < 0$ in QII and QIV. Therefore both conditions are met only in **quadrant II**.



b. $\cos\theta > 0$ in QI(**A**ll) and QIV(**C**osine) and $\sin\theta < 0$ in $QIII$ and QIV. Therefore both conditions are met only in **quadrant IV**.



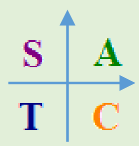
Example 6 ► **Identifying Signs of Trigonometric Functions of Any Angle**

Using the **CAST** rule, identify the sign of each function value.

a. $\cos 150°$ **b.** $\tan 225°$

Solution ► **a.** Since $150° \in Q\text{II}$ and cosine is negative in $Q\text{II}$, then $\cos 150°$ is **negative**.

b. Since $225° \in Q\text{III}$ and tangent is positive in $Q\text{III}$, then $\tan 225°$ is **positive**.



To find the exact value of a trigonometric function $\boldsymbol{T}$ of an angle $\boldsymbol{\theta}$ with the reference angle $\boldsymbol{\theta_{ref}}$ being a special angle, we follow the rule:

$$\boldsymbol{T(\theta) = \pm T(\theta_{ref})},$$

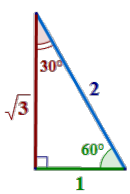
where the final sign is determined according to the quadrant of angle θ and the **CAST** rule.

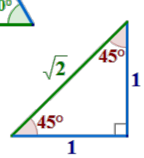
Example 7 ► **Finding Exact Function Values Using Reference Angles**

Find the exact values of the following expressions.

a. $\sin 240°$ **b.** $\cos 315°$

Solution ►





a. The reference angle of $240°$ is $240° - 180° = 60°$. Since $240° \in Q\text{III}$ and sine in the third quadrant is **positive**, we have

$$\sin 240° = \sin 60° = \frac{\sqrt{3}}{2}$$

b. The reference angle of $315°$ is $360° - 315° = 45°$. Since $315° \in Q\text{IV}$ and cosine in the fourth quadrant is **negative**, we have

$$\cos 315° = -\cos 45° = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

Finding Special Angles in Various Quadrants when Given Trigonometric Function Value

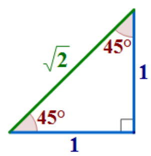
Now that it has been shown how to find exact values of trigonometric functions of angles that have a reference angle of one of the special angles ($\boldsymbol{30°}, \boldsymbol{45°}$, or $\boldsymbol{60°}$), we can work at reversing this process. Familiarity with values of trigonometric functions of the special angles, in combination with the idea of reference angles and quadrantal sign analysis, should help us in solving equations of the type $\boldsymbol{T(\theta)} = \boldsymbol{exact\ value}$, where $\boldsymbol{T}$ represents any trigonometric function.

Example 8 ► **Finding Angles with a Given Exact Function Value, in Various Quadrants**

Find all angles θ satisfying the following conditions.

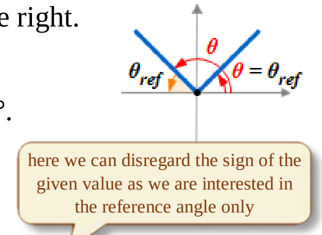
a. $\sin\theta = \frac{\sqrt{2}}{2};\ \theta \in [0°, 180°)$ **b.** $\cos\theta = -\frac{1}{2};\ \theta \in [0°, 360°)$

Solution ► **a.** Refering to the half of a square triangle, we recognize that $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$ represents the ratio of sine of 45°. Thus, the reference angle $\theta_{ref} = 45°$. Moreover, we are searching for an angle θ from the interval $[0°, 180°)$ and we know that $\sin\theta > 0$. Therefore, θ must lie in the first or second quadrant and have the reference angle of 45°. Each quadrant gives us one solution, as shown in the figure on the right.

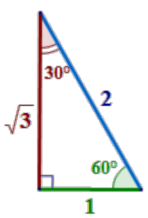


If θ is in the first quadrant, then $\theta = \theta_{ref} = 45°$.
If θ is in the second quadrant, then $\theta = 180° - 45° = 135°$.

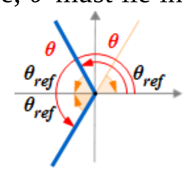
So the solution set of the above problem is $\{\mathbf{45°}, \mathbf{135°}\}$.



b. Refering to the half of an equlateral triangle, we recognize that $\frac{1}{2}$ represents the ratio of cosine of 60°. Thus, the reference angle $\theta_{ref} = 60°$. We are searching for an angle θ from the interval $[0°, 360°)$ and we know that $\cos\theta < 0$. Therefore, θ must lie in the second or third quadrant and have the reference angle of 60°.



If θ is in the second quadrant, then $\theta = 180° - 60° = 120°$.
If θ is in the third quadrant, then $\theta = 180° + 60° = 240°$.



So the solution set of the above problem is $\{\mathbf{120°}, \mathbf{240°}\}$.

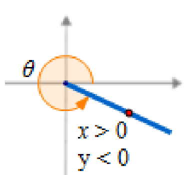
Finding Other Trigonometric Function Values

Example 9 ► **Finding Other Function Values Using a Known Value, Quadrant Analysis, and the x, y, r Definition of Trigonometric Ratios**

Find values of the remaining trigonometric functions of the angle satisfying the given conditions.

a. $\sin\theta = -\frac{7}{13}$; $\theta \in QIV$ **b.** $\tan\theta = \frac{15}{8}$; $\theta \in QIII$

Solution ► **a.** We know that $\sin\theta = -\frac{7}{13} = \frac{y}{r}$. Hence, the terminal side of angle $\theta \in QIV$ contains a point $P(x, y)$ satisfying the condition $\frac{y}{r} = -\frac{7}{13}$. Since r must be positive, we will assign $y = -7$ and $r = 13$, to model the situation. Using the Pythagorean equation and the fact that the x-coordinate of any point in the fourth quadrant is positive, we determine the corresponding x-value to be



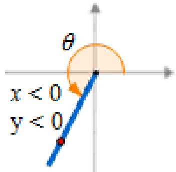
$$x = \sqrt{r^2 - y^2} = \sqrt{13^2 - (-7)^2} = \sqrt{169 - 49} = \sqrt{120} = 2\sqrt{30}.$$

Now, we are ready to state the remaining function values of angle θ:

$$\cos\theta = \frac{x}{r} = \frac{\mathbf{2\sqrt{30}}}{\mathbf{13}}$$

and

$$\tan\theta = \frac{y}{x} = \frac{-7}{2\sqrt{30}} \cdot \frac{\sqrt{30}}{\sqrt{30}} = \frac{\mathbf{-7\sqrt{30}}}{\mathbf{60}}.$$



b. We know that $\tan\theta = \frac{15}{8} = \frac{y}{x}$. Similarly as above, we would like to determine x, y, and r values that would model the situation. Since angle $\theta \in Q\text{III}$, both x and y values must be negative. So we assign $y = -15$ and $x = -8$. Therefore,

$$r = \sqrt{x^2 + y^2} = \sqrt{(-15)^2 + (-8)^2} = \sqrt{225 + 64} = \sqrt{289} = 17$$

Now, we are ready to state the remaining function values of angle θ:

$$\sin\theta = \frac{y}{r} = \boldsymbol{\frac{-15}{17}}$$

and

$$\cos\theta = \frac{x}{r} = \boldsymbol{\frac{-8}{17}}.$$

T.3 Exercises

Vocabulary Check *Fill in each blank.*

1. When a scientific or graphing calculator is used to find a trigonometric function value, in most cases the result is an _________________ value.

2. Angles $30°, 45°, 60°$ are called ______________, because we can find the ___________ trigonometric function values of those angles. This is done by using relationships between the length of sides of _____________ triangles.

3. For any angle θ, its ______________ angle θ_{ref} is the positive ___________ angle formed by the terminal side of θ and the ______________.

4. The trigonometric function values of $150°$ can be found by taking the corresponding function values of the $30°$ ______________ angle and assigning signs based on the _________________ of the ______________ side of the angle θ.

*Use a calculator to **approximate** each value to **four** decimal places.*

5. $\sin 36°52'05''$ **6.** $\tan 57.125°$ **7.** $\cos 204°25'$

*Give the **exact** function value, **without** the aid of a calculator. Rationalize denominators when applicable.*

8. $\cos 30°$ **9.** $\sin 45°$ **10.** $\tan 60°$ **11.** $\sin 60°$

12. $\tan 30°$ **13.** $\cos 60°$ **14.** $\sin 30°$ **15.** $\tan 45°$

*Give the equivalent expression using the **cofunction** relationship.*

16. $\cos 50°$ **17.** $\sin 22.5°$ **18.** $\sin 10°$

Concept Check *For each angle, find the* ***reference angle****.*

19. $98°$ **20.** $212°$ **21.** $13°$ **22.** $297°$ **23.** $186°$

Concept Check *Identify the quadrant or quadrants for each angle satisfying the given conditions.*

24. $\cos\alpha > 0$ **25.** $\sin\beta < 0$ **26.** $\tan\gamma > 0$

27. $\sin\theta > 0;\ \cos\theta < 0$ **28.** $\cos\alpha < 0;\ \tan\alpha > 0$ **29.** $\sin\alpha < 0;\ \tan\alpha < 0$

Identify the sign of each function value by quadrantal analysis.

30. $\cos 74°$ **31.** $\sin 245°$ **32.** $\tan 129°$ **33.** $\sin 183°$

34. $\tan 298°$ **35.** $\cos 317°$ **36.** $\sin 285°$ **37.** $\tan 215°$

Analytic Skills *Using reference angles, quadrantal analysis, and special triangles, find the exact values of the expressions. Rationalize denominators when applicable.*

38. $\cos 225°$ **39.** $\sin 120°$ **40.** $\tan 150°$ **41.** $\sin 150°$

42. $\tan 240°$ **43.** $\cos 210°$ **44.** $\sin 330°$ **45.** $\tan 225°$

Analytic Skills *Find all values of $\theta \in [0°, 360°)$ satisfying the given condition.*

46. $\sin\theta = -\frac{1}{2}$ **47.** $\cos\theta = \frac{1}{2}$ **48.** $\tan\theta = -1$ **49.** $\sin\theta = \frac{\sqrt{3}}{2}$

50. $\tan\theta = \sqrt{3}$ **51.** $\cos\theta = -\frac{\sqrt{2}}{2}$ **52.** $\sin\theta = 0$ **53.** $\tan\theta = -\frac{\sqrt{3}}{3}$

Analytic Skills *Find values of the remaining trigonometric functions of the angle satisfying the given conditions.*

54. $\sin\theta = \frac{\sqrt{5}}{7};\ \theta \in QII$ **55.** $\cos\alpha = \frac{3}{5};\ \alpha \in QIV$ **56.** $\tan\beta = \sqrt{3};\ \beta \in QIII$