TRIGONOMETRY

Trigonometry is the branch of mathematics that studies the relations between the sides and angles of triangles. The word "**trigonometry**" comes from the Greek **trigōnon** (triangle) and **metron** (measure). It was first studied by the Babylonians, Greeks, and Egyptians, and used in surveying, navigation, and astronomy. Trigonometry is a powerful tool that allows us to find the measures of angles and sides of triangles, without physically measuring them, and areas of plots of land. We begin our study of trigonometry by studying angles and their degree measures.

T.1 Angles and Degree Measure

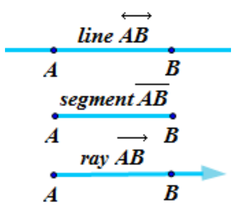


Figure 1a

Two distinct points $\boldsymbol{A}$ and $\boldsymbol{B}$ determine a line denoted $\overleftrightarrow{\boldsymbol{AB}}$. The portion of the line between $\boldsymbol{A}$ and $\boldsymbol{B}$, including the points $\boldsymbol{A}$ and $\boldsymbol{B}$, is called a **line segment** (or simply, a **segment**) $\overline{\boldsymbol{AB}}$. The portion of the line $\overleftrightarrow{\boldsymbol{AB}}$ that starts at $\boldsymbol{A}$ and continues past $\boldsymbol{B}$ is called the **ray** $\overrightarrow{\boldsymbol{AB}}$ (see *Figure 1a*). Point $\boldsymbol{A}$ is the **endpoint** of this ray.

Two rays $\overrightarrow{\boldsymbol{AB}}$ and $\overrightarrow{\boldsymbol{AC}}$ sharing the same endpoint $\boldsymbol{A}$, cut the plane into two separate regions. The union of the two rays and one of those regions is called an **angle**, the common endpoint $\boldsymbol{A}$ is called a **vertex**, and the two **rays** are called **sides** or **arms** of this angle. Customarily, we draw a small arc connecting the two rays to indicate which of the two regions we have in mind.

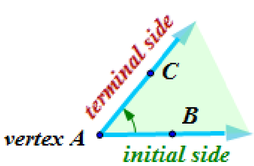


Figure 1b

In trigonometry, an **angle** is often identified with its **measure**, which is the **amount of rotation** that a ray in its initial position (called the **initial side**) needs to turn about the vertex to come to its final position (called the **terminal side**), as in *Figure 1b*. If the rotation from the initial side to the terminal side is *counterclockwise*, the angle is considered to be *positive*. If the rotation is *clockwise*, the angle is *negative* (see *Figure 1c*).

An angle is named either after its vertex, its rays, or the amount of rotation between the two rays. For example, an angle can be denoted $\angle\boldsymbol{A}$, $\angle\boldsymbol{BAC}$, or $\angle\boldsymbol{\theta}$, where the sign $\angle$ (or $\measuredangle$) simply means *an angle*. Notice that in the case of naming an angle with the use of more than one letter, like $\angle\boldsymbol{BAC}$, the middle letter ($\boldsymbol{A}$) is associated with the vertex and the angle is oriented from the ray containing the first point ($\boldsymbol{B}$) to the ray containing the third point ($\boldsymbol{C}$). Customarily, angles (often identified with their measures) are denoted by Greek letters such as $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\theta}$, etc.

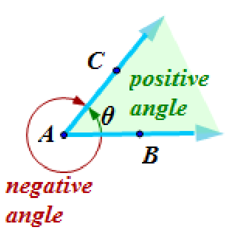


Figure 1c

An angle formed by rotating a ray counterclockwise (in short, **ccw**) exactly one **complete revolution** around its vertex is defined to have a measure of 360 degrees, which is abbreviated as **360°**.

Definition 1.1 ► One **degree (1°)** is the measure of an angle that is $\frac{1}{360}$ part of a complete revolution.
One **minute** $(\boldsymbol{1'})$, is the measure of an angle that is $\frac{1}{60}$ part of a degree.
One **second** $(\boldsymbol{1''})$ is the measure of an angle that is $\frac{1}{60}$ part of a minute.

Therefore $\boldsymbol{1° = 60'}$ and $\boldsymbol{1' = 60''}$.

A fractional part of a degree can be expressed in decimals (e.g. 29.68°) or in minutes and seconds (e.g. 29°40'48"). We say that the first angle is given in **decimal form,** while the second angle is given in **DMS** (**D**egree, **M**inute, **S**econd) **form**.

Example 1 ► **Converting Between Decimal and DMS Form**

Convert as indicated.

a. $29.68°$ to DMS form

b. $46°18'21''$ to decimal degree form

Solution ► **a.** $29.68°$ can be converted to DMS form, using any calculator with DMS or ° ′ ″ key. To do it by hand, separate the fractional part of a degree and use the conversion factor $1° = 60'$.

$$\begin{aligned} \mathbf{29.68}° &= 29° + 0.68° \\ &= 29° + 0.68 \cdot 60' = 29° + 40.8' \end{aligned}$$

Similarly, to convert the fractional part of a minute to seconds, separate it and use the conversion factor $1' = 60''$. So we have

$$29.68° = 29° + 40' + 0.8 \cdot 60'' = \mathbf{29°40'48''}$$

b. Similarly, $46°18'21''$ can be converted to the decimal form, using the DMS or ° ′ ″ key. To do it by hand, we use the conversions $1' = \left(\frac{1}{60}\right)^{\circ}$ and $1'' = \left(\frac{1}{3600}\right)^{\circ}$.

$$\mathbf{46°18'21''} = \left[46 + 18 \cdot \frac{1}{60} + 21 \cdot \frac{1}{3600}\right]^{\circ} \cong \mathbf{46.3058}°$$

Example 2 ► **Adding and Subtracting Angles in DMS Form**

Perform the indicated operations.

a. $36°58'21'' + 5°06'45''$ **b.** $36°17' - 15°46'15''$

Solution ► **a.** First, we add degrees, minutes, and seconds separately. Then, we convert each $60''$ into $1'$ and each $60'$ into $1°$. Finally, we add the degrees, minutes, and seconds again.

$$\begin{aligned} 36°58'21'' + 5°06'45'' &= 41° + 64' + 66'' \\ &= 41° + 1°04' + 1'06'' = \mathbf{42°05'06''} \end{aligned}$$

b. We can subtract within each denomination, degrees, minutes, and seconds, even if the answer is negative. Then, if we need more minutes or seconds to perform the remaining subtraction, we convert $1°$ into $60'$ or $1'$ into $60''$ to finish the calculation.

$$\begin{aligned} 36°17' - 15°46'15'' &= 21° - 29' - 15'' \\ &= 20° + 60' - 29' - 15'' = 20° + 31' - 15'' \\ &= 20° + 30' + 60'' - 15'' = \mathbf{20°30'45''} \end{aligned}$$

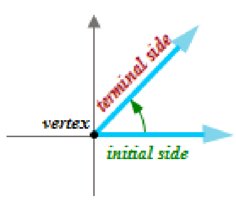


Figure 2

Angles in Standard Position

In trigonometry, we often work with angles in **standard position**, which means angles located in a rectangular system of coordinates with the vertex at the origin and the initial

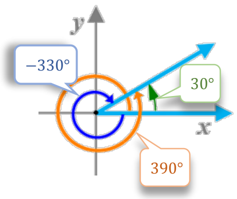


Figure 3

side on the positive x-axis, as in *Figure 2*. With the notion of angle as an amount of rotation of a ray to move from the initial side to the terminal side of an angle, the standard position allows us to represent infinitely many angles with the same terminal side. Those are the angles produced by rotating a ray from the initial side by full revolutions beyond the terminal side, either in a positive or negative direction. Such angles share the same initial and terminal sides and are referred to as **coterminal** angles.

For example, angles $-330°, 30°, 390°, 750°$, and so on, are coterminal.

Definition 1.2 ► Angles $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are **coterminal**, if and only if there is an integer $\boldsymbol{k}$, such that

$$\boldsymbol{\alpha} = \boldsymbol{\beta} + \boldsymbol{k} \cdot \mathbf{360°}$$

Example 3 ► **Finding Coterminal Angles**

Find one positive and one negative angle that is closest to $0°$ and coterminal with

a. $80°$

b. $-530°$

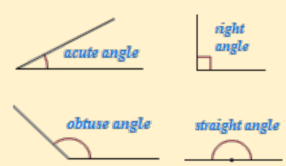
Solution ►

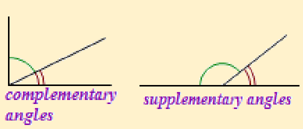
a. To find the closest to $0°$ positive angle coterminal with $80°$ we add one complete revolution, so we have $80° + 360° = \mathbf{440°}$.
Similarly, to find the closest to $0°$ negative angle coterminal with $80°$ we subtract one complete revolution, so we have $80° - 360° = \mathbf{-280°}$.

b. This time, to find the closest to $0°$ positive angle coterminal with $-530°$ we need to add two complete revolutions: $-530° + 2 \cdot 360° = \mathbf{190°}$.
To find the closest to $0°$ negative angle coterminal with $-530°$, it is enough to add one revolution: $-530° + 360° = \mathbf{-170°}$.

Definition 1.3 ► Let $\boldsymbol{\alpha}$ be the measure of an angle. Such an angle is called

acute, if $\boldsymbol{\alpha} \in (\mathbf{0°}, \mathbf{90°})$;
right, if $\boldsymbol{\alpha} = \mathbf{90°}$; (right angle is marked by the symbol ∟)
obtuse, if $\boldsymbol{\alpha} \in (\mathbf{90°}, \mathbf{180°})$; and
straight, if $\boldsymbol{\alpha} = \mathbf{180°}$.





Angles that sum to $\mathbf{90°}$ are called **complementary**.
Angles that sum to $\mathbf{180°}$ are called **supplementary**.

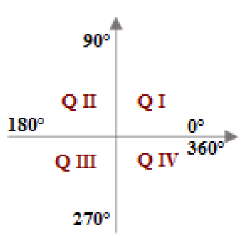


Figure 4

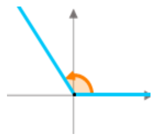
The two axes divide the plane into 4 regions, called **quadrants**. They are numbered counterclockwise, starting with the top right one, as in *Figure 4*.
An angle in standard position is said to lie in the quadrant in which its terminal side lies. For example, an **acute** angle is in *quadrant I* and an **obtuse** angle is in *quadrant II*.
Angles in standard position with their terminal sides along the x-axis or y-axis, such as $\mathbf{0°}, \mathbf{90°}, \mathbf{180°}, \mathbf{270°}$, and so on, are called **quadrantal angles**.

Example 4 ► **Classifying Angles by Quadrants**

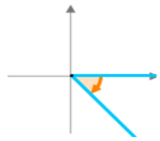
Draw each angle in standard position. Determine the quadrant in which each angle lies or classify the angle as quadrantal.

a. $125°$ **b.** $-50°$ **c.** $270°$ **d.** $210°$

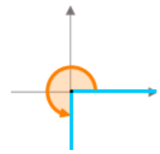
Solution ►

a. 

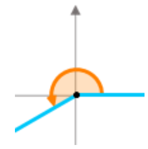
$125°$ is in **QII**

b. 

$-50°$ is in **QIV**

c. 

quadrantal angle

d. 

$210°$ is in **QIII**

Example 5 ► **Finding Complementary and Supplementary Angles**

Find the complement and the supplement of $57°$.

Solution ► Since complementary angles add to $90°$, the complement of $57°$ is $90° - 57° = 33°$. Since supplementary angles add to $180°$, the supplement of $57°$ is $180° - 57° = 123°$.

T.1 Exercises

Vocabulary Check *Complete each blank with the most appropriate term or number from the given list:* ***Complementary, coterminal, quadrantal, standard, 180, 360****.*

1. _______________ angles sum to $90°$. Supplementary angles sum to _____°.

2. The initial side of an angle in ______________ position lines up with the positive part of the x-axis.

3. Angles in standard position that share their terminal sides are called ________________ angles. These angles always differ by multiples of _______°.

4. Angles in standard position with the terminal side on one of the axes are called _______________ angles.

Convert each angle measure to decimal degrees. Round the answer to the nearest thousandth of a degree.

5. $20°04'30''$ **6.** $71°45'$ **7.** $274°18'15''$

8. $34°41'07''$ **9.** $15°10'05''$ **10.** $64°51'35''$

Convert each angle measure to degrees, minutes, and seconds. Round the answer to the nearest second.

11. $18.0125°$ **12.** $89.905°$ **13.** $65.0015°$

14. $184.3608°$ **15.** $175.3994°$ **16.** $102.3771°$

Perform each calculation.

17. $62°18' + 21°41'$ **18.** $71°58' + 47°29'$ **19.** $65°15' - 31°25'$

20. $90° - 51°28'$ **21.** $15°57'45'' + 12°05'18''$ **22.** $90° - 36°18'47''$

Give the complement and the supplement of each angle.

23. $30°$ **24.** $60°$ **25.** $45°$ **26.** $86.5°$ **27.** $15°30'$

28. Give an expression representing the complement of a $\boldsymbol{\theta}°$ angle.

29. Give an expression representing the supplement of a $\boldsymbol{\theta}°$ angle.

Concept check *Sketch each angle in standard position. Draw an arrow representing the correct amount of rotation. Give the quadrant of each angle or identify it as a quadrantal angle.*

30. $75°$ **31.** $135°$ **32.** $-60°$ **33.** $270°$ **34.** $390°$

35. $315°$ **36.** $510°$ **37.** $-120°$ **38.** $240°$ **39.** $-180°$

Find the angle of least positive measure coterminal with each angle.

40. $-30°$ **41.** $375°$ **42.** $-203°$ **43.** $855°$ **44.** $1020°$

Give an expression that generates all angles coterminal with the given angle. Use $\boldsymbol{k}$ *to represent any integer.*

45. $30°$ **46.** $45°$ **47.** $0°$ **48.** $90°$ **49.** $\alpha°$

Analytic Skills *Find the degree measure of the smaller angle formed by the hands of a clock at the following times.*

50.

51. $3:15$ **52.** $1:45$