T.5 The Law of Sines and Cosines and Its Applications

The concepts of solving triangles developed in *section T4* can be extended to all triangles. A triangle that is not right-angled is called an **oblique triangle**. Many application problems involve solving oblique triangles. Yet, we can not use the SOH-CAH-TOA rules when solving those triangles since **SOH-CAH-TOA** definitions **apply only to right triangles!** So, we need to search for other rules that will allow us to solve oblique triangles.

The Sine Law

Observe that all triangles can be classified with respect to the size of their angles as **acute** (with all acute angles), **right** (with one right angle), or **obtuse** (with one obtuse angle). Therefore, oblique triangles are either acute or obtuse.

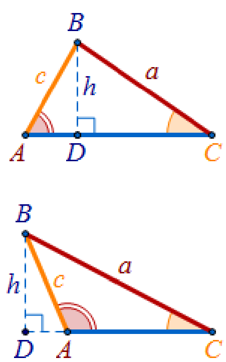


Figure 1

Let's consider both cases of an oblique $\triangle ABC$, as in *Figure 1*. In each case, let's drop the height h from vertex B onto the line $\overleftrightarrow{AC}$, meeting this line at point D. This way, we obtain two more right triangles, $\triangle ADB$ with hypotenuse c, and $\triangle BDC$ with hypotenuse a. Applying the ratio of sine to both of these triangles, we have:

$$\sin\angle A = \frac{h}{c}, \text{ so } h = c\sin\angle A$$

and

$$\sin\angle C = \frac{h}{a}, \text{ so } h = a\sin\angle C.$$

Thus,

$$a\sin\angle C = c\sin\angle A,$$

and we obtain

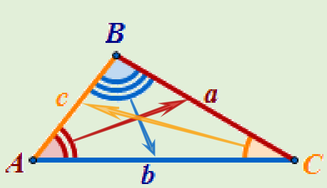
$$\frac{\boldsymbol{a}}{\boldsymbol{\sin\angle A}} = \frac{\boldsymbol{c}}{\boldsymbol{\sin\angle C}}.$$

Similarly, by dropping heights from the other two vertices, we can show that

$$\frac{\boldsymbol{a}}{\boldsymbol{\sin\angle A}} = \frac{\boldsymbol{b}}{\boldsymbol{\sin\angle B}} \text{ and } \frac{\boldsymbol{b}}{\boldsymbol{\sin\angle B}} = \frac{\boldsymbol{c}}{\boldsymbol{\sin\angle C}}.$$

This result is known as the law of sines.

The Sine Law ► In any triangle ABC, the lengths of the **sides are proportional to the sines of the opposite angles**. This fact can be expressed in any of the following, equivalent forms:



$$\frac{\boldsymbol{a}}{\boldsymbol{b}} = \frac{\boldsymbol{\sin\angle A}}{\boldsymbol{\sin\angle B}}, \quad \frac{\boldsymbol{b}}{\boldsymbol{c}} = \frac{\boldsymbol{\sin\angle B}}{\boldsymbol{\sin\angle C}}, \quad \frac{\boldsymbol{c}}{\boldsymbol{a}} = \frac{\boldsymbol{\sin\angle C}}{\boldsymbol{\sin\angle A}}$$

or

$$\frac{\boldsymbol{a}}{\boldsymbol{\sin\angle A}} = \frac{\boldsymbol{b}}{\boldsymbol{\sin\angle B}} = \frac{\boldsymbol{c}}{\boldsymbol{\sin\angle C}}$$

or

$$\frac{\boldsymbol{\sin\angle A}}{\boldsymbol{a}} = \frac{\boldsymbol{\sin\angle B}}{\boldsymbol{b}} = \frac{\boldsymbol{\sin\angle C}}{\boldsymbol{c}}$$

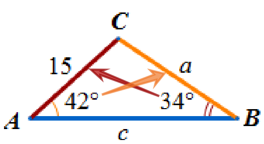
Observation: *As with any other proportion, to solve for one variable, we need to know the three remaining values. Notice that when using the Sine Law proportions, the three known values must include* ***one pair of opposite data****: a side and its opposite angle.*

Example 1 ► **Solving Oblique Triangles with the Aid of The Sine Law**

Given the information, solve each triangle ABC.

a. $\angle A = 42°,\ \angle B = 34°,\ b = 15$ **b.** $\angle A = 35°,\ a = 12,\ b = 9$

Solution ►



a. First, we will sketch a triangle ABC that models the given data. Since the sum of angles in any triangle equals 180°, we have

$$\angle \boldsymbol{C} = 180° - 42° - 34° = \mathbf{104°}.$$

Then, to find length a, we will use the pair $(a, \angle A)$ of opposite data, side a and $\angle A$, and the given pair $(b, \angle B)$. From the Sine Law proportion, we have

$$\frac{a}{\sin 42°} = \frac{15}{\sin 34°},$$

which gives

$$\boldsymbol{a} = \frac{15 \cdot \sin 42°}{\sin 34°} \simeq \mathbf{17.9}$$

To find length c, we will use the pair $(c, \angle C)$ and the given pair of opposite data $(b, \angle B)$. From the Sine Law proportion, we have

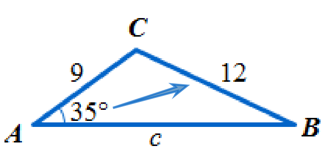
$$\frac{c}{\sin 104°} = \frac{15}{\sin 34°},$$

for easier calculations, keep the unknown in the numerator

which gives

$$\boldsymbol{c} = \frac{15 \cdot \sin 104°}{\sin 34°} \simeq \mathbf{26}$$

So the triangle is solved.



b. As before, we will start by sketching a triangle ABC that models the given data. Using the pair $(9, \angle B)$ and the given pair of opposite data $(12, 35°)$, we can set up a proportion

$$\frac{\sin \angle B}{9} = \frac{\sin 35°}{12}.$$

Then, solving it for $\sin \angle B$, we have

$$\sin \angle B = \frac{9 \cdot \sin 35°}{12} \simeq 0.4302,$$

which, after applying the inverse sine function, gives us

$$\angle \boldsymbol{B} \simeq \mathbf{25.5°}$$

Now, we are ready to find $\angle \boldsymbol{C} = 180° - 35° - 25.5° = \mathbf{119.5}°$,

and finally, from the proportion

$$\frac{c}{\sin 119.5°} = \frac{12}{\sin 35°},$$

we have

$$\boldsymbol{c} = \frac{12 \cdot \sin 119.5°}{\sin 35°} \simeq \mathbf{18.2}$$

Thus, the triangle is solved.

Ambiguous Case

Observe that the size of one angle and the length of two sides does not always determine a unique triangle. For example, there are two different triangles that can be constructed with $\angle A = 35°$, $a = 9$, $b = 12$.

Such a situation is called an **ambiguous case**. It occurs when the opposite side to the given angle is shorter than the other given side but long enough to complete the construction of an oblique triangle, as illustrated in *Figure 2*.

In application problems, if the given information does not determine a unique triangle, both possibilities should be considered in order for the solution to be complete.

On the other hand, not every set of data allows for the construction of a triangle. For example (see *Figure 3*), if $\angle A = 35°$, $a = 5$, $b = 12$, the side a is too short to complete a triangle, or if $a = 2$, $b = 3$, $c = 6$, the sum of lengths of a and b is smaller than the length of c, which makes impossible to construct a triangle fitting the data.

Note that in any triangle, the **sum of lengths of any two sides** is always **bigger than the length of the third side**.

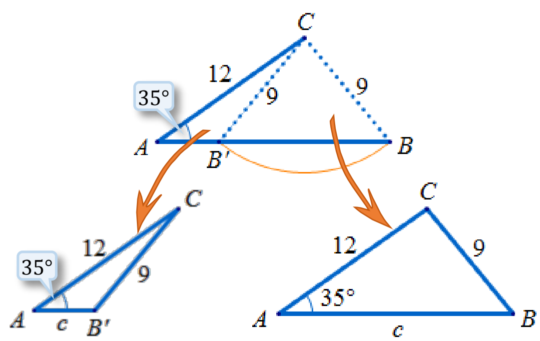


Figure 2

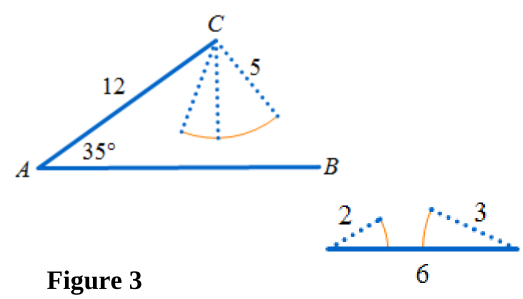
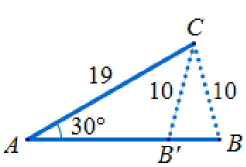


Figure 3

Example 2 ► **Using the Sine Law in an Ambiguous Case**

Solve triangle ABC, knowing that $\angle A = 30°$, $a = 10$, $b = 16$.

Solution ►



When sketching a diagram, we notice that there are two possible triangles, $\triangle ABC$ and $\triangle AB'C$, complying with the given information. $\triangle ABC$ can be solved in the same way as the triangle in *Example 1b*. In particular, one can calculate that in $\triangle ABC$, we have $\angle \boldsymbol{B} \simeq \mathbf{71.8}°$, $\angle \boldsymbol{C} \simeq \mathbf{78.2}°$, and $\boldsymbol{c} \simeq \mathbf{19.6}$.

Let's see how to solve $\triangle AB'C$ then. As before, to find $\angle B'$, we will use the proportion

$$\frac{\sin \angle B'}{19} = \frac{\sin 30°}{10},$$

which gives us $\sin \angle B' = \frac{19 \cdot \sin 30°}{10} = 0.95$. However, when applying the inverse sine function to the number 0.95, a calculator returns the approximate angle of 71.8°. Yet, we know that angle B' is obtuse. So, we should look for an angle in the second quadrant, with the reference angle of 71.8°. Therefore, $\angle \boldsymbol{B'} = 180° - 71.8° = \mathbf{108.2}°$.

Now, $\angle \boldsymbol{C} = 180° - 30° - 108.2° = \mathbf{41.8}°$

and finally, from the proportion

$$\frac{c}{\sin 41.8°} = \frac{10}{\sin 30°},$$

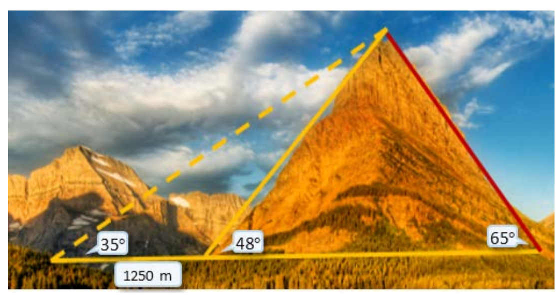
we have

$$\boldsymbol{c} = \frac{10 \cdot \sin 41.8°}{\sin 30°} \simeq \mathbf{13.3}$$

Thus, $\triangle AB'C$ is solved.

Example 3 ► **Solving an Application Problem Using the Sine Law**

Approaching from the west, a group of hikers records the angle of elevation to the summit of a steep mountain to be 35° at a distance of 1250 meters from the base of the mountain. Arriving at the base of the mountain, the hikers estimate that this side of the mountain has an average slope of 48°.



a. Find the slant height of the mountain's west side.

b. Find the slant height of the east side of the mountain, if the east side has an average slope of 65°.

c. How tall is the mountain?

Solution ►

First, let's draw a diagram that models the situation and label its important parts, as in *Figure 3*.

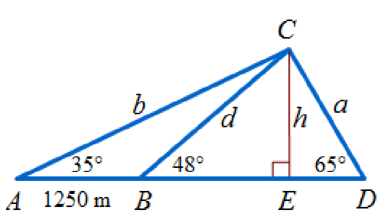


Figure 3

a. To find the slant height d, consider $\triangle ABC$. Observe that one can easily find the remaining angles of this triangle, as shown below:

$$\angle ABC = 180° - 48° = 135°$$ supplementary angles

and

$$\angle ACB = 180° - 35° - 135° = 10°$$ sum of angles in a $\triangle$

Therefore, applying the law of sines, we have

$$\frac{d}{\sin 35°} = \frac{1250}{\sin 10°},$$

which gives

$$\boldsymbol{d} = \frac{1250 \sin 35°}{\sin 10°} \simeq \mathbf{4128.9}\ \boldsymbol{m}.$$

b. To find the slant height a, we can apply the law of sines to $\varDelta BDC$ using the pair $(4128.9, 65°)$ to have

$$\frac{a}{\sin 48°} = \frac{4128.9}{\sin 65°},$$

which gives

$$\boldsymbol{a} = \frac{4128.9 \sin 48°}{\sin 65°} \simeq \mathbf{3385.6}\ \boldsymbol{m}.$$

c. To find the height h of the mountain, we can use the right triangle BCE. Using the definition of sine, we have

$$\frac{h}{4128.9} = \sin 48°,$$

so $\boldsymbol{h} = 4128.9 \sin 48° = \mathbf{3068.4}\ \boldsymbol{m}$.

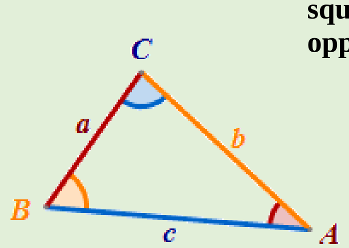
The Cosine Law

The above examples show how the **Sine Law** can help in solving oblique triangles when one **pair of opposite data** is given. However, the Sine Law is not enough to solve a triangle if the given information is

- the length of the **three sides** (but no angles), or
- the length of **two sides** and the **enclosed angle**.

Both of the above cases can be solved with the use of another property of a triangle, called the Cosine Law.

The Cosine Law ► In any triangle ABC, **the square of a side of a triangle is equal to the sum of the squares of the other two sides, minus twice their product times the cosine of the opposite angle**.



$$a^2 = b^2 + c^2 - 2bc \cos \angle A$$

$$b^2 = a^2 + c^2 - 2ac \cos \angle B$$

$$c^2 = a^2 + b^2 - 2ab \cos \angle C$$

note the opposite side and angle

Observation: *If the angle of interest in any of the above equations is right, since* $\cos 90° = 0$, *the equation becomes Pythagorean. So the* ***Cosine Law*** *can be seen as an* ***extension of the Pythagorean Theorem****.*

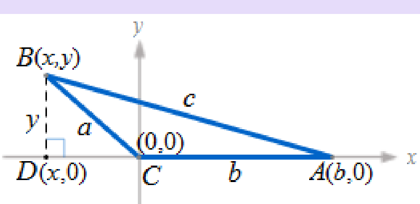


Figure 3

To derive this law, let's place an oblique triangle ABC in the system of coordinates so that vertex C is at the origin, side AC lies along the positive x-axis, and vertex B is above the x-axis, as in *Figure 3*.

Thus $C = (0,0)$ and $A = (b, 0)$. Suppose point B has coordinates (x, y). By *Definition 2.2*, we have

$$\sin \angle C = \frac{y}{a} \quad \text{and} \quad \cos \angle C = \frac{x}{a},$$

which gives us

$$y = a \sin \angle C \quad \text{and} \quad x = a \cos \angle C .$$

Let $D = (x, 0)$ be the perpendicular projection of the vertex B onto the x-axis. After applying the Pythagorean equation to the right triangle ABD, with $\angle D = 90°$, we obtain

here we use the Pythagorean identity developed in *section T2*:
$\boldsymbol{\sin^2 \theta + \cos^2 \theta = 1}$

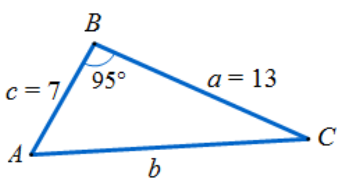
$$\begin{aligned} \boldsymbol{c^2} &= y^2 + (b - x)^2 \\ &= (a \sin \angle C)^2 + (b - a \cos \angle C)^2 \\ &= a^2 \sin^2 \angle C + b^2 - 2ab \cos \angle C + a^2 \cos^2 \angle C \\ &= a^2 (\sin^2 \angle C + \cos^2 \angle C) + b^2 - 2ab \cos \angle C \\ &= \boldsymbol{a^2 + b^2 - 2ab \cos \angle C} \end{aligned}$$

Similarly, by placing the vertices A or B at the origin, one can develop the remaining two forms of the Cosine Law.

Example 4 ► **Solving Oblique Triangles Given Two Sides and the Enclosed Angle**

Solve triangle ABC, given that $\angle B = 95°$, $a = 13$, and $c = 7$.

Solution ►



First, we will sketch an oblique triangle ABC to model the situation. Since there is no pair of opposite data given, we cannot use the law of sines. However, applying the law of cosines with respect to side b and $\angle B$ allows for finding the length b. From

$$b^2 = 13^2 + 7^2 - 2 \cdot 13 \cdot 7 \cos 95° \simeq 233.86,$$

we have $\boldsymbol{b \simeq 15.3}$.

watch the order of operations here!

Now, since we already have the pair of opposite data $(15.3, 95°)$, we can apply the law of sines to find, for example, $\angle C$. From the proportion

$$\frac{\sin \angle C}{7} = \frac{\sin 95°}{15.3},$$

we have

$$\sin \angle C = \frac{7 \cdot \sin 95°}{15.3} \simeq 0.4558,$$

thus $\angle \boldsymbol{C} = \sin^{-1} 0.4558 \simeq \boldsymbol{27.1}°$.
Finally, $\angle \boldsymbol{A} = 180° - 95° - 27.1° = \boldsymbol{57.9}°$ and the triangle is solved.

When applying the law of cosines in the above example, there was no other choice but to start with the pair of opposite data $(b, \angle B)$. However, in the case of three given sides, one could apply the law of cosines corresponding to any pair of opposite data. Is there any preference as to which pair to start with? Actually, yes. Observe that after using the law of cosines, we often use the **law of sines** to complete the solution since the **calculations are usually easier** to perform this way. Unfortunately, when solving a sine proportion for an obtuse angle, one would need to change the angle obtained from a calculator to its supplementary one. This is because calculators are programmed to return angles from the first quadrant when applying $\sin^{-1}$ to positive ratios. If we look for an obtuse angle, we need to employ the fact that $\sin\alpha = \sin(180° - \alpha)$ and take the supplement of the calculator's answer. To avoid this ambiguity, it is recommended to **apply the cosine law** to the pair of the **longest side and largest angle** first. This will guarantee that the law of sines will be used to find only acute angles and thus it will not cause ambiguity.

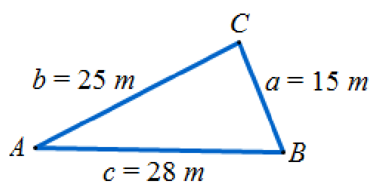
Recommendations:

- ***apply the Cosine Law only when it is absolutely necessary*** *(SAS or SSS)*
- ***apply the Cosine Law to find the largest angle first****, if applicable*

Example 5 ► **Solving Oblique Triangles Given Three Sides**

Solve triangle ABC, given that $a = 15\ m$, $b = 25\ m$, and $c = 28\ m$.

Solution ►



First, we will sketch a triangle ABC to model the situation. As before, there is no pair of opposite data given, so we cannot use the law of sines. So, we will apply the law of cosines with respect to the pair $(28, \angle C)$, as the side $c = 28$ is the longest. To solve the equation

$$28^2 = 15^2 + 25^2 - 2 \cdot 15 \cdot 25 \cos\angle C$$

watch the order of operations when solving for cosine

for $\angle C$, we will first solve it for $\cos\angle C$, and have

$$\cos\angle C = \frac{28^2 - 15^2 - 25^2}{-2 \cdot 15 \cdot 25} = \frac{-66}{-750} = 0.088,$$

which, after applying $\cos^{-1}$, gives $\angle \boldsymbol{C} \simeq \mathbf{85}°$.

Since now we already have the pair of opposite data $(28, 85°)$, we can apply the law of sines to find, for example, $\angle A$. From the proportion

$$\frac{\sin\angle A}{15} = \frac{\sin 85°}{28},$$

we have

$$\sin\angle A = \frac{15 \cdot \sin 85°}{28} \simeq 0.5337,$$

thus $\angle \boldsymbol{A} = \sin^{-1} 0.5337 \simeq \mathbf{32.3}°$.
Finally, $\angle \boldsymbol{B} = 180° - 85° - 32.3° = \mathbf{62.7}°$ and the triangle is solved.

Example 6 ► **Solving an Application Problem Using the Cosine Law**

Two planes leave an airport at the same time and fly for two hours. Plane A flies in the direction of 165° at 385 km/h and plane B flies in the direction of 250° at 410 km/h. How far apart are the planes after two hours?

Solution ► As usual, we start the solution by sketching a diagram appropriate to the situation. Assume the notation as in *Figure 4*.

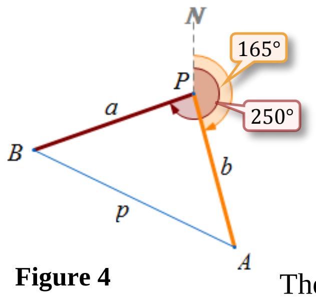


Figure 4

Since plane A flies at 385 km/h for two hours, we can find the distance

$$b = 2 \cdot 385 = 770\ km.$$

Similarly, since plane B flies at 410 km/h for two hours, we have

$$a = 2 \cdot 410 = 820\ km.$$

The measure of the enclosed angle APB can be obtained as a difference between the given directions. So we have

$$\angle APB = 250° - 165° = 85°.$$

Now, we are ready to apply the law of cosines in reference to the pair $(p, 85°)$. From the equation

$$p^2 = 820^2 + 770^2 - 2 \cdot 820 \cdot 770 \cos 85°,$$

we have $\boldsymbol{p} \simeq \sqrt{1155239.7} \simeq \mathbf{1074.8}\ \boldsymbol{km}$.

So we know that after two hours, the two planes are about 1074.8 kilometers apart.

Area of a Triangle

The method used to derive the law of sines can also be used to derive a handy formula for finding the area of a triangle, without knowing its height.

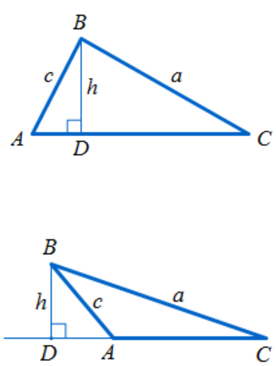


Figure 5

Let ABC be a triangle with height h dropped from the vertex B onto the line $\overleftrightarrow{AC}$, meeting $\overleftrightarrow{AC}$ at the point D, as shown in *Figure 5*. Using the right $\triangle ABD$, we have

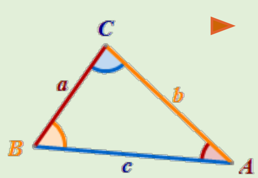
$$\sin \angle A = \frac{h}{c},$$

and equivalently $h = c \sin \angle A$, which after substituting into the well known formula for area of a triangle $[\boldsymbol{ABC}] = \frac{1}{2}\boldsymbol{bh}$, gives us

$$[\boldsymbol{ABC}] = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{bc}\ \mathbf{sin}\ \angle \boldsymbol{A}$$

Starting the proof with dropping a height from a different vertex would produce two more versions of this formula, as stated below.

The Sine Formula for Area of a Triangle

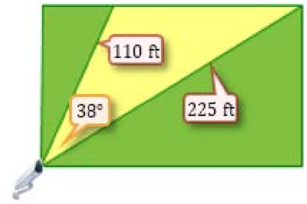


The **area** $[ABC]$ of a triangle ABC can be calculated by taking **half of a product of the lengths of two sides and the sine of the enclosed angle.** We have

$$[\boldsymbol{ABC}] = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{bc}\ \mathbf{sin}\ \angle \boldsymbol{A}, \quad [\boldsymbol{ABC}] = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{ac}\ \mathbf{sin}\ \angle \boldsymbol{B}, \quad \text{or} \quad [\boldsymbol{ABC}] = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{ab}\ \mathbf{sin}\ \angle \boldsymbol{C}.$$

Example 7 ► **Finding Area of a Triangle Given Two Sides and the Enclosed Angle**

A stationary surveillance camera is set up to monitor activity in the parking lot of a shopping mall. If the camera has a 38° field of vision, how many square feet of the parking lot can it tape using the given dimensions?



Solution ► We start with sketching an appropriate diagram. Assume the notation as in *Figure 6.*

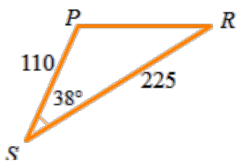


Figure 6

From the sine formula for area of a triangle, we have

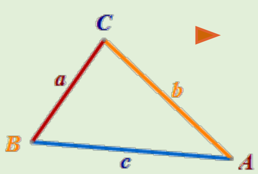
$$[\boldsymbol{PRS}] = \frac{1}{2} \cdot 110 \cdot 225 \sin 38° \simeq \boldsymbol{7619\ ft^2}.$$

The surveillance camera monitors approximately 7619 square feet of the parking lot.

Heron's Formula

The law of cosines can be used to derive a formula for the area of a triangle when only the lengths of the three sides are known. This formula is known as Heron's formula, named after the Greek mathematician Heron of Alexandria.

Heron's Formula for Area of a Triangle



► The **area** $[\boldsymbol{ABC}]$ of a triangle ABC with sides $a, b, c,$ and **semiperimeter** $\boldsymbol{s} = \frac{\boldsymbol{a+b+c}}{\boldsymbol{2}}$ can be calculated using the formula

$$[\boldsymbol{ABC}] = \sqrt{\boldsymbol{s}(\boldsymbol{s}-\boldsymbol{a})(\boldsymbol{s}-\boldsymbol{b})(\boldsymbol{s}-\boldsymbol{c})}$$

Example 8 ► **Finding Area of a Triangle Given Three Sides**

A New York City developer wants to build condominiums on the triangular lot formed by Greenwich, Watts, and Canal Streets. How many square meters does the developer have to work with if the frontage along each street is approximately 34.1 m, 43.5 m, and 62.4 m, respectively?

Solution ► To find the area of the triangular lot with given sides, we would like to use Heron's Formula. For this reason, we first calculate the semiperimeter

$$s = \frac{34.1 + 43.5 + 62.4}{2} = 70.$$

Then, the area equals

$$\sqrt{70(70-34.1)(70-43.5)(70-62.4)} = \sqrt{506118.2} \simeq \boldsymbol{711\ m^2}.$$

Thus, the developer has approximately 711 square meters to work with in the lot.

T.5 Exercises

Vocabulary Check *Fill in each blank.*

1. A triangle that is not right-angled is called an ______________ triangle.

2. When solving a triangle, we apply the law of sines only when a pair of ______________ data is given.

3. To solve triangles with all __________ sides or two sides and the _________________angle given, we use the law of _______________ .

4. The ambiguous case refers to the situation when _______ _________________ satisfying the given data can be constructed.

5. In any triangle, the _______________ side is always opposite the largest _____________.

6. In any triangle the _________ of lengths of any pair of sides is bigger than the _____________ of the third __________ .

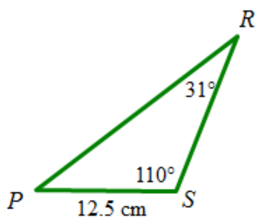
7. To avoid dealing with the __________________ case, we should use the law of _____________ when solving for the ____________ angle.

8. The Cosine Law can be considered as an extension of the ____________________ Theorem.

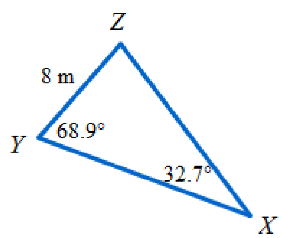
9. The _____________ of a triangle with three given ____________ can be calculated by using the Heron's formula.

Concept check *Use the law of sines to solve each triangle.*

10.



11.



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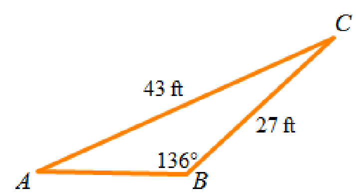

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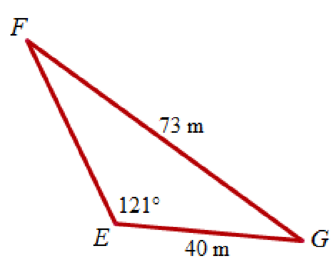

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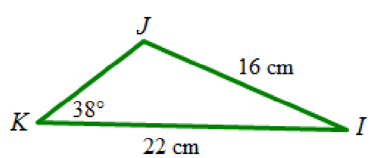

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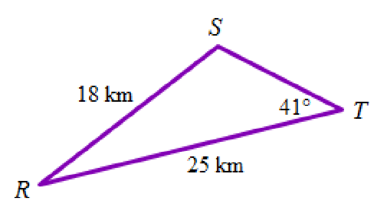


16. $\angle A = 30°,\ \angle B = 30°,\ a = 10$

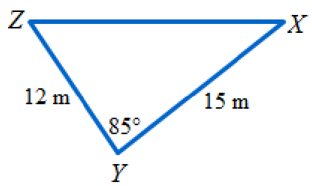
17. $\angle A = 150°,\ \angle C = 20°,\ a = 200$

18. $\angle C = 145°,\ b = 4,\ c = 14$

19. $\angle A = 110°15',\ a = 48,\ b = 16$

Concept check *Use the law of cosines to solve each triangle.*

20.



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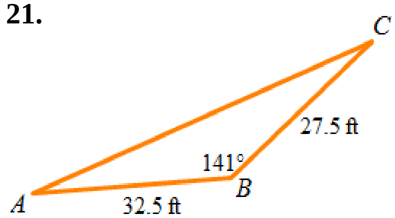

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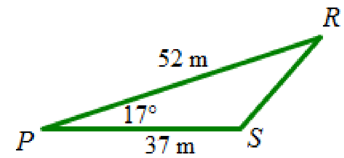

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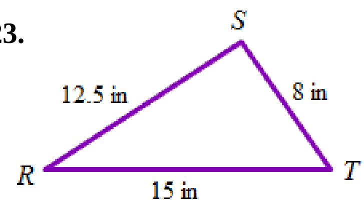

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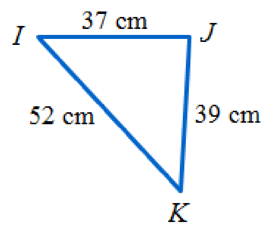

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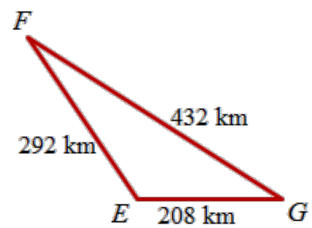


26. $\angle C = 60°,\ a = 3,\ b = 10$

27. $\angle B = 112°,\ a = 23,\ c = 31$

28. $a = 2,\ b = 3,\ c = 4$

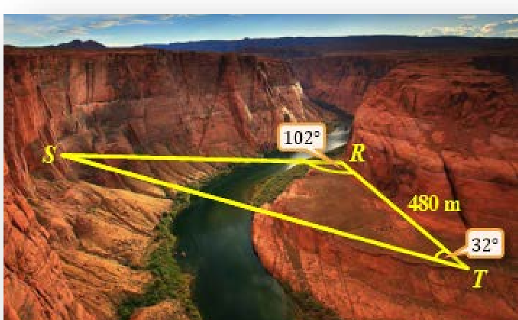
29. $a = 34,\ b = 12,\ c = 17.5$

Concept check

30. If side a is twice as long as side b, is $\angle A$ necessarily twice as large as $\angle B$?

Use the appropriate law to solve each application problem.

31. To find the distance AB across a river, a surveyor laid off a distance BC = 354 meters on one side of the river, as shown in the accompanying figure. It is found that $\angle B = 112°10'$ and $\angle C = 15°20'$. Find the distance AB.



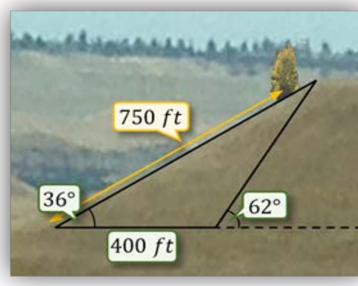
32. To determine the distance RS across a deep canyon (*see the accompanying figure*), Peter lays off a distance TR = 480 meters. Then he finds that $\angle T = 32°$ and $\angle R = 102°$. Find the distance RS.

33. A ship is sailing due north. At a certain point, the captain of the ship notices a lighthouse 12.5 km away from the ship, at the bearing of $\boldsymbol{N}38.8°\boldsymbol{E}$. Later on, the bearing of the lighthouse becomes $\boldsymbol{S}44.2°\boldsymbol{E}$. In meters, how far did the ship travel between the two observations of the lighthouse?

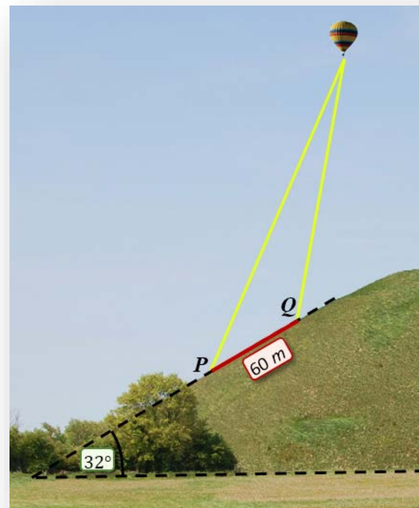
34. The bearing of a lighthouse from a ship was found to be $\boldsymbol{N}37°\boldsymbol{E}$. After the ship sailed 2.5 mi due south, the new bearing was $\boldsymbol{N}25°\boldsymbol{E}$. Find the distance between the ship and the lighthouse at each location.

35. Joe and Jill set sail from the same point, with Joe sailing in the direction of S4°E and Jill sailing in the direction S9°W. After 4 hr, Jill was 2 mi due west of Joe. How far had Jill sailed?

36. A hill has an angle of inclination of 36°, as shown in the accompanying figure. A study completed by a state's highway commission showed that the placement of a highway requires that 400 ft of the hill, measured horizontally, be removed. The engineers plan to leave a slope alongside the highway with an angle of inclination of 62°, as shown in the figure. Located 750 ft up the hill measured from the base is a tree containing the nest of an endangered hawk. Will this tree be removed in the excavation?



37. Radio direction finders are placed at points A and B, which are 3.46 mi apart on an east-west line, with A west of B. A radio transmitter is found to be at the direction of 47.7° from A and 302.5° from B. Find the distance of the transmitter from A, to the nearest hundredth of a mile.



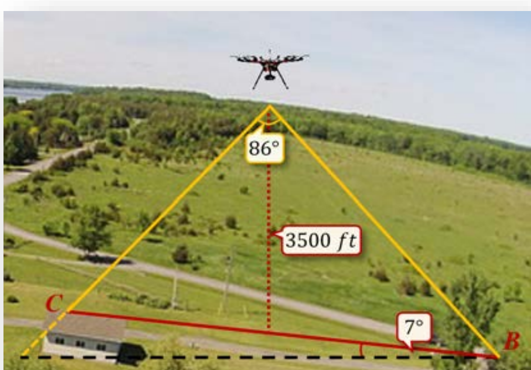
38. Observers at P and Q are located on the side of a hill that is inclined 32° to the horizontal, as shown in the accompanying figure. The observer at P determines the angle of elevation to a hot-air balloon to be 62°. At the same instant, the observer at Q measures the angle of elevation to the balloon to be 71°. If P is 60 meters down the hill from Q, find the distance from Q to the balloon.

39. What is the length of the chord subtending a central angle of 19° in a circle of radius 30 ft?

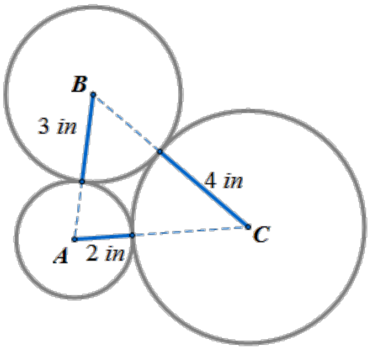
40. A pilot flies her plane on a heading of 35° from point X to point Y, which is 400 mi from X. Then she turns and flies on a heading of 145° to point Z, which is 400 mi from her starting point X. What is the heading of Z from X, and what is the distance YZ?

41. A painter is going to apply a special coating to a triangular metal plate on a new building. Two sides measure 16.1 m and 15.2 m. She knows that the angle between these sides is 125°. What is the area of the surface she plans to cover with the coating?

42. A camera lens with a 6-in. focal length has an angular coverage of 86°. Suppose an aerial photograph is taken vertically with no tilt at an altitude of 3500 ft over ground with an increasing slope of 7°, as shown in the accompanying figure. Calculate the ground distance CB that will appear in the resulting photograph.



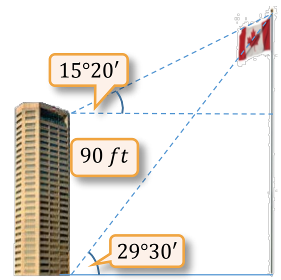
43. A solar panel with a width of 1.2 m is positioned on a flat roof, as shown in the accompanying figure. What is the angle of elevation α of the solar panel?



44. An engineer wants to position three pipes so that they are tangent to each other. A perpendicular cross section of the structure is shown in the accompanying figure. If pipes with centers A, B, and C have radii 2 in., 3 in., and 4 in., respectively, then what are the angles of the triangle ABC?

45. A flagpole 95 ft tall is on the top of a building. From a point on level ground, the angle of elevation of the top of the flagpole is 35°, and the angle of elevation of the bottom of the flagpole is 26°. Find the height of the building.

46. The angle of elevation (*see the figure to the right*) from the top of a building 90 ft high to the top of a nearby mast is $15°20'$. From the base of the building, the angle of elevation of the tower is $29°30'$. Find the height of the mast.

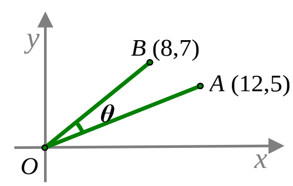


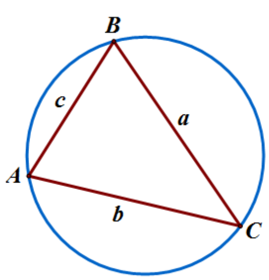
47. A real estate agent wants to find the area of a triangular lot. A surveyor takes measurements and finds that two sides are 52.1 m and 21.3 m, and the angle between them is 42.2°. What is the area of the triangular lot?

48. A painter needs to cover a triangular region with sides of lengths 75 meters, 68 meters, and 85 meters. A can of paint covers 75 square meters of area. How many cans will be needed?

Analytic Skills

49. Find the measure of angle θ enclosed by the segments OA and OB, as on the accompanying diagram.





50. Prove that for a triangle inscribed in a circle of radius r (*see the diagram to the left*), the law of sine ratios $\frac{a}{\sin\angle A}$, $\frac{b}{\sin\angle B}$, and $\frac{c}{\sin\angle C}$ have value $2r$. Then confirm that in a circle of diameter 1, the following equations hold: $\sin\angle A = a$, $\sin\angle B = b$, and $\sin\angle C = c$.
(This provides an alternative way to define the sine function for angles between 0° and 180°. It was used nearly 2000 years ago by the mathematician Ptolemy to construct one of the earliest trigonometric tables.)

51. Josie places her lawn sprinklers at the vertices of a triangle that has sides of 9 m, 10 m, and 11 m. The sprinklers water in circular patterns with radii of 4, 5, and 6 m. No area is watered by more than one sprinkler. What amount of area inside the triangle is not watered by any of the three sprinklers? *Round the answer to the nearest hundredth of a square meter.*

52. The Pentagon in Washington D.C. is 921 ft on each side, as shown in the accompanying figure. What is the distance r from a vertex to the center of the Pentagon?