Trigonometry

Trigonometry is the branch of mathematics that studies the relations between the sides and angles of triangles. The word "**trigonometry**" comes from the Greek **trigōnon** (triangle) and **metron** (measure.) It was first studied by the Babylonians, Greeks, and Egyptians, and used in surveying, navigation, and astronomy. Trigonometry is a powerful tool that allows us to find the measures of angles and sides of triangles, without physically measuring them, and areas of plots of land. We begin our study of trigonometry by studying angles and their degree measures.

T.1 Angles and Degree Measure

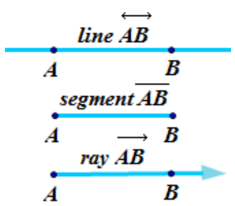


Figure 1a

Two distinct points $\boldsymbol{A}$ and $\boldsymbol{B}$ determine a line denoted $\overleftrightarrow{\boldsymbol{AB}}$. The portion of the line between $\boldsymbol{A}$ and $\boldsymbol{B}$, including the points $\boldsymbol{A}$ and $\boldsymbol{B}$, is called a **line segment** (or simply, a **segment**) $\overline{\boldsymbol{AB}}$. The portion of the line $\overleftrightarrow{\boldsymbol{AB}}$ that starts at $\boldsymbol{A}$ and continues past $\boldsymbol{B}$ is called the **ray** $\overrightarrow{\boldsymbol{AB}}$ (see *Figure 1a*.) Point $\boldsymbol{A}$ is the **endpoint** of this ray.

Two rays $\overrightarrow{\boldsymbol{AB}}$ and $\overrightarrow{\boldsymbol{AC}}$ sharing the same endpoint $\boldsymbol{A}$, cut the plane into two separate regions. The union of the two rays and one of those regions is called an **angle**, the common endpoint $\boldsymbol{A}$ is called a **vertex**, and the two **rays** are called **sides** or **arms** of this angle. Customarily, we draw a small arc connecting the two rays to indicate which of the two regions we have in mind.

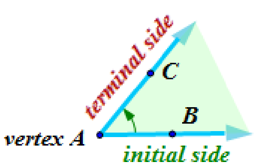


Figure 1b

In trigonometry, an **angle** is often identified with its **measure**, which is the **amount of rotation** that a ray in its initial position (called the **initial side**) needs to turn about the vertex to come to its final position (called the **terminal side**), as in *Figure 1b*. If the rotation from the initial side to the terminal side is *counterclockwise*, the angle is considered to be *positive*. If the rotation is *clockwise*, the angle is *negative* (see *Figure 1c*).

An angle is named either after its vertex, its rays, or the amount of rotation between the two rays. For example, an angle can be denoted $\angle\boldsymbol{A}$, $\angle\boldsymbol{BAC}$, or $\angle\boldsymbol{\theta}$, where the sign $\angle$ (or $\measuredangle$) simply means *an angle*. Notice that in the case of naming an angle with the use of more than one letter, like $\angle\boldsymbol{BAC}$, the middle letter ($\boldsymbol{A}$) is associated with the vertex and the angle is oriented from the ray containing the first point ($\boldsymbol{B}$) to the ray containing the third point ($\boldsymbol{C}$). Customarily, angles (often identified with their measures) are denoted by Greek letters such as $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\theta}$, etc.

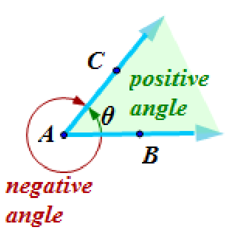


Figure 1c

An angle formed by rotating a ray counterclockwise (in short, **ccw**) exactly one **complete revolution** around its vertex is defined to have a measure of 360 degrees, which is abbreviated as **360°**.

Definition 1.1 ► One **degree (1°)** is the measure of an angle that is $\frac{1}{360}$ part of a complete revolution.
One **minute** $(\mathbf{1'})$, is the measure of an angle that is $\frac{1}{60}$ part of a degree.
One **second** $(\mathbf{1''})$ is the measure of an angle that is $\frac{1}{60}$ part of a minute.

Therefore $\mathbf{1° = 60'}$ and $\mathbf{1' = 60''}$.

A fractional part of a degree can be expressed in decimals (e.g. 29.68°) or in minutes and seconds (e.g. 29°40'48"). We say that the first angle is given in **decimal form,** while the second angle is given in **DMS** (**D**egree, **M**inute, **S**econd) **form**.

Example 1 ► **Converting Between Decimal and DMS Form**

Convert as indicated.

a. $29.68°$ to DMS form

b. $46°18'21''$ to decimal degree form

Solution ► **a.** $29.68°$ can be converted to DMS form, using any calculator with **DMS** or **° ′ ″** key. To do it by hand, separate the fractional part of a degree and use the conversion factor $1° = 60'$.

$$\mathbf{29.68°} = 29° + 0.68°$$
$$= 29° + 0.68 \cdot 60' = 29° + 40.8'$$

Similarly, to convert the fractional part of a minute to seconds, separate it and use the conversion factor $1' = 60''$. So we have

$$29.68° = 29° + 40' + 0.8 \cdot 60'' = \mathbf{29°40'48''}$$

b. Similarly, $46°18'21''$ can be converted to the decimal form, using the **DMS** or **° ′ ″** key. To do it by hand, we use the conversions $1' = \left(\frac{1}{60}\right)^{°}$ and $1'' = \left(\frac{1}{3600}\right)^{°}$.

$$\mathbf{46°18'21''} = \left[46 + 18 \cdot \frac{1}{60} + 21 \cdot \frac{1}{3600}\right]^{°} \cong \mathbf{46.3058°}$$

Example 2 ► **Adding and Subtracting Angles in DMS Form**

Perform the indicated operations.

a. $36°58'21'' + 5°06'45''$ **b.** $36°17' - 15°46'15''$

Solution ► **a.** First, we add degrees, minutes, and seconds separately. Then, we convert each $60''$ into $1'$ and each $60'$ into $1°$. Finally, we add the degrees, minutes, and seconds again.

$$36°58'21'' + 5°06'45'' = 41° + 64' + 66''$$
$$= 41° + 1°04' + 1'06'' = \mathbf{42°05'06''}$$

b. We can subtract within each denomination, degrees, minutes, and seconds, even if the answer is negative. Then, if we need more minutes or seconds to perform the remaining subtraction, we convert $1°$ into $60'$ or $1'$ into $60''$ to finish the calculation.

$$36°17' - 15°46'15'' = 21° - 29' - 15''$$
$$= 20° + 60' - 29' - 15'' = 20° + 31' - 15''$$
$$= 20° + 30' + 60'' - 15'' = \mathbf{20°30'45''}$$

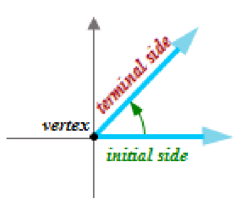


Figure 2

Angles in Standard Position

In trigonometry, we often work with angles in **standard position**, which means angles located in a rectangular system of coordinates with the vertex at the origin and the initial

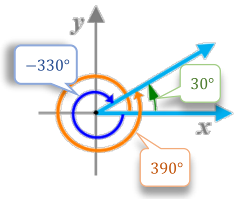


Figure 3

side on the positive x-axis, as in *Figure 2*. With the notion of angle as an amount of rotation of a ray to move from the initial side to the terminal side of an angle, the standard position allows us to represent infinitely many angles with the same terminal side. Those are the angles produced by rotating a ray from the initial side by full revolutions beyond the terminal side, either in a positive or negative direction. Such angles share the same initial and terminal sides and are referred to as **coterminal** angles.

For example, angles $-330°, 30°, 390°, 750°$, and so on, are coterminal.

Definition 1.2 ► Angles $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are **coterminal**, if and only if there is an integer $\boldsymbol{k}$, such that

$$\boldsymbol{\alpha = \beta + k \cdot 360°}$$

Example 3 ► **Finding Coterminal Angles**

Find one positive and one negative angle that is closest to $0°$ and coterminal with

a. $80°$

b. $-530°$

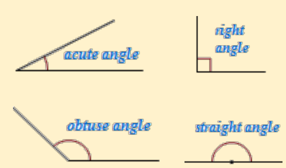
Solution ►

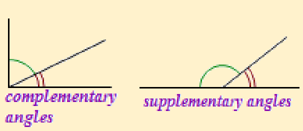
a. To find the closest to $0°$ positive angle coterminal with $80°$ we add one complete revolution, so we have $80° + 360° = \mathbf{440°}$.
Similarly, to find the closest to $0°$ negative angle coterminal with $80°$ we subtract one complete revolution, so we have $80° - 360° = \mathbf{-280°}$.

b. This time, to find the closest to $0°$ positive angle coterminal with $-530°$ we need to add two complete revolutions: $-530° + 2 \cdot 360° = \mathbf{190°}$.
To find the closest to $0°$ negative angle coterminal with $-530°$, it is enough to add one revolution: $-530° + 360° = \mathbf{-170°}$.

Definition 1.3 ► Let $\boldsymbol{\alpha}$ be the measure of an angle. Such an angle is called

acute, if $\boldsymbol{\alpha} \in (\mathbf{0°, 90°})$;
right, if $\boldsymbol{\alpha} = \mathbf{90°}$; (right angle is marked by the symbol ∟)
obtuse, if $\boldsymbol{\alpha} \in (\mathbf{90°, 180°})$; and
straight, if $\boldsymbol{\alpha} = \mathbf{180°}$.





Angles that sum to $\mathbf{90°}$ are called **complementary**.
Angles that sum to $\mathbf{180°}$ are called **supplementary**.

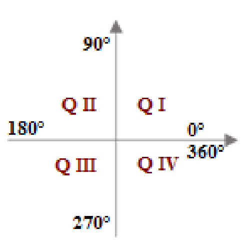


Figure 4

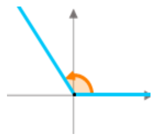
The two axes divide the plane into 4 regions, called **quadrants**. They are numbered counterclockwise, starting with the top right one, as in *Figure 4*.
An angle in standard position is said to lie in the quadrant in which its terminal side lies.
For example, an **acute** angle is in *quadrant I* and an **obtuse** angle is in *quadrant II*.
Angles in standard position with their terminal sides along the x-axis or y-axis, such as $\mathbf{0°, 90°, 180°, 270°}$, and so on, are called **quadrantal angles**.

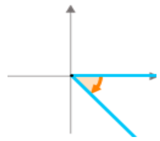
Example 4 ► **Classifying Angles by Quadrants**

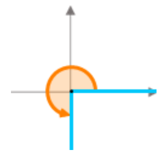
Draw each angle in standard position. Determine the quadrant in which each angle lies or classify the angle as quadrantal.

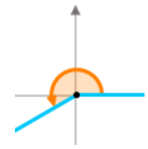
a. $125°$ **b.** $-50°$ **c.** $270°$ **d.** $210°$

Solution ►

a.

$125°$ is in **QII**

b.

$-50°$ is in **QIV**

c.

quadrantal angle

d.

$210°$ is in **QIII**

Example 5 ► **Finding Complementary and Supplementary Angles**

Find the complement and the supplement of $57°$.

Solution ► Since complementary angles add to $90°$, the complement of $57°$ is $90° - 57° = 33°$.
Since supplementary angles add to $180°$, the supplement of $57°$ is $180° - 57° = 123°$.

T.1 Exercises

Vocabulary Check *Complete each blank with the most appropriate term or number from the given list:* ***Complementary, coterminal, quadrantal, standard, 180, 360***.

1. _______________ angles sum to $90°$. Supplementary angles sum to _____°.

2. The initial side of an angle in ______________ position lines up with the positive part of the x-axis.

3. Angles in standard position that share their terminal sides are called ________________ angles. These angles always differ by multiples of _______°.

4. Angles in standard position with the terminal side on one of the axes are called _______________ angles.

Convert each angle measure to decimal degrees. Round the answer to the nearest thousandth of a degree.

5. $20°04'30''$ **6.** $71°45'$ **7.** $274°18'15''$

8. $34°41'07''$ **9.** $15°10'05''$ **10.** $64°51'35''$

Convert each angle measure to degrees, minutes, and seconds. Round the answer to the nearest second.

11. $18.0125°$ **12.** $89.905°$ **13.** $65.0015°$

14. $184.3608°$ **15.** $175.3994°$ **16.** $102.3771°$

Perform each calculation.

17. $62°18' + 21°41'$ **18.** $71°58' + 47°29'$ **19.** $65°15' - 31°25'$

20. $90° - 51°28'$ **21.** $15°57'45'' + 12°05'18''$ **22.** $90° - 36°18'47''$

Give the complement and the supplement of each angle.

23. $30°$ **24.** $60°$ **25.** $45°$ **26.** $86.5°$ **27.** $15°30'$

28. Give an expression representing the complement of a $\boldsymbol{\theta}°$ angle.

29. Give an expression representing the supplement of a $\boldsymbol{\theta}°$ angle.

Concept check *Sketch each angle in standard position. Draw an arrow representing the correct amount of rotation. Give the quadrant of each angle or identify it as a quadrantal angle.*

30. $75°$ **31.** $135°$ **32.** $-60°$ **33.** $270°$ **34.** $390°$

35. $315°$ **36.** $510°$ **37.** $-120°$ **38.** $240°$ **39.** $-180°$

Find the angle of least positive measure coterminal with each angle.

40. $-30°$ **41.** $375°$ **42.** $-203°$ **43.** $855°$ **44.** $1020°$

Give an expression that generates all angles coterminal with the given angle. Use $\boldsymbol{k}$ *to represent any integer.*

45. $30°$ **46.** $45°$ **47.** $0°$ **48.** $90°$ **49.** $\alpha°$

Analytic Skills *Find the degree measure of the smaller angle formed by the hands of a clock at the following times.*

50. 

51. $3:15$ **52.** $1:45$

T.2 Trigonometric Ratios of an Acute Angle and of Any Angle

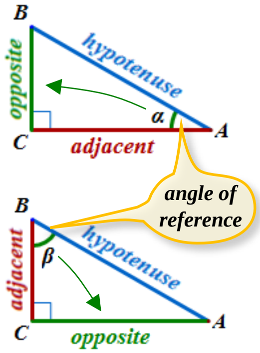


Figure 2.1

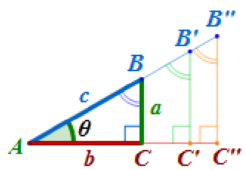


Figure 2.2

Generally, trigonometry studies ratios between sides in right angle triangles. When working with right triangles, it is convenient to refer to the side **opposite** to an angle, the side **adjacent** to (next to) an angle, and the **hypotenuse**, which is the longest side, opposite to the right angle. Notice that the opposite and adjacent sides depend on the **angle of reference** (one of the two acute angles.) However, the hypotenuse stays the same, regardless of the choice of the angle or reference. See *Figure 2.1*.

Notice that any two right triangles with the same acute angle $\boldsymbol{\theta}$ are **similar**. See *Figure 2.2.* **Similar** means that their corresponding angles are **congruent** and their corresponding sides are **proportional**. For instance, assuming notation as on *Figure 2.2*, we have

$$\frac{AB}{AB'} = \frac{AC}{AC'} = \frac{BC}{B'C'},$$

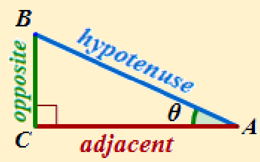
or equivalently

$$\frac{BC}{AB} = \frac{B'C'}{AB'}, \qquad \frac{AC}{AB} = \frac{AC'}{AB'}, \qquad \frac{BC}{AC} = \frac{B'C'}{AC'}.$$

Therefore, the ratios of any two sides of a right triangle does not depend on the size of the triangle but only on the size of the angle of reference. See the following demonstration. This means that we can study those **ratios** of sides as **functions** of an acute angle.

Trigonometric Functions of Acute Angles

Definition 2.1 ►



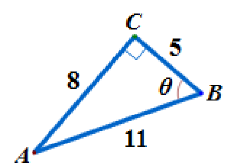
Given a **right angle triangle** with an **acute** angle $\boldsymbol{\theta}$, the three **primary trigonometric ratios** of the angle $\boldsymbol{\theta}$, called ***sine***, ***cosine***, and ***tangent*** (abbreviation: *sin*, *cos*, *tan*) are defined as follows:

$$\sin\theta = \frac{Opposite}{Hypotenuse}, \qquad \cos\theta = \frac{Adjacent}{Hypotenuse}, \qquad \tan\theta = \frac{Opposite}{Adjacent}$$

For easier memorization, we can use the acronym **SOH – CAH – TOA** (read: *so - ka - toe - ah*), formed from the first letter of the function and the corresponding ratio.

Example 1 ► **Identifying Sides of a Right Triangle to Form Trigonometric Ratios**

Identify the hypotenuse, opposite, and adjacent side of angle θ and state values of the three trigonometric ratios.

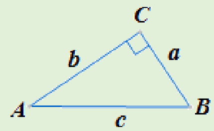


Solution ►

Side AB is the hypotenuse, as it lies across the right angle.
Side BC is the adjacent, as it is part of the angle θ, other than hypotenuse.
Side AC is the opposite, as it lies across angle θ.
Therefore, $\sin\theta = \frac{opp.}{hyp.} = \frac{8}{11}$, $\cos\theta = \frac{adj.}{hyp.} = \frac{5}{11}$, and $\tan\theta = \frac{opp.}{adj.} = \frac{8}{5}$.

The three **primary trigonometric ratios** together with the **Pythagorean Theorem** allow us to **solve** any right angle triangle. That means that given the measurements of two sides, or one side and one angle, with a little help of algebra, we can find the measurements of all remaining sides and angles of any right triangle. See *section T.4*.

Pythagorean Theorem ► A **triangle** $\boldsymbol{ABC}$ **is right** with $\angle C = 90°$ if and only if $\boldsymbol{a^2 + b^2 = c^2}$.

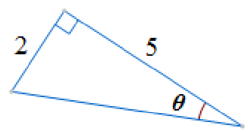


Convention: *The* ***side opposite*** *the given* ***vertex*** *(or angle) is named after the vertex, except that by a small rather than a capital letter. For example, the side opposite vertex* ***A*** *is called* $\boldsymbol{a}$.

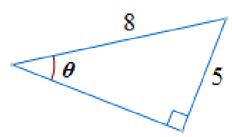
Example 2 ► **Finding Values of Trigonometric Ratios With the Aid of Pythagorean Theorem**

Given the triangle, find the exact values of the sine, cosine, and tangent ratios for angle θ.

a.



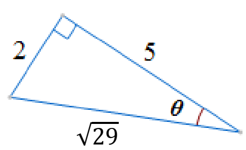
b.



Solution ► **a.** Let $\boldsymbol{h}$ denote the hypotenuse. By **Pythagorean Theorem**, we have

$$h^2 = 2^2 + 5^2$$
$$h = \sqrt{4 + 25} = \sqrt{29}$$

Now, we are ready to state the exact values of the three trigonometric ratios:



$$\sin\theta = \frac{2}{\sqrt{29}} \cdot \frac{\sqrt{29}}{\sqrt{29}} = \boldsymbol{\frac{2\sqrt{29}}{29}}$$

$$\cos\theta = \frac{5}{\sqrt{29}} \cdot \frac{\sqrt{29}}{\sqrt{29}} = \boldsymbol{\frac{5\sqrt{29}}{29}}$$

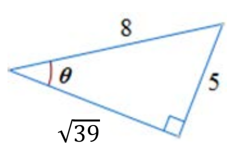
Note:
It is customary to rationalize the denominator.

$$\tan\theta = \boldsymbol{\frac{2}{5}}$$

b. Let $\boldsymbol{a}$ denote the adjacent side. By the **Pythagorean Theorem**, we have

$$a^2 + 5^2 = 8^2$$
$$a = \sqrt{8^2 - 5^2} = \sqrt{64 - 25} = \sqrt{39}$$

Now, we are ready to state the exact values of the three trigonometric ratios:



$$\sin\theta = \boldsymbol{\frac{5}{8}}$$

$$\cos\theta = \boldsymbol{\frac{\sqrt{39}}{8}}$$

$$\tan\theta = \frac{5}{\sqrt{39}} \cdot \frac{\sqrt{39}}{\sqrt{39}} = \boldsymbol{\frac{5\sqrt{39}}{39}}$$

Trigonometric Functions of Any Angle

Notice that any angle of a right triangle, other than the right angle, is acute. Thus, the "SOH – CAH – TOA" definition of the trigonometric ratios refers to acute angles only. However, we can extend this definition to include all angles. This can be done by observing our right triangle within the Cartesian Coordinate System.

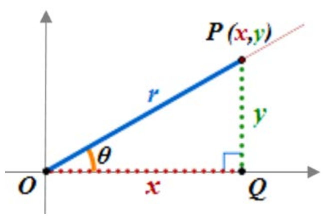


Figure 2.3

Let triangle OPQ with $\angle Q = 90°$ be placed in the coordinate system so that O coincides with the origin, Q lies on the positive part of the x-axis, and P lies in the first quadrant. See *Figure 2.3*. Let (x, y) be the coordinates of the point P, and let θ be the measurement of $\angle QOP$. This way, angle θ is in standard position and the triangle OPQ is obtained by **projecting** point P perpendicularly onto the x-axis. Thus in this setting, the position of point P actually determines both the angle θ and the ΔOPQ. Observe that the coordinates of point P ($\boldsymbol{x}$ and $\boldsymbol{y}$) really represent the length of the **adjacent** and the **opposite** side, correspondingly. Since the length of the **hypotenuse** represents the distance of the point P from the origin, it is often denoted by $\boldsymbol{r}$ (from *radius*.)

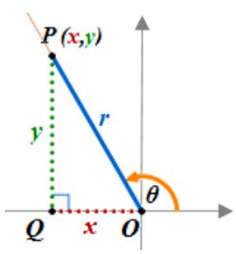


Figure 2.4

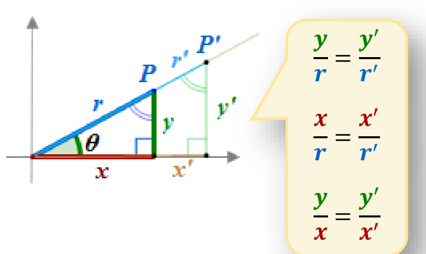
By rotating the radius $\boldsymbol{r}$ and projecting the point P perpendicularly onto x-axis (follow the green dotted line from P to Q in *Figure 2.4*), we can obtain a right triangle corresponding to any angle θ, not only an acute angle. Since the coordinates of a point in a plane can be negative, to establish a correcpondence between the coordinates x and y of the point P, and the distances OQ and QP, it is convenient to think of **directed distances** rather than just distances. Distance becomes directed if we assign a sign to it. So, lets assign a positive sign to horizontal or vertical distances that follow the directions of the corresponding number lines, and a negative sign otherwise. For example, the directed distance $OQ = x$ in *Figure 2.3* is positive because the direction from O to Q follows the order on the x-axis while the directed distance $OQ = x$ in *Figure 2.4* is negative because the direction from O to Q is against the order on the x-axis. Likewise, the directed distance $QP = y$ is positive for angles in the first and second quadrant (as in *Figure 2.3* and *2.4*), and it is negative for angles in the third and fourth quadrant (convince yourself by drawing a diagram).

Definition 2.2 ► Let $P(\boldsymbol{x}, \boldsymbol{y})$ be any point, different than the origin, on the terminal side of an angle $\boldsymbol{\theta}$ in standard position. Also, let $\boldsymbol{r} = \sqrt{\boldsymbol{x}^2 + \boldsymbol{y}^2}$ be the distance of the point P from the origin. We define

$$\sin\boldsymbol{\theta} = \frac{\boldsymbol{y}}{\boldsymbol{r}}, \quad \cos\boldsymbol{\theta} = \frac{\boldsymbol{x}}{\boldsymbol{r}}, \quad \tan\boldsymbol{\theta} = \frac{\boldsymbol{y}}{\boldsymbol{x}} \text{ (for } x \neq 0)$$

Observations:

- For acute angles, definition 2.2 agrees with the "**SOH** – **CAH** – **TOA**" definition 2.1.



- Proportionality of similar triangles guarantees that each point of the same terminal ray defines the same trigonometric ratio. This means that the above definition assigns a unique value to each trigonometric ratio for any given angle regardless of the point chosen on the terminal side of this angle. Thus, the above trigonometric ratios are in fact **functions of any real angle** and these functions are properly defined in terms of x, y, and r.

- Since $r > 0$, the first two trigonometric functions, **sine** $\left(\frac{y}{r}\right)$ and **cosine** $\left(\frac{x}{r}\right)$, are defined for any real angle θ.

- The third trigonometric function, **tangent** $\left(\frac{y}{x}\right)$, is defined for all real angles θ except for angles with terminal sides on the y-axis. This is because the x-coordinate of any point on the y-axis equals zero, which cannot be used to create the ratio $\frac{y}{x}$. Thus, tangent is a function of all real angles, except for $90°, 270°$, and so on (generally, except for angles of the form $\mathbf{90° + k \cdot 180°}$, where $\boldsymbol{k}$ is an integer.)

- Notice that after dividing both sides of the Pythagorean equation $\boldsymbol{x^2 + y^2 = r^2}$ by $\boldsymbol{r^2}$, we have

$$\left(\frac{\boldsymbol{x}}{\boldsymbol{r}}\right)^{\mathbf{2}} + \left(\frac{\boldsymbol{y}}{\boldsymbol{r}}\right)^{\mathbf{2}} = \mathbf{1}.$$

Since $\frac{x}{r} = \cos\theta$ and $\frac{y}{r} = \sin\theta$, we obtain the following **Pythagorean Identity**:

$$\boldsymbol{\sin^2\theta + \cos^2\theta = 1}$$

- Also, observe that as long as $x \neq 0$, the quatient of the first two ratios gives us the third ratio:

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{\boldsymbol{y}}{\boldsymbol{r}}}{\frac{\boldsymbol{x}}{\boldsymbol{r}}} = \frac{\boldsymbol{y}}{\cancel{\boldsymbol{r}}} \cdot \frac{\cancel{\boldsymbol{r}}}{\boldsymbol{x}} = \frac{\boldsymbol{y}}{\boldsymbol{x}} = \tan\theta.$$

Thus, we have the identity

$$\boldsymbol{\tan\theta = \frac{\sin\theta}{\cos\theta}}$$

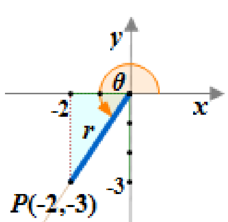
for all angles θ in the domain of the tangent.

Example 3 ► **Evaluating Trigonometric Functions of any Angle in Standard Position**

Find the exact value of the three primary trigonometric functions of an angle θ in standard position whose terminal side contains the point

a. $P(-2,-3)$ **b.** $P(0,1)$

Solution ►



a. To ilustrate the situation, lets sketch the least positive angle θ in standard position with the point $P(-2,-3)$ on its terminal side.

To find values of the three trigonometric functions, first, we will determine the length of r:

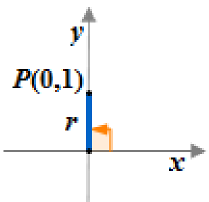
$$r = \sqrt{(-2)^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}$$

Now, we can state the exact values of the three trigonometric functions:

$$\sin\theta = \frac{y}{r} = \frac{-3}{\sqrt{13}} = \mathbf{\frac{-3\sqrt{13}}{13}}$$

$$\cos\theta = \frac{x}{r} = \frac{-2}{\sqrt{13}} = \mathbf{\frac{-2\sqrt{13}}{13}}$$

$$\tan\theta = \frac{y}{x} = \frac{-3}{-2} = \mathbf{\frac{3}{2}}$$



b. Since $x = 0,\ y = 1,\ r = \sqrt{0^2 + 1^2} = 1,$ then

$$\sin\theta = \frac{y}{r} = \frac{1}{1} = \mathbf{1}$$

$$\cos\theta = \frac{x}{r} = \frac{0}{1} = \mathbf{0}$$

$$\tan\theta = \frac{y}{x} = \frac{1}{0} = \textbf{undefined}$$

we can't divide by zero!

Notice that the measure of the least positive angle θ in standard position with the point $P(0,1)$ on its terminal side is 90°. Therefore, we have

$$\sin 90° = 1,\quad \cos 90° = 0,\quad \tan 90° = \text{undefined}$$

The values of trigonometric functions of other commonly used quadrantal angles, such as $0°, 180°, 270°$, and $360°$, can be found similary as in *Example 3b*. These values are summarized in the table below.

Table 2.1 Function Values of Quadrantal Angles

function \ $\theta =$	$0°$	$90°$	$180°$	$270°$	$360°$
$\sin\theta$	0	1	0	-1	0
$\cos\theta$	1	0	-1	0	1
$\tan\theta$	0	undefined	0	undefined	0

Example 4 ► **Evaluating Trigonometric Functions Using Basic Identities**

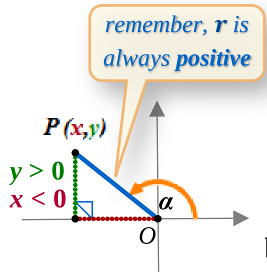
Knowing that $\cos\alpha = -\frac{3}{4}$ and the angle α is in quadrant II, find

a. $\sin\alpha$ **b.** $\tan\alpha$

Solution ► **a.** To find the value of $\sin\alpha$, we can use the Pythagorean Identity $\sin^2\alpha + \cos^2\alpha = 1$. After substituting $\cos\alpha = -\frac{3}{4}$, we have

$$\sin^2\alpha + \left(-\frac{3}{4}\right)^2 = 1$$

$$\sin^2\alpha = 1 - \frac{9}{16} = \frac{7}{16}$$



$$\sin \alpha = \pm\sqrt{\frac{7}{16}} = \pm\frac{\sqrt{7}}{4}$$

Since α is in in the second quadrant, $\sin\theta = \frac{y}{r}$ must be positive (as $y > 0$ in QII), so

$$\sin \alpha = \frac{\sqrt{\mathbf{7}}}{\mathbf{4}}.$$

b. To find the value of $\tan\alpha$, since we already know the value of $\sin\alpha$, we can use the identity $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$. After substituting values $\sin\alpha = \frac{\sqrt{7}}{4}$ and $\cos\alpha = -\frac{3}{4}$, we obtain

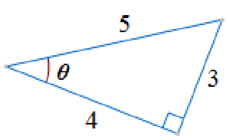
$$\tan\alpha = \frac{\frac{\sqrt{7}}{4}}{-\frac{3}{4}} = \frac{\sqrt{7}}{4}\cdot\left(-\frac{4}{3}\right) = -\frac{\sqrt{\mathbf{7}}}{\mathbf{3}}.$$

To confirm that the sign of $\tan\alpha = \frac{y}{x}$ in the second quadrant is indeed negative, observe that $y > 0$ and $x < 0$ in QII.

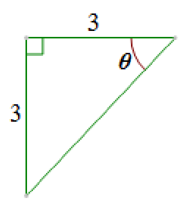
T.2 Exercises

Concept Check *Find the exact values of the three trigonometric functions for the indicated angle θ. Rationalize denominators when applicable.*

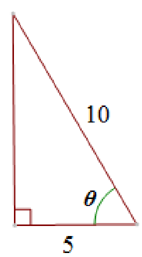
1.



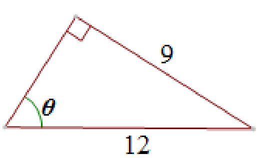
2.



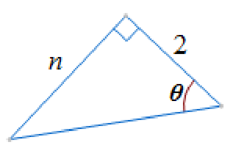
3.



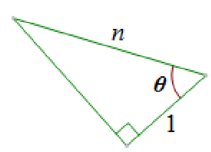
4.



5.



6.



Concept Check *Sketch an angle θ in standard position such that θ has the least positive measure, and the given point is on the terminal side of θ. Then find the values of the three trigonometric functions for each angle. Rationalize denominators when applicable.*

7. $(-3,4)$ **8.** $(-4,-3)$ **9.** $(5,-12)$ **10.** $(0,3)$ **11.** $(-4,0)$

12. $(1,\sqrt{3})$ **13.** $(3,5)$ **14.** $(0,-8)$ **15.** $(-2\sqrt{3},-2)$ **16.** $(5,0)$

17. If the terminal side of an angle θ is in quadrant III, what is the sign of each of the trigonometric function values of θ?

Suppose that the point (x, y) *is in the indicated quadrant. Decide whether the given ratio is* ***positive*** *or* ***negative****.*

18. $QI,\ \frac{y}{x}$ **19.** $QII,\ \frac{y}{x}$ **20.** $QII,\ \frac{y}{r}$ **21.** $QIII,\ \frac{x}{r}$ **22.** $QIV,\ \frac{y}{x}$

23. $QIII,\ \frac{y}{x}$ **24.** $QIV,\ \frac{y}{r}$ **25.** $QI,\ \frac{y}{r}$ **26.** $QIV,\ \frac{x}{r}$ **27.** $QII,\ \frac{x}{r}$

Concept Check *Use the definition of trigonometric functions in terms of x, y, and r to determine each value. If it is undefined, say so.*

28. $\sin 90°$ **29.** $\cos 0°$ **30.** $\tan 180°$ **31.** $\cos 180°$ **32.** $\tan 270°$

33. $\cos 270°$ **34.** $\sin 270°$ **35.** $\cos 90°$ **36.** $\sin 0°$ **37.** $\tan 90°$

Analytic Skills *Use basic identities to determine values of the remaining two trigonometric functions of the angle satisfying given conditions. Rationalize denominators when applicable.*

39. $\sin\alpha = \frac{\sqrt{2}}{4};\ \alpha \in QII$ **40.** $\sin\beta = -\frac{2}{3};\ \beta \in QIII$ **41.** $\cos\theta = \frac{2}{5};\ \theta \in QIV$

T.3 Evaluation of Trigonometric Functions

In the previous section, we defined sine, cosine, and tangent as functions of real angles. In this section, we will take interest in finding values of these functions for angles $\theta \in [0°,\ 360°)$. As shown before, one can find exact values of trigonometric functions of an angle θ with the aid of a right triangle with the acute angle θ and given side lengths, or by using coordinates of a given point on the terminal side of the angle θ in standard position. What if such data is not given? Then, one could consider approximating trigonometric function values by measuring sides of a right triangle with the desired angle θ and calculating corresponding ratios. However, this could easily prove to be a cumbersome process, with inaccurate results. Luckily, we can rely on calculators, which are programmed to return approximated values of the three primary trigonometric functions for any angle.

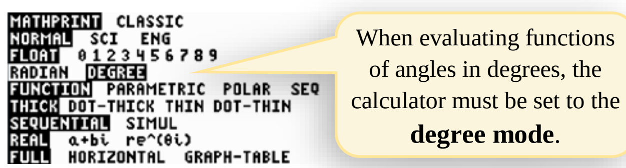
Attention: *In this section, any calculator instruction will refer to graphing calculators such as* ***TI-83*** *or* ***TI-84***.

Example 1 ► **Evaluating Trigonometric Functions Using a Calculator**

Find each function value up to four decimal places.

a. $\sin 39°12'10''$ **b.** $\tan 102.6°$

Solution ► **a.** Before entering the expression into the calculator, we need to check if the calculator is in degree mode by pressing the MODE key and highlighting the "degree" option, with the aid of arrows and the ENTER key.



To go back to the main screen, we press 2nd MODE, and now we can enter $\sin 39°12'10''$. The degree (°) and minute (′) signs can be found under the "ANGLE" list. To get the second (″) sign, we press ALPHA + .

Thus $\sin 39°12'10'' \approx \mathbf{0.6321}$ when rounded to four decimal places.

b. When evaluating trigonometric functions of angles in decimal degrees, it is not necessary to write the degree (°) sign when in degree mode. We simply key in TAN 102.6 ENTER to obtain $\tan 102.6° \approx \mathbf{-4.4737}$ when rounded to four decimal places.

Special Angles

It has already been discussed how to find the **exact values** of trigonometric functions of **quadrantal angles** using the definitions in terms of x, y, and r. *See section T.2, Example 3.b, and table 2.1.*

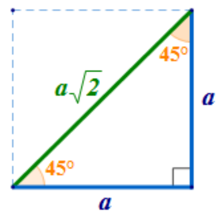


Figure 3.1

Are there any other angles for which the trigonometric functions can be evaluated exactly? Yes, we can find the exact values of trigonometric functions of any angle that can be modelled by a right triangle with known sides. For example, angles such as 30°, 45°, or 60° can be modeled by half of a square or half of an equilateral triangle. In each triangle, the relations between the lengths of sides are easy to establish.

In the case of half a square *(see Figure 3.1)*, we obtain a right triangle with two acute angles of **45°**, and two equal sides of certain length $\boldsymbol{a}$.
Hence, by The Pythagorean Theorem, the diagonal $d = \sqrt{a^2 + a^2} = \sqrt{2a^2} = \boldsymbol{a\sqrt{2}}$.

Summary: The sides of any $\mathbf{45° - 45° - 90°}$ triangle are in the relation $\boldsymbol{a - a - a\sqrt{2}}$.

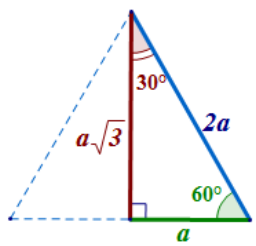


Figure 3.2

By dividing an equilateral triangle *(see Figure 3.1)* along its height, we obtain a right triangle with acute angles of **30°** and **60°**. If the length of the side of the original triangle is denoted by $\boldsymbol{2a}$, then the length of half a side is $\boldsymbol{a}$, and the length of the height can be calculated by applying The Pythagorean Theorem, $h = \sqrt{(2a)^2 - a^2} = \sqrt{3a^2} = \boldsymbol{a\sqrt{3}}$.

Summary: The sides of any $\mathbf{30° - 60° - 90°}$ triangle are in the relation $\boldsymbol{a - 2a - a\sqrt{3}}$.

Since the trigonometric ratios do not depend on the size of a triangle, for simplicity, we can assume that $a = 1$ and work with the following **special triangles**:

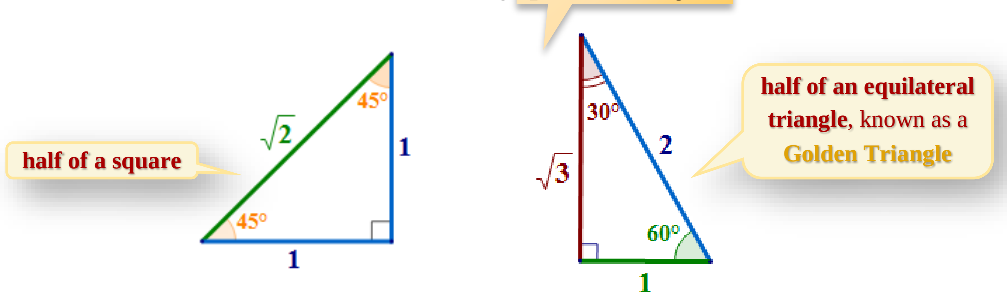


Figure 3.3

Special angles such as **30°, 45°,** and **60°** are frequently seen in applications. We will often refer to the exact values of trigonometric functions of these angles. Special triangles give us a tool for finding those values.

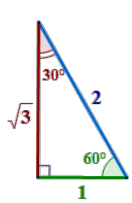
Advice: *Make sure that you can **recreate the special triangles** by taking half of a square or half of an equilateral triangle, anytime you wish to **recall the relations** between their sides.*

Example 2 ► **Finding Exact Values of Trigonometric Functions of Special Angles**

Find the <u>*exact*</u> value of each expression.

a. $\cos 60°$ **b.** $\tan 30°$ **c.** $\sin 45°$ **d.** $\tan 45°$

Solution ►

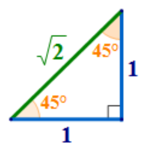


a. Refer to the $30° - 60° - 90°$ triangle and follow the SOH-CAH-TOA definition of sine:

$$\cos 60° = \frac{adj.}{hyp.} = \frac{1}{2}$$

b. Refer to the same triangle as above:

$$\tan 30° = \frac{opp.}{adj.} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$



c. Refer to the $45° - 45° - 90°$ triangle:

$$\sin 45° = \frac{opp.}{hyp.} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

d. Refer to the $45° - 45° - 90°$ triangle:

$$\tan 45° = \frac{opp.}{adj.} = \frac{1}{1} = 1$$

The exact values of trigonometric functions of special angles are summarized in the table below.

Table 3.1 Function Values of Special Angles

function \ $\theta =$	$30°$	$45°$	$60°$
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Observations:

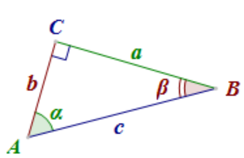
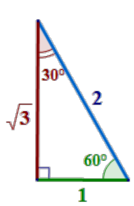


Figure 3.4

- Notice that $\sin 30° = \cos 60°$, $\sin 60° = \cos 30°$, and $\sin 45° = \cos 45°$. Is there any general rule to explain this fact? Lets look at a right triangle with acute angles $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ (*see Figure 3.4*). Since the sum of angles in any triangle is $180°$ and $\angle C = 90°$, then $\boldsymbol{\alpha} + \boldsymbol{\beta} = 90°$, therefore they are **complementary angles**. From the definition, we have $\sin\boldsymbol{\alpha} = \frac{\boldsymbol{a}}{\boldsymbol{b}} = \cos\boldsymbol{\beta}$. Since angle $\boldsymbol{\alpha}$ was chosen arbitrarily, this rule applies to any pair of acute complementary angles. It happens that this rule actually applies to all complementary angles. So we have the following claim:

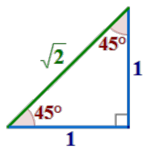
$\sin\alpha = \cos(90° - \alpha)$

The **cofunctions** (like sine and cosine) of **complementary** angles are equal.



- Notice that $\tan 30° = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}} = \frac{1}{\tan 60°}$, or equivalently, $\tan 30° \cdot \tan 60° = 1$. This is because of the previously observed rules:

$$\tan\theta \cdot \tan(90° - \theta) = \frac{\sin\theta}{\cos\theta} \cdot \frac{\sin(90° - \theta)}{\cos(90° - \theta)} = \frac{\sin\theta}{\cos\theta} \cdot \frac{\cos\theta}{\sin\theta} = 1$$



In general, we have:

$$\tan\theta = \frac{1}{\tan(90° - \theta)}$$

- Observe that $\tan 45° = 1$. This is an easy, but very useful value to memorize.

Example 3 ► **Using the Cofunction Relationship**

Rewrite $\cos 75°$ in terms of the cofunction of the complementary angle.

Solution ► Since the complement of $75°$ is $90° - 75° = 15°$, then $\cos 75° = \mathbf{\sin 15°}$.

Reference Angles

Can we determine exact values of trigonometric functions of nonquadrantal angles that are larger than 90°?

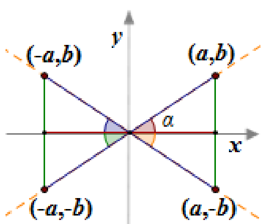
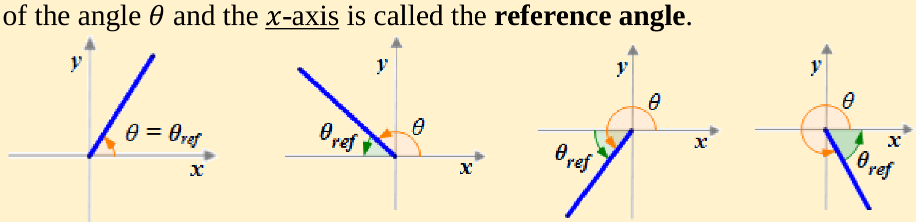


Figure 3.5

Assume that point (a, b) lies on the terminal side of acute angle α. By definition 2.2, the values of trigonometric functions of angles with terminals containing points $(-a, b), (-a, -b)$, and $(a, -b)$ are the same as the values of corresponding functions of the angle α, except for their signs.

Therefore, to find the value of a trigonometric function of any angle θ, it is enough to evaluate this function at the corresponding acute angle θ_{ref}, called the **reference angle**, and apply the sign appropriate to the quadrant of the terminal side of θ.

Definition 3.1 ► Let θ be an angle in standard position. The acute angle $\boldsymbol{\theta_{ref}}$ formed by the <u>terminal side</u> of the angle θ and the <u>x-axis</u> is called the **reference angle**.



Attention: *Think of a **reference angle** as the smallest rotation of the terminal arm required to line it up with the **x-axis**.*

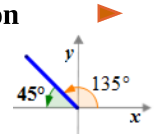
Example 4 ► **Finding the Reference Angle**

Find the **reference angle** for each of the given angles.

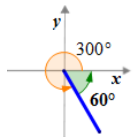
a. 40° **b.** 135° **c.** 210° **d.** 300°

Solution ►

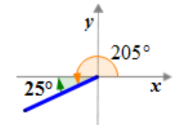
a. Since $\mathbf{40°} \in QI$, this is already the reference angle.



b. Since $135° \in QII$, the reference angle equals $180° - 135° = \mathbf{45°}$.



c. Since $205° \in QIII$, the reference angle equals $205° - 180° = \mathbf{25°}$.

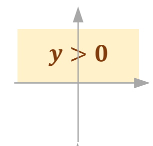


d. Since $300° \in QIV$, the reference angle equals $360° - 300° = \mathbf{60°}$.

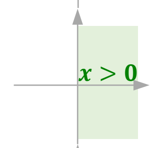
CAST Rule

Using the x, y, r definition of trigonometric functions, we can determine and summarize the signs of those functions in each of the quadrants.

Since $\sin\theta = \frac{y}{r}$ and r is positive, then the sign of the sine ratio is the same as the sign of the y-value. This means that the values of sine are positive only in quadrants where y is positive, thus in QI and QII.



Since $\cos\theta = \frac{x}{r}$ and r is positive, then the sign of the cosine ratio is the same as the sign of the x-value. This means that the values of cosine are positive only in quadrants where x is positive, thus in QI and QIV.



Since $\tan\theta = \frac{y}{x}$, then the values of the tangent ratio are positive only in quadrants where both x and y have the same signs, thus in QI and $QIII$.

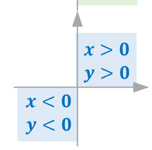


Table 3.2 Signs of Trigonometric Functions in Quadrants

function \ $\theta \in$	QI	QII	$QIII$	QIV
$\sin\theta$	$+$	$+$	$-$	$-$
$\cos\theta$	$+$	$-$	$-$	$+$
$\tan\theta$	$+$	$-$	$+$	$-$

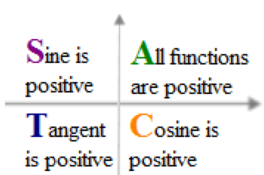


Figure 3.6

Since we will be making frequent decisions about signs of trigonometric function values, it is convenient to have an acronym helping us memorizing these signs in different quadrants. The first letters of the names of functions that are positive in particular quadrants, starting from the fourth quadrant and going counterclockwise, spells **CAST**, which is very helpful when working with trigonometric functions of any angles.

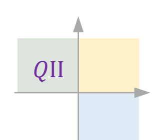
Example 5 ► **Identifying the Quadrant of an Angle**

Identify the quadrant or quadrants for each angle satisfying the given conditions.

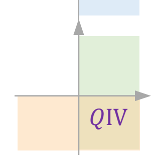
a. $\sin\theta > 0$; $\tan\theta < 0$ **b.** $\cos\theta > 0$; $\sin\theta < 0$

Solution ►

a. Using **CAST**, we have $\sin\theta > 0$ in QI(**A**ll) and QII(**S**ine) and $\tan\theta < 0$ in QII and QIV. Therefore both conditions are met only in **quadrant II**.



b. $\cos\theta > 0$ in QI(**A**ll) and QIV(**C**osine) and $\sin\theta < 0$ in $QIII$ and QIV. Therefore both conditions are met only in **quadrant IV**.



Example 6 ► **Identifying Signs of Trigonometric Functions of Any Angle**

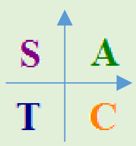
Using the **CAST** rule, identify the sign of each function value.

a. $\cos 150°$ **b.** $\tan 225°$

Solution ►

a. Since $150° \in Q\text{II}$ and cosine is negative in $Q\text{II}$, then $\cos 150°$ is **negative**.

b. Since $225° \in Q\text{III}$ and tangent is positive in $Q\text{III}$, then $\tan 225°$ is **positive**.



To find the exact value of a trigonometric function $\boldsymbol{T}$ of an angle $\boldsymbol{\theta}$ with the reference angle $\boldsymbol{\theta_{ref}}$ being a special angle, we follow the rule:

$$\boldsymbol{T(\theta) = \pm T(\theta_{ref})},$$

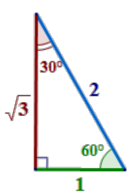
where the final sign is determined according to the quadrant of angle θ and the **CAST** rule.

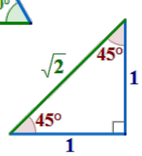
Example 7 ► **Finding Exact Function Values Using Reference Angles**

Find the exact values of the following expressions.

a. $\sin 240°$ **b.** $\cos 315°$

Solution ►





a. The reference angle of $240°$ is $240° - 180° = 60°$. Since $240° \in Q\text{III}$ and sine in the third quadrant is **positive**, we have

$$\sin 240° = \sin 60° = \frac{\sqrt{3}}{2}$$

b. The reference angle of $315°$ is $360° - 315° = 45°$. Since $315° \in Q\text{IV}$ and cosine in the fourth quadrant is **negative**, we have

$$\cos 315° = -\cos 45° = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

Finding Special Angles in Various Quadrants when Given Trigonometric Function Value

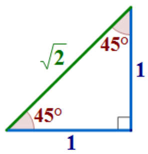
Now that it has been shown how to find exact values of trigonometric functions of angles that have a reference angle of one of the special angles ($\mathbf{30°}, \mathbf{45°}$, or $\mathbf{60°}$), we can work at reversing this process. Familiarity with values of trigonometric functions of the special angles, in combination with the idea of reference angles and quadrantal sign analysis, should help us in solving equations of the type $\boldsymbol{T(\theta)} = \boldsymbol{exact\ value}$, where $\boldsymbol{T}$ represents any trigonometric function.

Example 8 ► **Finding Angles with a Given Exact Function Value, in Various Quadrants**

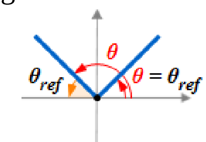
Find all angles θ satisfying the following conditions.

a. $\sin\theta = \frac{\sqrt{2}}{2};\ \theta \in [0°, 180°)$ **b.** $\cos\theta = -\frac{1}{2};\ \theta \in [0°, 360°)$

Solution ►

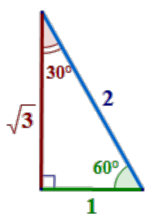


a. Refering to the half of a square triangle, we recognize that $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$ represents the ratio of sine of 45°. Thus, the reference angle $\theta_{ref} = 45°$. Moreover, we are searching for an angle θ from the interval $[0°, 180°)$ and we know that $\sin\theta > 0$. Therefore, θ must lie in the first or second quadrant and have the reference angle of 45°. Each quadrant gives us one solution, as shown in the figure on the right.

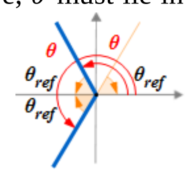


If θ is in the first quadrant, then $\theta = \theta_{ref} = 45°$.
If θ is in the second quadrant, then $\theta = 180° - 45° = 135°$.

So the solution set of the above problem is $\{\mathbf{45°}, \mathbf{135°}\}$.



b. Refering to the half of an equlateral triangle, we recognize that $\frac{1}{2}$ represents the ratio of cosine of 60°. Thus, the reference angle $\theta_{ref} = 60°$. We are searching for an angle θ from the interval $[0°, 360°)$ and we know that $\cos\theta < 0$. Therefore, θ must lie in the second or third quadrant and have the reference angle of 60°.



If θ is in the second quadrant, then $\theta = 180° - 60° = 120°$.
If θ is in the third quadrant, then $\theta = 180° + 60° = 240°$.

So the solution set of the above problem is $\{\mathbf{120°}, \mathbf{240°}\}$.

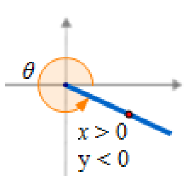
Finding Other Trigonometric Function Values

Example 9 ► **Finding Other Function Values Using a Known Value, Quadrant Analysis, and the x, y, r Definition of Trigonometric Ratios**

Find values of the remaining trigonometric functions of the angle satisfying the given conditions.

a. $\sin\theta = -\frac{7}{13}$; $\theta \in QIV$ **b.** $\tan\theta = \frac{15}{8}$; $\theta \in QIII$

Solution ►



a. We know that $\sin\theta = -\frac{7}{13} = \frac{y}{r}$. Hence, the terminal side of angle $\theta \in QIV$ contains a point $P(x, y)$ satisfying the condition $\frac{y}{r} = -\frac{7}{13}$. Since r must be positive, we will assign $y = -7$ and $r = 13$, to model the situation. Using the Pythagorean equation and the fact that the x-coordinate of any point in the fourth quadrant is positive, we determine the corresponding x-value to be

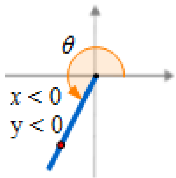
$$x = \sqrt{r^2 - y^2} = \sqrt{13^2 - (-7)^2} = \sqrt{169 - 49} = \sqrt{120} = 2\sqrt{30}.$$

Now, we are ready to state the remaining function values of angle θ:

$$\cos\theta = \frac{x}{r} = \mathbf{\frac{2\sqrt{30}}{13}}$$

and

$$\tan\theta = \frac{y}{x} = \frac{-7}{2\sqrt{30}} \cdot \frac{\sqrt{30}}{\sqrt{30}} = \mathbf{\frac{-7\sqrt{30}}{60}}.$$



b. We know that $\tan\theta = \frac{15}{8} = \frac{y}{x}$. Similarly as above, we would like to determine x, y, and r values that would model the situation. Since angle $\theta \in Q\text{III}$, both x and y values must be negative. So we assign $y = -15$ and $x = -8$. Therefore,

$$r = \sqrt{x^2 + y^2} = \sqrt{(-15)^2 + (-8)^2} = \sqrt{225 + 64} = \sqrt{289} = 17$$

Now, we are ready to state the remaining function values of angle θ:

$$\sin\theta = \frac{y}{r} = \mathbf{\frac{-15}{17}}$$

and

$$\cos\theta = \frac{x}{r} = \mathbf{\frac{-8}{17}}.$$

T.3 Exercises

Vocabulary Check *Fill in each blank with the most appropriate term from the given list:* ***acute, approximated, exact, quadrant, reference, special, terminal, triangles, x-axis,*** $\mathbf{30°}$.

1. When a scientific or graphing calculator is used to find a trigonometric function value, in most cases the result is an ________________ value.

2. Angles $30°, 45°, 60°$ are called ______________, because we can find the ___________ trigonometric function values of those angles. This is done by using relationships between the length of sides of special ____________.

3. For any angle θ, its ______________ angle θ_{ref} is the positive ___________ angle formed by the terminal side of θ and the ______________.

4. The trigonometric function values of $150°$ can be found by taking the corresponding function values of the ______ reference angle and assigning signs based on the ________________ of the ______________ side of the angle θ.

Use a calculator to ***approximate*** *each value to* ***four*** *decimal places.*

5. $\sin 36°52'05''$ **6.** $\tan 57.125°$ **7.** $\cos 204°25'$

Give the ***exact*** *function value,* ***without*** *the aid of a calculator. Rationalize denominators when applicable.*

8. $\cos 30°$ **9.** $\sin 45°$ **10.** $\tan 60°$ **11.** $\sin 60°$

12. $\tan 30°$ **13.** $\cos 60°$ **14.** $\sin 30°$ **15.** $\tan 45°$

Give the equivalent expression using the ***cofunction*** *relationship.*

16. $\cos 50°$ **17.** $\sin 22.5°$ **18.** $\sin 10°$

Concept Check *For each angle, find the **reference angle**.*

19. $98°$ **20.** $212°$ **21.** $13°$ **22.** $297°$ **23.** $186°$

Concept Check *Identify the quadrant or quadrants for each angle satisfying the given conditions.*

24. $\cos\alpha > 0$ **25.** $\sin\beta < 0$ **26.** $\tan\gamma > 0$

27. $\sin\theta > 0; \cos\theta < 0$ **28.** $\cos\alpha < 0; \tan\alpha > 0$ **29.** $\sin\alpha < 0; \tan\alpha < 0$

Identify the sign of each function value by quadrantal analysis.

30. $\cos 74°$ **31.** $\sin 245°$ **32.** $\tan 129°$ **33.** $\sin 183°$

34. $\tan 298°$ **35.** $\cos 317°$ **36.** $\sin 285°$ **37.** $\tan 215°$

Analytic Skills *Using reference angles, quadrantal analysis, and special triangles, find the exact values of the expressions. Rationalize denominators when applicable.*

38. $\cos 225°$ **39.** $\sin 120°$ **40.** $\tan 150°$ **41.** $\sin 150°$

42. $\tan 240°$ **43.** $\cos 210°$ **44.** $\sin 330°$ **45.** $\tan 225°$

Analytic Skills *Find all values of* $\theta \in [0°, 360°)$ *satisfying the given condition.*

46. $\sin\theta = -\frac{1}{2}$ **47.** $\cos\theta = \frac{1}{2}$ **48.** $\tan\theta = -1$ **49.** $\sin\theta = \frac{\sqrt{3}}{2}$

50. $\tan\theta = \sqrt{3}$ **51.** $\cos\theta = -\frac{\sqrt{2}}{2}$ **52.** $\sin\theta = 0$ **53.** $\tan\theta = -\frac{\sqrt{3}}{3}$

Analytic Skills *Find values of the remaining trigonometric functions of the angle satisfying the given conditions.*

54. $\sin\theta = \frac{\sqrt{5}}{7};\ \theta \in QII$ **55.** $\cos\alpha = \frac{3}{5};\ \alpha \in QIV$ **56.** $\tan\beta = \sqrt{3};\ \beta \in QIII$

T.4 Applications of Right Angle Trigonometry

Solving Right Triangles

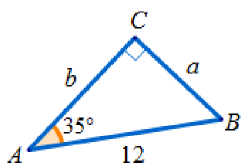
Geometry of right triangles has many applications in the real world. It is often used by carpenters, surveyors, engineers, navigators, scientists, astronomers, etc. Since many application problems can be modelled by a right triangle and trigonometric ratios allow us to find different parts of a right triangle, it is essential that we learn how to apply trigonometry to solve such triangles first.

Definition 4.1 ► To **solve a triangle** means to find the measures of all the unknown **sides** and **angles** of the triangle.

Example 1 ► **Solving a Right Triangle Given an Angle and a Side**

Given the information, solve triangle ABC, assuming that $\angle C = 90°$.

a.



b. $\angle B = 11.4°,\ b = 6\ cm$

Solution ► **a.** To find the length a, we want to relate it to the given length of 12 and the angle of 35°. Since a is opposite angle 35° and 12 is the length of the hypotenuse, we can use the ratio of sine:

$$\frac{a}{12} = \sin 35°$$

Then, after multiplying by 12, we have

$$a = 12 \sin 35° \simeq \mathbf{6.9}$$

round **lengths** to **one decimal place**

Attentin: To be more accurate, if possible, **use the given data** rather than the previously calculated ones, which are most likely already rounded

Since we already have the value of a, the length b can be determined in two ways: by applying the Pythagorean Theorem, or by using the cosine ratio. For better accuracy, we will apply the cosine ratio:

$$\frac{b}{12} = \cos 35°$$

which gives

$$b = 12 \cos 35° \simeq \mathbf{9.8}$$

Finally, since the two acute angles are complementary, $\angle B = 90° - 35° = \mathbf{55}°$.

We have found the three missing measurements, $a \simeq 6.9$, $b \simeq 9.8$, and $\angle B = 55°$, so the triangle is solved.

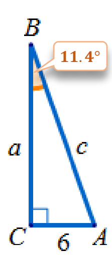


Figure 1

b. To visualize the situation, let's sketch a right triangle with $\angle B = 11.4°$ and $b = 6$ (see *Figure 1*). To find side a, we we would like to set up an equation that relates 6, a, and

$11.4°$. Since $b = 6$ is the opposite and a is the adjacent with respect to $\angle B = 11.4°$, we will use the ratio of tangent:

$$\tan 11.4° = \frac{6}{a}$$

To solve for a, we may want to multiply both sides of the equation by a and divide by $\tan 11.4°$. Observe that this will cause a and $\tan 11.4°$ to interchange (swap) their positions. So, we obtain

$$a = \frac{6}{\tan 11.4°} \simeq \mathbf{29.8}$$

To find side c, we will set up an equation that relates 6, c, and $11.4°$. Since $b = 6$ is the opposite to $\angle B = 11.4°$ and c is the hypothenuse, the ratio of sine applies. So, we have

$$\sin 11.4° = \frac{6}{c}$$

Similarly as before, to solve for c, we can simply interchange the position of $\sin 11.4°$ and c to obtain

$$c = \frac{6}{\sin 11.4°} \simeq \mathbf{30.4}$$

Finally, $\angle A = 90° - 11.4° = \mathbf{78.6}°$, which completes the solution.

In summary, $\angle A = 78.6°$, $a \simeq 29.8$, and $c \simeq 30.4$.

Observation: Notice that after approximated length a was found, we could have used the Pythagorean Theoreom to find length c. However, this could decrease the accuracy of the result. For this reason, it is advised that we use the given rather than approximated data, if possible.

Finding an Angle Given a Trigonometric Function Value

So far we have been evaluating trigonometric functions for a given angle. Now, what if we wish to reverse this process and try to recover an angle that corresponds to a given trigonometric function value?

Example 2 ► **Finding an Angle Given a Trigonometric Function Value**

Find an angle θ, satisfying the given equation. *Round to one decimal place, if needed.*

a. $\sin\theta = 0.7508$ **b.** $\cos\theta = -0.5$

Solution ► **a.** Since 0.7508 is not a special value, we will not be able to find θ by relating the equation to a special triangle as we did in *section T3, example 8*. This time, we will need to rely on a calculator. To find θ, we want to "undo" the sine. The function that can "undo" the sine is called **arcsine**, or **inverse sine**, and it is often abbreviated by $\mathbf{sin^{-1}}$. By applying the $\sin^{-1}$ to both sides of the equation

$$\sin\theta = 0.7508,$$

we have

$$\sin^{-1}(\sin\theta) = \sin^{-1}(0.7508)$$

Since $\sin^{-1}$ "undoes" the sine function, we obtain

$$\theta = \sin^{-1} 0.7508 \simeq \mathbf{48.7}°$$

round **angles** to **one decimal place**

On most calculators, to find this value, we follow the sequence of keys:

2nd or INV or Shift , SIN , 0.7508, ENTER or =

b. In this example, the absolute value of cosine is a special value. This means that θ can be found by referring to the **golden triangle** properties and the **CAST** rule of signs as in *section T3, example 8b*. The other way of finding θ is via a calculator

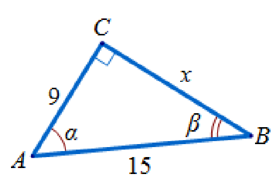
$$\theta = \cos^{-1}(-0.5) = \mathbf{120}°$$

Note: *Calculators are programed to return*
$\mathbf{sin^{-1}}$ *and* $\mathbf{tan^{-1}}$ *as angles from the interval* $[-\mathbf{90}°, \mathbf{90}°]$ *and*
$\mathbf{cos^{-1}}$ *as angles from the interval* $[\mathbf{0}°, \mathbf{180}°]$.
That implies that when looking for an obtuse angle, it is easier to work with $\mathbf{cos^{-1}}$*, if possible, as our calculator will return the actual angle. When using* $\mathbf{sin^{-1}}$ *or* $\mathbf{tan^{-1}}$*, we might need to search for a corresponding angle in the second quadrant on our own.*

More on Solving Right Triangles

Example 3 ► **Solving a Right Triangle Given Two Sides**

Solve the triangle.



Solution ► Since $\triangle ABC$ is a right triangle, to find the length x, we can use the Pythagorean Theorem.

$$x^2 + 9^2 = 15^2$$

so

$$\boldsymbol{x} = \sqrt{225 - 81} = \sqrt{144} = \mathbf{12}$$

To find the angle α, we can relate either $x = 12$, 9, and α, or 12, 15, and α. We will use the second triple and the ratio of sine. Thus, we have

$$\sin\alpha = \frac{12}{15},$$

therefore

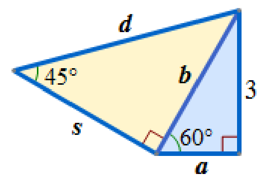
$$\boldsymbol{\alpha} = \sin^{-1}\frac{12}{15} \simeq \mathbf{53.1}°$$

Finally, $\boldsymbol{\beta} = 90° - \alpha \simeq 90° - 53.1° = \mathbf{36.9}°$.

In summary, $\alpha = 53.1°$, $\beta \simeq 36.9°$, and $x = 12$.

Example 4 ► **Using Relationships Between Sides of Special Triangles**

Find the **exact** value of each unknown in the figure.



Solution ► First, consider the blue right triangle. Since one of the acute angles is 60°, the other must be 30°. Thus the blue triangle represents half of an equilateral triangle with the side $\boldsymbol{b}$ and the height of 3 units. Using the relation $h = a\sqrt{3}$ between the height h and half a side a of an equilateral triangle, we obtain

$$\boldsymbol{a}\sqrt{3} = 3,$$

which gives us $\boldsymbol{a} = \frac{3}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{\mathbf{3}}$. Consequently, $\boldsymbol{b} = 2\boldsymbol{a} = \mathbf{2}\sqrt{\mathbf{3}}$.

Now, considering the yellow right triangle, we observe that both acute angles are equal to 45° and therefore the triangle represents half of a square with the side $\boldsymbol{s} = \boldsymbol{b} = \mathbf{2}\sqrt{\mathbf{3}}$.

Finally, using the relation between the diagonal and a side of a square, we have

$$\boldsymbol{d} = \boldsymbol{s}\sqrt{2} = 2\sqrt{3}\sqrt{2} = \mathbf{2}\sqrt{\mathbf{6}}.$$

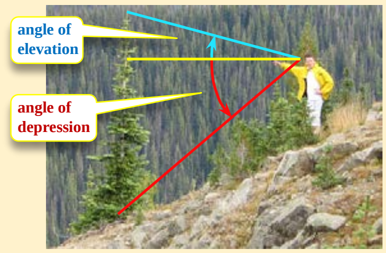
Angles of Elevation or Depression in Applications

The method of solving right triangles is widely adopted in solving many applied problems. One of the critical steps in the solution process is sketching a triangle that models the situation, and labeling the parts of this triangle correctly.

In trigonometry, many applied problems refer to angles of **elevation** or **depression**, or include some navigation terminology, such as **direction** or **bearing**.

Definition 4.2 ► **Angle of elevation** (or **inclination**) is the acute angle formed by a **horizontal** line and the line of sight to an object **above** the horizontal line.

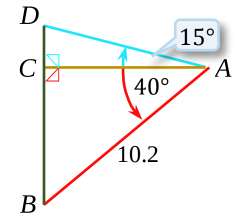
Angle of depression (or **declination**) is the acute angle formed by a **horizontal** line and the line of sight to an object **below** the horizontal line.



Example 5 ► Applying Angles of Elevation or Depression

Find the height of the tree in the picture given next to *Definition 4.2*, assuming that the observer sees the top of the tree at an angle of elevation of 15°, the base of the tree at an angle of depression of 40°, and the distance from the base of the tree to the observer's eyes is 10.2 meters.

Solution ► First, let's draw a diagram to model the situation, label the vertices, and place the given data. Then, observe that the height of the tree BD can be obtained as the sum of distances BC and CD.



BC can be found from $\triangle ABC$, by using the ratio of sine of 40°.
From the equation

$$\frac{BC}{10.2} = \sin 40°,$$

we have

$$BC = 10.2 \sin 40° \simeq \mathbf{6.56}$$

To calculate the length DC, we would need to have another piece of information about $\triangle ADC$ first. Notice that the side AC is common for the two triangles. This means that we can find it from $\triangle ABC$, and use it for $\triangle ADC$ in subsequent calculations.
From the equation

$$\frac{CA}{10.2} = \cos 40°,$$

we have

$$CA = 10.2 \cos 40° \simeq 7.8137$$

since we use this result in further calculations, four decimals of accuracy is advised

Now, employing tangent of 15° in $\triangle ADC$, we have

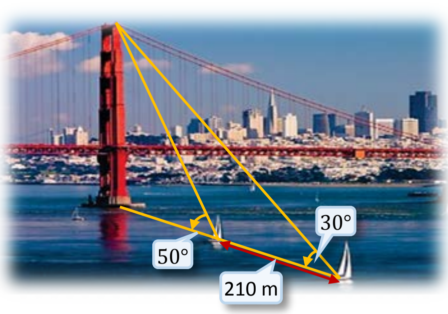
$$\frac{CD}{7.8137} = \tan 15°$$

which gives us

$$CD = 7.8137 \cdot \tan 15° \simeq \mathbf{2.09}$$

Hence the height of the tree is $BC \simeq 6.56 + 2.09 = 8.65 \simeq \mathbf{8.7}$ meters.

Example 6 ► Using Two Angles of Elevation at a Given Distance to Determine the Height



The Golden Gate Bridge has two main towers of equal height that support the two main cables. A visitor on a sailboat passing through San Francisco Bay views the top of one of the towers and estimates the angle of elevation to be 30°. After sailing 210 meters closer, the visitor estimates the angle of elevation to the same tower to be 50°. Approximate the height of the tower to the nearest meter.

Solution ► Let's draw the diagram to model the situation and adopt the notation as in *Figure 2*. We look for height $\boldsymbol{h}$, which is a part of the two right triangles $\triangle ABC$ and $\triangle DBC$.

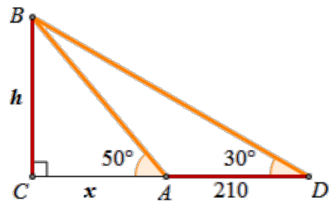


Figure 2

Since trigonometric ratios involve two sides of a triangle, and we already have length AD, a part of the side CD, it is reasonable to introduce another unknown, call it $\boldsymbol{x}$, to represent the remaining part CA. Then, applying the ratio of tangent to each of the right triangles, we produce the following system of equations:

$$\begin{cases} \dfrac{h}{x} = \tan 50° \\ \dfrac{h}{x+210} = \tan 30° \end{cases}$$

To solve the above system, we first solve each equation for h

$$\begin{cases} h \simeq 1.1918x \\ h \simeq 0.5774(x+210) \end{cases},$$

substitute to the top equation

and then by equating the right sides, we obtain

$$1.1918x = 0.5774(x+210)$$

$$1.1918x - 0.5774x = 121.254$$

$$0.6144x = 121.254$$

$$x = \frac{121.254}{0.6144} \simeq 197.4$$

Therefore, $\boldsymbol{h} \simeq 1.1918 \cdot 197.4 \simeq \mathbf{235}\ \boldsymbol{m}$.

The height of the tower is approximately 235 meters.

Direction or Bearing in Applications

A large group of applied problems in trigonometry refer to **direction** or **bearing** to describe the location of an object, usually a plane or a ship. The idea comes from following the behaviour of a compass. The magnetic needle in a compass points North. Therefore, the location of an object is described as a clockwise deviation from the SOUTH-NORTH line.

There are two main ways of describing directions:

- One way is by stating the angle θ that starts from the North and opens clockwise until the line of sight of an object. For example, we can say that the point $\boldsymbol{B}$ is seen in the **direction** of 108° from the point $\boldsymbol{A}$, as in *Figure 2a*.

- Another way is by stating the acute angle formed by the South-North line and the line of sight. Such an angle starts either from the North (**N**) or the South (**S**) and opens either towards the East (**E**) or the West (**W**). For instance, the position of the point $\boldsymbol{B}$ in *Figure 2b* would be described as being at a **bearing** of **S72°E** (*read:* South 72° towards the East) from the point $\boldsymbol{A}$.

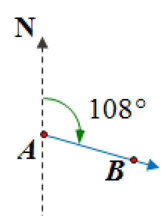


Figure 2a

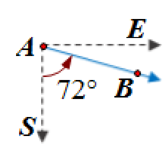


Figure 2b

This, for example, means that:

the direction of 195° can be seen as the bearing **S**15°**W**
and the direction of 290° means the same as **N**70°**W**.

Example 7 ► **Using Direction in Applications Involving Navigation**

An airplane flying at a speed of 400 mi/hr flies from a point A in the direction of 153° for one hour and then flies in the direction of 63° for another hour.

a. How long will it take the plane to get back to the point A?

b. What is the direction that the plane needs to fly in order to get back to the point A?

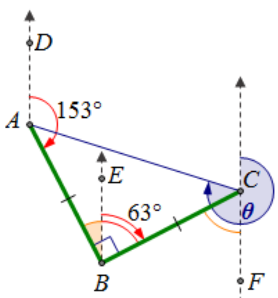


Figure 3

a. First, let's draw a diagram modeling the situation. Assume the notation as in *Figure 3*. Since the plane flies at 153° and the South-North lines $\overleftrightarrow{AD}$ and $\overleftrightarrow{BE}$ are parallel, by the property of interior angles, we have $\angle ABD = 180° - 153° = 27°$. This in turn gives us $\angle ABC = \angle ABE + \angle EBC = 27° + 63° = 90°$. So the $\triangle ABC$ is right angled with $\angle B = 90°$ and the two legs of length $AB = BC = 400\ mi$. This means that the $\triangle ABC$ is in fact a special triangle of the type $45° - 45° - 90°$.

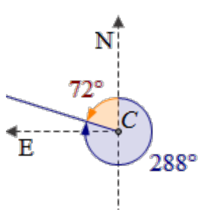
Therefore $AC = AB\sqrt{2} = 400\sqrt{2} \simeq 565.7\ mi$.

Now, solving the well-known motion formula $R \cdot T = D$ for the time T, we have

$$T = \frac{D}{R} \simeq \frac{\cancel{400}\sqrt{2}}{\cancel{400}} = \sqrt{2} \simeq 1.4142\ hr \simeq \mathbf{1\ hr\ 25\ min}$$

Thus, it will take the plane approximately 1 hour and 25 minutes to return to the starting point A.

b. To direct the plane back to the starting point, we need to find angle θ, marked in blue, rotating clockwise from the North to the ray $\overrightarrow{CA}$. By the property of alternating angles, we know that $\angle FCB = 63°$. We also know that $\angle BCA = 45°$, as $\angle ABD$ is the "half of a square" special triangle. Therefore,



$$\theta = 180° + 63° + 45° = \mathbf{288°}.$$

Thus, to get back to the point A, the plane should fly in the direction of 288°.
Notice that this direction can also be stated as **N**72°**W**.

T.4 Exercises

Vocabulary Check *Complete each blank with the most appropriate term or number from the given list:* ***depression, East, elevation, inverse, ratio, right, sides, sight, solve, South, three, 25.***

1. To _________ a triangle means to find the measure of all _______ angles and all three ________.

2. The value of a trigonometric function of an acute angle of a _____________ triangle represents the _____________ of lengths of appropriate sides of this triangle.

3. The value of an _____________ trigonometric function of a given number represents the angle.

4. The acute angle formed by a line of sight that falls *below* a horizontal line is called an angle of ________________. The acute angle formed by a line of __________ that rises *above* a horizontal line is called an angle of ________________.

5. The bearing **S**$25°$**E** indicates a line of sight that forms an angle of ____° to the __________ of a line heading ___________.

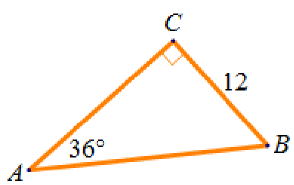
Concept check *Using a calculator, find an angle θ satisfying the given equation. Leave your answer in decimal degrees rounded to the nearest tenth of a degree if needed.*

6. $\sin\theta = 0.7906$ **7.** $\cos\theta = 0.7906$ **8.** $\tan\theta = 2.5302$

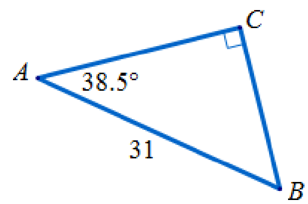
9. $\cos\theta = -0.75$ **10.** $\tan\theta = \sqrt{3}$ **11.** $\sin\theta = \frac{3}{4}$

Concept check *Given the data, solve each triangle ABC with $\angle C = 90°$.*

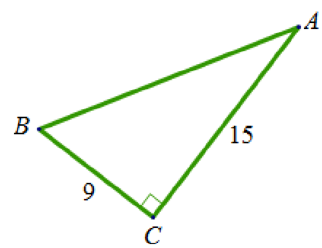
12.



13.



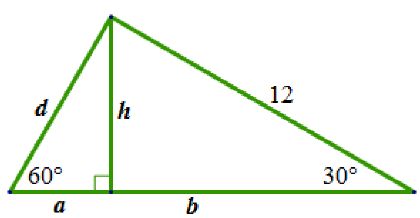
14.



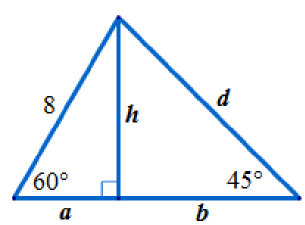
15. $\angle A = 42°,\ b = 17$ **16.** $a = 9.45,\ c = 9.81$ **17.** $\angle B = 63°12',\ b = 19.1$

*Find the **exact** value of each unknown in the figure.*

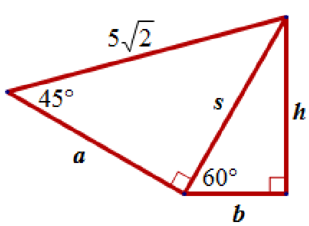
18.



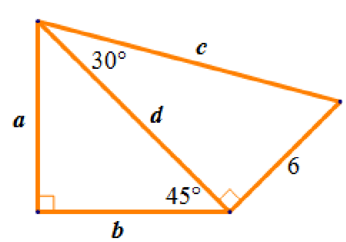
19.



20.



21.



22. A circle of radius 6 inches is inscribed in a regular hexagon. Find the exact length of one of its sides.

23. Find the perimeter of a regular hexagon that is inscribed in a circle of radius 8 meters.

24. A guy wire 77.4 meters long is attached to the top of an antenna mast that is 71.3 meters high. Find the angle that the wire makes with the ground.

25. A 100-foot guy wire is attached to the top of an antenna. The angle between the guy wire and the ground is 62°. How tall is the antenna to the nearest foot?

26. From the top of a lighthouse 52 m high, the angle of depression to a boat is $4°15'$. How far is the boat from the base of the lighthouse?

27. A security camera in a bank is mounted on a wall 9 feet above the floor. What angle of depression should be used if the camera is to be directed to a spot 6 feet above the floor and 12 feet from the wall?

28. For a person standing 100 meters from the center of the base of the Eiffel Tower, the angle of elevation to the top of the tower is 71.6°. How tall is the Eiffel Tower?

29. Find the altitude of an isosceles triangle having a base of 184.2 cm if the angle opposite the base is $68°44'$.

Analytic Skills

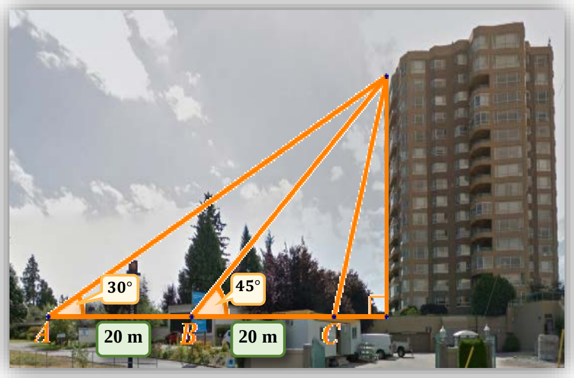
30. From city A to city B, a plane flies 650 miles at a bearing of **N**48°**E**. Then the plane flies 810 miles from city B to city C at a bearing of **S**42°**E**. Find the distance AC and the bearing directly from A to C.

31. A plane flies at 360 km/h for 30 minutes in the direction of 137°. Then, it changes its direction to 227° and flies for 45 minutes. How far and in what direction is the plane at that time from the starting point?

32. The tallest free-standing tower in the world is the CNN Tower in Toronto, Canada. The tower includes a rotating restaurant high above the ground. From a distance of 500 ft the angle of elevation to the pinnacle of the tower is 74.6°. The angle of elevation to the restaurant from the same vantage point is 66.5°. How tall is the CNN Tower including its pinnacle? How far below the tower is the restaurant located?

33. A hot air balloon is rising upward from the earth at a constant rate, as shown in the accompanying figure. An observer 250 meters away spots the balloon at an angle of elevation of 24°. Two minutes later, the angle of elevation of the balloon is 58°. At what rate is the balloon ascending? Answer to the nearest tenth of a meter per second.

34. A hot air balloon is between two spotters who are 1.2 mi apart. One spotter reports that the angle of elevation of the balloon is 76°, and the other reports that it is 68°. What is the altitude of the balloon in miles?



35. From point A the angle of elevation to the top of the building is 30°, as shown in the accompanying figure. From point B, 20 meters closer to the building, the angle of elevation is 45°. Find the angle of elevation of the building from point C, which is another 20 meters closer to the building.

36. For years the Woolworth skyscraper in New York held the record for the world's tallest office building. If the length of the shadow of the Woolworth building increases by 17.4 m as the angle of elevation of the sun changes from 44° to 42°, then how tall is the building?

37. A policeman has positioned himself 150 meters from the intersection of two roads. He has carefully measured the angles of the lines of sight to points A and B as shown in the drawing. If a car passes from A to B in 1.75 sec and the speed limit is 90 km/h, is the car speeding?

T.5 The Law of Sines and Cosines and Its Applications

The concepts of solving triangles developed in *section T4* can be extended to all triangles. A triangle that is not right-angled is called an **oblique triangle**. Many application problems involve solving oblique triangles. Yet, we can not use the SOH-CAH-TOA rules when solving those triangles since **SOH-CAH-TOA** definitions **apply only to right triangles!** So, we need to search for other rules that will allow us to solve oblique triangles.

The Sine Law

Observe that all triangles can be classified with respect to the size of their angles as **acute** (with all acute angles), **right** (with one right angle), or **obtuse** (with one obtuse angle). Therefore, oblique triangles are either acute or obtuse.

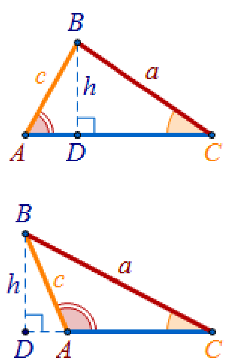


Figure 1

Let's consider both cases of an oblique $\triangle ABC$, as in *Figure 1*. In each case, let's drop the height h from vertex B onto the line $\overleftrightarrow{AC}$, meeting this line at point D. This way, we obtain two more right triangles, $\triangle ADB$ with hypotenuse c, and $\triangle BDC$ with hypotenuse a. Applying the ratio of sine to both of these triangles, we have:

$$\sin\angle A = \frac{h}{c}, \text{ so } h = c\sin\angle A$$

and

$$\sin\angle C = \frac{h}{a}, \text{ so } h = a\sin\angle C.$$

Thus,

$$a\sin\angle C = c\sin\angle A,$$

and we obtain

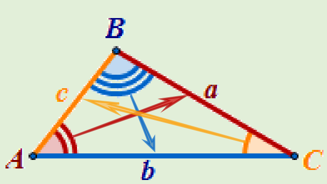
$$\frac{\boldsymbol{a}}{\sin\angle \boldsymbol{A}} = \frac{\boldsymbol{c}}{\sin\angle \boldsymbol{C}}.$$

Similarly, by dropping heights from the other two vertices, we can show that

$$\frac{\boldsymbol{a}}{\sin\angle \boldsymbol{A}} = \frac{\boldsymbol{b}}{\sin\angle \boldsymbol{B}} \text{ and } \frac{\boldsymbol{b}}{\sin\angle \boldsymbol{B}} = \frac{\boldsymbol{c}}{\sin\angle \boldsymbol{C}}.$$

This result is known as the law of sines.

The Sine Law ► In any triangle ABC, the lengths of the **sides are proportional to the sines of the opposite angles**. This fact can be expressed in any of the following, equivalent forms:



$$\frac{\boldsymbol{a}}{\boldsymbol{b}} = \frac{\sin\angle \boldsymbol{A}}{\sin\angle \boldsymbol{B}}, \quad \frac{\boldsymbol{b}}{\boldsymbol{c}} = \frac{\sin\angle \boldsymbol{B}}{\sin\angle \boldsymbol{C}}, \quad \frac{\boldsymbol{c}}{\boldsymbol{a}} = \frac{\sin\angle \boldsymbol{C}}{\sin\angle \boldsymbol{A}}$$

or

$$\frac{\boldsymbol{a}}{\sin\angle \boldsymbol{A}} = \frac{\boldsymbol{b}}{\sin\angle \boldsymbol{B}} = \frac{\boldsymbol{c}}{\sin\angle \boldsymbol{C}}$$

or

$$\frac{\sin\angle \boldsymbol{A}}{\boldsymbol{a}} = \frac{\sin\angle \boldsymbol{B}}{\boldsymbol{b}} = \frac{\sin\angle \boldsymbol{C}}{\boldsymbol{c}}$$

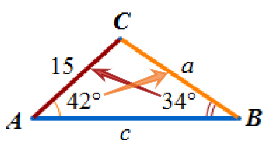
Observation: *As with any other proportion, to solve for one variable, we need to know the three remaining values. Notice that when using the Sine Law proportions, the three known values must include* ***one pair of opposite data****: a side and its opposite angle.*

Example 1 ► **Solving Oblique Triangles with the Aid of The Sine Law**

Given the information, solve each triangle ABC.

a. $\angle A = 42°,\ \angle B = 34°,\ b = 15$ **b.** $\angle A = 35°,\ a = 12,\ b = 9$

Solution ►



a. First, we will sketch a triangle ABC that models the given data. Since the sum of angles in any triangle equals 180°, we have

$$\angle \boldsymbol{C} = 180° - 42° - 34° = \mathbf{104°}.$$

Then, to find length a, we will use the pair $(a, \angle A)$ of opposite data, side a and $\angle A$, and the given pair $(b, \angle B)$. From the Sine Law proportion, we have

$$\frac{a}{\sin 42°} = \frac{15}{\sin 34°},$$

for easier calculations, keep the unknown in the numerator

which gives

$$\boldsymbol{a} = \frac{15 \cdot \sin 42°}{\sin 34°} \simeq \mathbf{17.9}$$

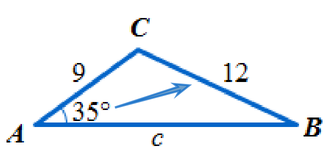
To find length c, we will use the pair $(c, \angle C)$ and the given pair of opposite data $(b, \angle B)$. From the Sine Law proportion, we have

$$\frac{c}{\sin 104°} = \frac{15}{\sin 34°},$$

which gives

$$\boldsymbol{c} = \frac{15 \cdot \sin 104°}{\sin 34°} \simeq \mathbf{26}$$

So the triangle is solved.



b. As before, we will start by sketching a triangle ABC that models the given data. Using the pair $(9, \angle B)$ and the given pair of opposite data $(12, 35°)$, we can set up a proportion

$$\frac{\sin \angle B}{9} = \frac{\sin 35°}{12}.$$

Then, solving it for $\sin \angle B$, we have

$$\sin \angle B = \frac{9 \cdot \sin 35°}{12} \simeq 0.4302,$$

which, after applying the inverse sine function, gives us

$$\angle \boldsymbol{B} \simeq \mathbf{25.5°}$$

Now, we are ready to find $\angle \boldsymbol{C} = 180° - 35° - 25.5° = \mathbf{119.5°}$,

and finally, from the proportion

$$\frac{c}{\sin 119.5°} = \frac{12}{\sin 35°},$$

we have

$$\boldsymbol{c} = \frac{12 \cdot \sin 119.5°}{\sin 35°} \simeq \mathbf{18.2}$$

Thus, the triangle is solved.

Ambiguous Case

Observe that the size of one angle and the length of two sides does not always determine a unique triangle. For example, there are two different triangles that can be constructed with $\angle A = 35°,\ a = 9$, $b = 12$.

Such a situation is called an **ambiguous case**. It occurs when the opposite side to the given angle is shorter than the other given side but long enough to complete the construction of an oblique triangle, as illustrated in *Figure 2*.

In application problems, if the given information does not determine a unique triangle, both possibilities should be considered in order for the solution to be complete.

On the other hand, not every set of data allows for the construction of a triangle. For example (see *Figure 3*), if $\angle A = 35°,\ a = 5$, $b = 12$, the side a is too short to complete a triangle, or if $a = 2$, $b = 3,\ c = 6$, the sum of lengths of a and b is smaller than the length of c, which makes impossible to construct a triangle fitting the data.

Note that in any triangle, the **sum of lengths of any two sides** is always **bigger than the length of the third side**.

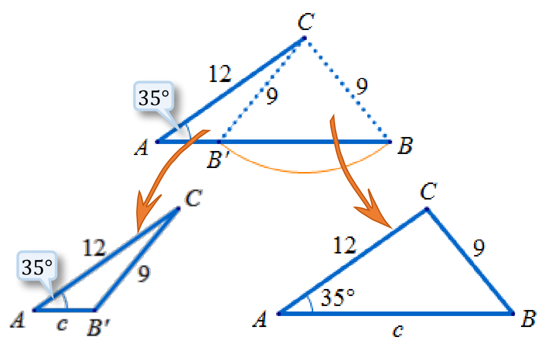


Figure 2

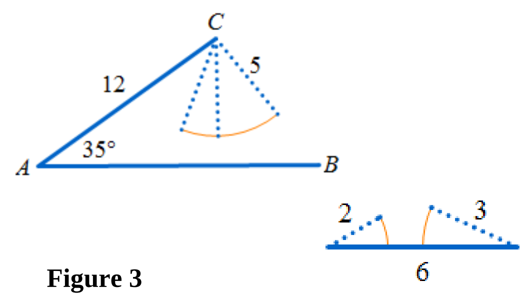
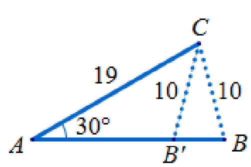


Figure 3

Example 2 ► **Using the Sine Law in an Ambiguous Case**

Solve triangle ABC, knowing that $\angle A = 30°,\ a = 10,\ b = 16$.

Solution ►



When sketching a diagram, we notice that there are two possible triangles, $\triangle ABC$ and $\triangle AB'C$, complying with the given information. $\triangle ABC$ can be solved in the same way as the triangle in *Example 1b*. In particular, one can calculate that in $\triangle ABC$, we have $\angle \boldsymbol{B} \simeq \mathbf{71.8°},\ \angle \boldsymbol{C} \simeq \mathbf{78.2°}$, and $\boldsymbol{c} \simeq \mathbf{19.6}$.

Let's see how to solve $\triangle AB'C$ then. As before, to find $\angle B'$, we will use the proportion

$$\frac{\sin \angle B'}{19} = \frac{\sin 30°}{10},$$

which gives us $\sin \angle B' = \frac{19 \cdot \sin 30°}{10} = 0.95$. However, when applying the inverse sine function to the number 0.95, a calculator returns the approximate angle of 71.8°. Yet, we know that angle B' is obtuse. So, we should look for an angle in the second quadrant, with the reference angle of 71.8°. Therefore, $\angle \boldsymbol{B'} = 180° - 71.8° = \mathbf{108.2}°$.

Now, $\angle \boldsymbol{C} = 180° - 30° - 108.2° = \mathbf{41.8}°$

and finally, from the proportion

$$\frac{c}{\sin 41.8°} = \frac{10}{\sin 30°},$$

we have

$$\boldsymbol{c} = \frac{10 \cdot \sin 41.8°}{\sin 30°} \simeq \mathbf{13.3}$$

Thus, $\varDelta AB'C$ is solved.

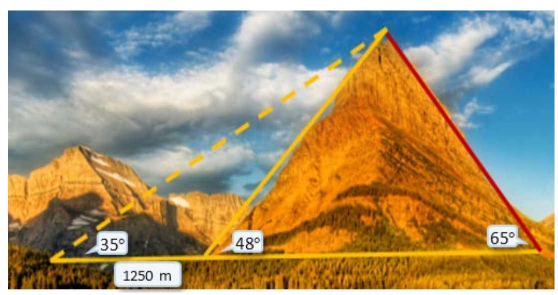
Example 3 ► **Solving an Application Problem Using the Sine Law**

Approaching from the west, a group of hikers records the angle of elevation to the summit of a steep mountain to be 35° at a distance of 1250 meters from the base of the mountain. Arriving at the base of the mountain, the hikers estimate that this side of the mountain has an average slope of 48°.

a. Find the slant height of the mountain's west side.

b. Find the slant height of the east side of the mountain, if the east side has an average slope of 65°.

c. How tall is the mountain?



Solution ► First, let's draw a diagram that models the situation and label its important parts, as in *Figure 3*.

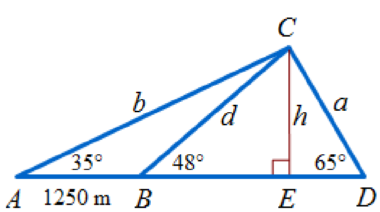


Figure 3

a. To find the slant height d, consider $\varDelta ABC$. Observe that one can easily find the remaining angles of this triangle, as shown below:

$$\angle ABC = 180° - 48° = 135°$$ supplementary angles

and

$$\angle ACB = 180° - 35° - 135° = 10°$$ sum of angles in a ⊿

Therefore, applying the law of sines, we have

$$\frac{d}{\sin 35°} = \frac{1250}{\sin 10°},$$

which gives

$$d = \frac{1250 \sin 35°}{\sin 10°} \simeq \mathbf{4128.9\ m}.$$

b. To find the slant height a, we can apply the law of sines to $\triangle BDC$ using the pair $(4128.9, 65°)$ to have

$$\frac{a}{\sin 48°} = \frac{4128.9}{\sin 65°},$$

which gives

$$a = \frac{4128.9 \sin 48°}{\sin 65°} \simeq \mathbf{3385.6\ m}.$$

c. To find the height h of the mountain, we can use the right triangle BCE. Using the definition of sine, we have

$$\frac{h}{4128.9} = \sin 48°,$$

so $\boldsymbol{h} = 4128.9 \sin 48° = \mathbf{3068.4\ m}$.

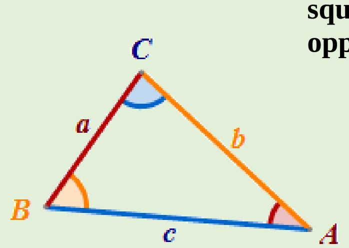
The Cosine Law

The above examples show how the **Sine Law** can help in solving oblique triangles when one **pair of opposite data** is given. However, the Sine Law is not enough to solve a triangle if the given information is

- the length of the **three sides** (but no angles), or
- the length of **two sides** and the **enclosed angle**.

Both of the above cases can be solved with the use of another property of a triangle, called the Cosine Law.

The Cosine Law ► In any triangle ABC, **the square of a side of a triangle is equal to the sum of the squares of the other two sides, minus twice their product times the cosine of the opposite angle**.



$$a^2 = b^2 + c^2 - 2bc \cos \angle A$$

$$b^2 = a^2 + c^2 - 2ac \cos \angle B$$

$$c^2 = a^2 + b^2 - 2ab \cos \angle C$$

note the opposite side and angle

Observation: *If the angle of interest in any of the above equations is right, since* $\cos 90° = 0$, *the equation becomes Pythagorean. So the* ***Cosine Law*** *can be seen as an* ***extension of the Pythagorean Theorem****.*

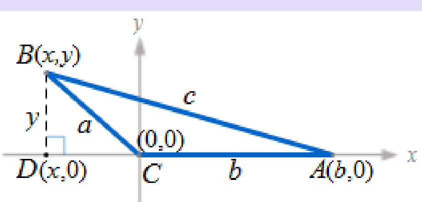


Figure 3

To derive this law, let's place an oblique triangle ABC in the system of coordinates so that vertex C is at the origin, side AC lies along the positive x-axis, and vertex B is above the x-axis, as in *Figure 3*.

Thus $C = (0,0)$ and $A = (b, 0)$. Suppose point B has coordinates (x, y). By *Definition 2.2*, we have

$$\sin\angle C = \frac{y}{a} \quad \text{and} \quad \cos\angle C = \frac{x}{a},$$

which gives us

$$y = a\sin\angle C \quad \text{and} \quad x = a\cos\angle C\,.$$

Let $D = (x, 0)$ be the perpendicular projection of the vertex B onto the x-axis. After applying the Pythagorean equation to the right triangle ABD, with $\angle D = 90°$, we obtain

here we use the Pythagorean identity developed in *section T2*: $\boldsymbol{\sin^2\theta + \cos^2\theta = 1}$

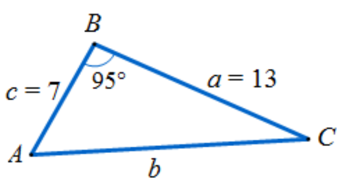
$$\begin{aligned}\boldsymbol{c^2} &= y^2 + (b-x)^2 \\ &= (a\sin\angle C)^2 + (b - a\cos\angle C)^2 \\ &= a^2\sin^2\angle C + b^2 - 2ab\cos\angle C + a^2\cos^2\angle C \\ &= a^2(\sin^2\angle C + \cos^2\angle C) + b^2 - 2ab\cos\angle C \\ &= \boldsymbol{a^2 + b^2 - 2ab\cos\angle C}\end{aligned}$$

Similarly, by placing the vertices A or B at the origin, one can develop the remaining two forms of the Cosine Law.

Example 4 ► **Solving Oblique Triangles Given Two Sides and the Enclosed Angle**

Solve triangle ABC, given that $\angle B = 95°$, $a = 13$, and $c = 7$.

Solution ►



First, we will sketch an oblique triangle ABC to model the situation. Since there is no pair of opposite data given, we cannot use the law of sines. However, applying the law of cosines with respect to side b and $\angle B$ allows for finding the length b. From

$$b^2 = 13^2 + 7^2 - 2\cdot 13\cdot 7\cos 95° \simeq 233.86,$$

we have $\boldsymbol{b \simeq 15.3}$.

watch the order of operations here!

Now, since we already have the pair of opposite data $(15.3, 95°)$, we can apply the law of sines to find, for example, $\angle C$. From the proportion

$$\frac{\sin\angle C}{7} = \frac{\sin 95°}{15.3},$$

we have

$$\sin\angle C = \frac{7\cdot\sin 95°}{15.3} \simeq 0.4558,$$

thus $\angle\boldsymbol{C} = \sin^{-1} 0.4558 \simeq \boldsymbol{27.1}°$.
Finally, $\angle\boldsymbol{A} = 180° - 95° - 27.1° = \boldsymbol{57.9}°$ and the triangle is solved.

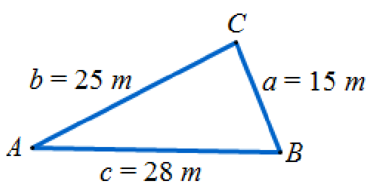
When applying the law of cosines in the above example, there was no other choice but to start with the pair of opposite data $(b, \angle B)$. However, in the case of three given sides, one could apply the law of cosines corresponding to any pair of opposite data. Is there any preference as to which pair to start with? Actually, yes. Observe that after using the law of cosines, we often use the **law of sines** to complete the solution since the **calculations are usually easier** to perform this way. Unfortunately, when solving a sine proportion for an obtuse angle, one would need to change the angle obtained from a calculator to its supplementary one. This is because calculators are programmed to return angles from the first quadrant when applying $\sin^{-1}$ to positive ratios. If we look for an obtuse angle, we need to employ the fact that $\sin\alpha = \sin(180° - \alpha)$ and take the supplement of the calculator's answer. To avoid this ambiguity, it is recommended to **apply the cosine law** to the pair of the **longest side and largest angle** first. This will guarantee that the law of sines will be used to find only acute angles and thus it will not cause ambiguity.

Recommendations:
- ***apply the Cosine Law only when it is absolutely necessary*** *(SAS or SSS)*
- ***apply the Cosine Law to find the largest angle first****, if applicable*

Example 5 ► **Solving Oblique Triangles Given Three Sides**

Solve triangle ABC, given that $a = 15\ m$, $b = 25\ m$, and $c = 28\ m$.

Solution ►



First, we will sketch a triangle ABC to model the situation. As before, there is no pair of opposite data given, so we cannot use the law of sines. So, we will apply the law of cosines with respect to the pair $(28, \angle C)$, as the side $c = 28$ is the longest. To solve the equation

$$28^2 = 15^2 + 25^2 - 2 \cdot 15 \cdot 25 \cos\angle C$$

watch the order of operations when solving for cosine

for $\angle C$, we will first solve it for $\cos\angle C$, and have

$$\cos\angle C = \frac{28^2 - 15^2 - 25^2}{-2 \cdot 15 \cdot 25} = \frac{-66}{-750} = 0.088,$$

which, after applying $\cos^{-1}$, gives $\angle \boldsymbol{C} \simeq \mathbf{85}°$.

Since now we already have the pair of opposite data $(28, 85°)$, we can apply the law of sines to find, for example, $\angle A$. From the proportion

$$\frac{\sin\angle A}{15} = \frac{\sin 85°}{28},$$

we have

$$\sin\angle A = \frac{15 \cdot \sin 85°}{28} \simeq 0.5337,$$

thus $\angle \boldsymbol{A} = \sin^{-1} 0.5337 \simeq \mathbf{32.3}°$.
Finally, $\angle \boldsymbol{B} = 180° - 85° - 32.3° = \mathbf{62.7}°$ and the triangle is solved.

Example 6 ► **Solving an Application Problem Using the Cosine Law**

Two planes leave an airport at the same time and fly for two hours. Plane A flies in the direction of 165° at 385 km/h and plane B flies in the direction of 250° at 410 km/h. How far apart are the planes after two hours?

Solution ► As usual, we start the solution by sketching a diagram appropriate to the situation. Assume the notation as in *Figure 4*.

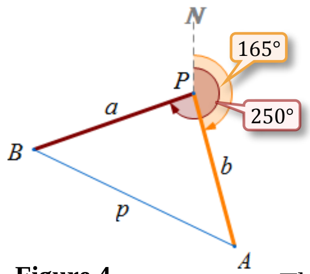


Figure 4

Since plane A flies at 385 km/h for two hours, we can find the distance

$$b = 2 \cdot 385 = 770\ km.$$

Similarly, since plane B flies at 410 km/h for two hours, we have

$$a = 2 \cdot 410 = 820\ km.$$

The measure of the enclosed angle APB can be obtained as a difference between the given directions. So we have

$$\angle APB = 250° - 165° = 85°.$$

Now, we are ready to apply the law of cosines in reference to the pair $(p, 85°)$. From the equation

$$p^2 = 820^2 + 770^2 - 2 \cdot 820 \cdot 770 \cos 85°,$$

we have $\boldsymbol{p} \simeq \sqrt{1155239.7} \simeq \mathbf{1074.8}\ \boldsymbol{km}$.

So we know that after two hours, the two planes are about 1074.8 kilometers apart.

Area of a Triangle

The method used to derive the law of sines can also be used to derive a handy formula for finding the area of a triangle, without knowing its height.

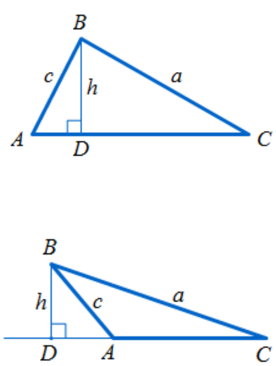


Figure 5

Let ABC be a triangle with height h dropped from the vertex B onto the line $\overleftrightarrow{AC}$, meeting $\overleftrightarrow{AC}$ at the point D, as shown in *Figure 5*. Using the right $\triangle ABD$, we have

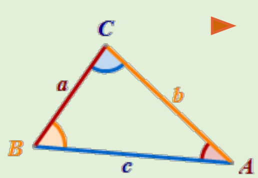
$$\sin \angle A = \frac{h}{c},$$

and equivalently $h = c \sin \angle A$, which after substituting into the well known formula for area of a triangle $[\boldsymbol{ABC}] = \frac{1}{2}\boldsymbol{bh}$, gives us

$$[\boldsymbol{ABC}] = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{bc}\ \mathbf{sin}\ \angle\boldsymbol{A}$$

Starting the proof with dropping a height from a different vertex would produce two more versions of this formula, as stated below.

The Sine Formula for Area of a Triangle

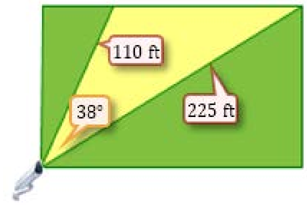


► The **area** $[ABC]$ of a triangle ABC can be calculated by taking **half of a product of the lengths of two sides and the sine of the enclosed angle.** We have

$$[\boldsymbol{ABC}] = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{bc}\ \mathbf{sin}\ \angle\boldsymbol{A}, \quad [\boldsymbol{ABC}] = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{ac}\ \mathbf{sin}\ \angle\boldsymbol{B}, \quad \text{or} \quad [\boldsymbol{ABC}] = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{ab}\ \mathbf{sin}\ \angle\boldsymbol{C}.$$

Example 7 ► **Finding Area of a Triangle Given Two Sides and the Enclosed Angle**

A stationary surveillance camera is set up to monitor activity in the parking lot of a shopping mall. If the camera has a 38° field of vision, how many square feet of the parking lot can it tape using the given dimensions?



Solution ► We start with sketching an appropriate diagram. Assume the notation as in *Figure 6.*

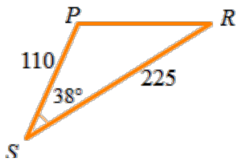


Figure 6

From the sine formula for area of a triangle, we have

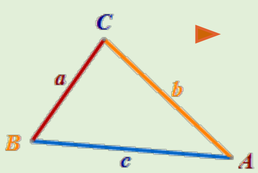
$$[\boldsymbol{PRS}] = \frac{1}{2} \cdot 110 \cdot 225 \sin 38° \simeq \mathbf{7619}\ \boldsymbol{ft^2}.$$

The surveillance camera monitors approximately 7619 square feet of the parking lot.

Heron's Formula

The law of cosines can be used to derive a formula for the area of a triangle when only the lengths of the three sides are known. This formula is known as Heron's formula (as mentioned in *section RD1*), named after the Greek mathematician Heron of Alexandria.

Heron's Formula for Area of a Triangle



► The **area** $[\boldsymbol{ABC}]$ of a triangle ABC with sides $a, b, c,$ and **semiperimeter** $\boldsymbol{s} = \frac{\boldsymbol{a+b+c}}{\mathbf{2}}$ can be calculated using the formula

$$[\boldsymbol{ABC}] = \sqrt{\boldsymbol{s}(\boldsymbol{s}-\boldsymbol{a})(\boldsymbol{s}-\boldsymbol{b})(\boldsymbol{s}-\boldsymbol{c})}$$

Example 8 ► **Finding Area of a Triangle Given Three Sides**

A New York City developer wants to build condominiums on the triangular lot formed by Greenwich, Watts, and Canal Streets. How many square meters does the developer have to work with if the frontage along each street is approximately 34.1 m, 43.5 m, and 62.4 m, respectively?

Solution ► To find the area of the triangular lot with given sides, we would like to use Heron's Formula. For this reason, we first calculate the semiperimeter

$$s = \frac{34.1 + 43.5 + 62.4}{2} = 70.$$

Then, the area equals

$$\sqrt{70(70-34.1)(70-43.5)(70-62.4)} = \sqrt{506118.2} \simeq \mathbf{711}\ \boldsymbol{m^2}.$$

Thus, the developer has approximately 711 square meters to work with in the lot.

T.5 Exercises

Vocabulary Check *Complete each blank with the most appropriate term from the given list:* ***ambiguous, angle, area, cosines, enclosed, largest, length, longest, oblique, opposite, Pythagorean, side, sides, sum, three, triangles, two****.*

1. A triangle that is not right-angled is called an _____________ triangle.

2. When solving a triangle, we apply the law of sines only when a pair of _____________ data is given.

3. To solve triangles with all _________ sides or two sides and the ________________angle given, we use the law of ______________ .

4. The ambiguous case refers to the situation when ______ ________________ satisfying the given data can be constructed.

5. In any triangle, the ______________ side is always opposite the largest ____________.

6. In any triangle the ________ of lengths of any pair of sides is bigger than the ____________ of the third _________ .

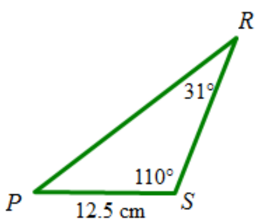
7. To avoid dealing with the _________________ case, we should use the law of cosines when solving for the ___________ angle.

8. The Cosine Law can be considered as an extension of the ___________________ Theorem.

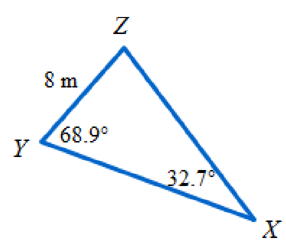
9. The ____________ of a triangle with three given ___________ can be calculated by using the Heron's formula.

Concept check *Use the law of sines to solve each triangle.*

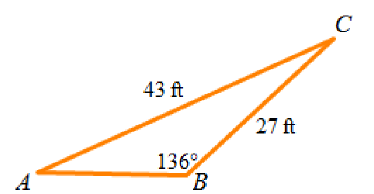
10.



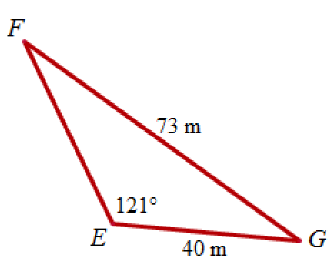
11.



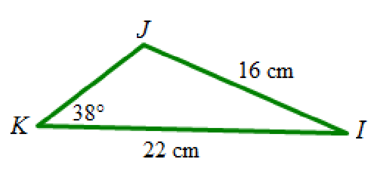
12.



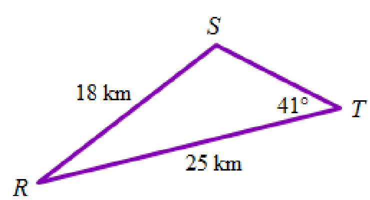
13.



14.



15.



16. $\angle A = 30°,\ \angle B = 30°,\ a = 10$

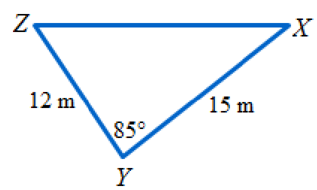
17. $\angle A = 150°,\ \angle C = 20°,\ a = 200$

18. $\angle C = 145°,\ b = 4,\ c = 14$

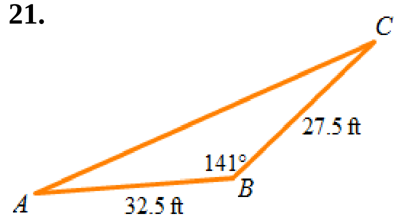
19. $\angle A = 110°15',\ a = 48,\ b = 16$

Concept check *Use the law of cosines to solve each triangle.*

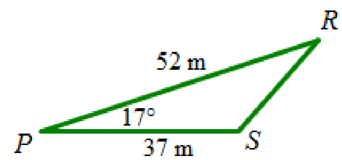
20.



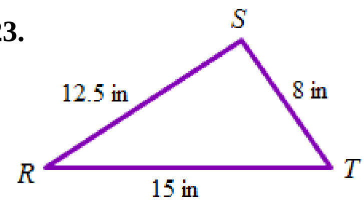
21.



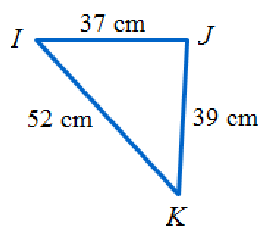
22.



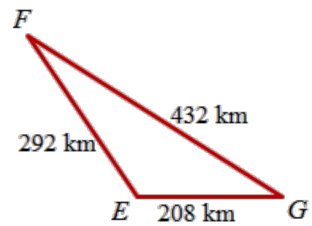
23.



24.



25.



26. $\angle C = 60°,\ a = 3,\ b = 10$

27. $\angle B = 112°,\ a = 23,\ c = 31$

28. $a = 2,\ b = 3,\ c = 4$

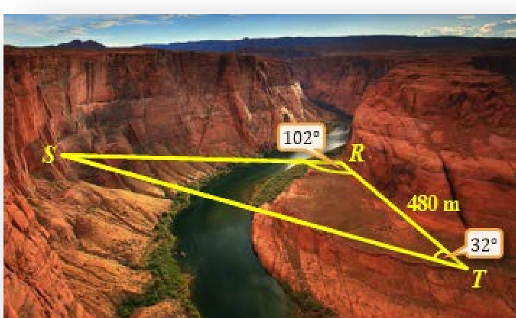
29. $a = 34,\ b = 12,\ c = 17.5$

Concept check

30. If side a is twice as long as side b, is $\angle A$ necessarily twice as large as $\angle B$?

Use the appropriate law to solve each application problem.

31. To find the distance AB across a river, a surveyor laid off a distance BC = 354 meters on one side of the river, as shown in the accompanying figure. It is found that $\angle B = 112°10'$ and $\angle C = 15°20'$. Find the distance AB.



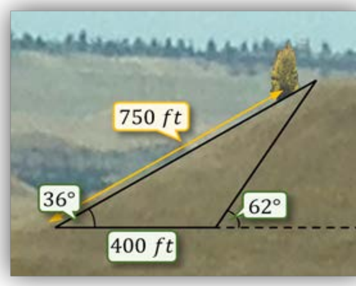
32. To determine the distance RS across a deep canyon (*see the accompanying figure*), Peter lays off a distance TR = 480 meters. Then he finds that $\angle T = 32°$ and $\angle R = 102°$. Find the distance RS.

33. A ship is sailing due north. At a certain point, the captain of the ship notices a lighthouse 12.5 km away from the ship, at the bearing of $\boldsymbol{N}38.8°\boldsymbol{E}$. Later on, the bearing of the lighthouse becomes $\boldsymbol{S}44.2°\boldsymbol{E}$. In meters, how far did the ship travel between the two observations of the lighthouse?

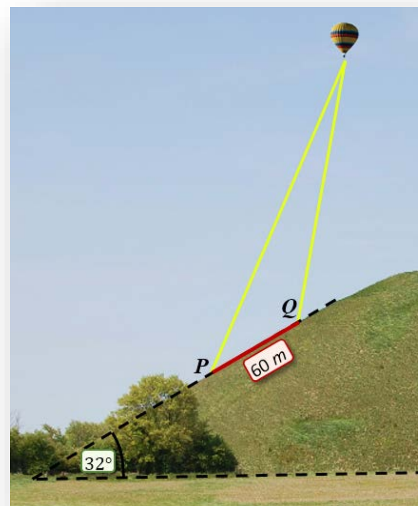
34. The bearing of a lighthouse from a ship was found to be $\boldsymbol{N}37°\boldsymbol{E}$. After the ship sailed 2.5 mi due south, the new bearing was $\boldsymbol{N}25°\boldsymbol{E}$. Find the distance between the ship and the lighthouse at each location.

35. Joe and Jill set sail from the same point, with Joe sailing in the direction of S4°E and Jill sailing in the direction S9°W. After 4 hr, Jill was 2 mi due west of Joe. How far had Jill sailed?

36. A hill has an angle of inclination of 36°, as shown in the accompanying figure. A study completed by a state's highway commission showed that the placement of a highway requires that 400 ft of the hill, measured horizontally, be removed. The engineers plan to leave a slope alongside the highway with an angle of inclination of 62°, as shown in the figure. Located 750 ft up the hill measured from the base is a tree containing the nest of an endangered hawk. Will this tree be removed in the excavation?



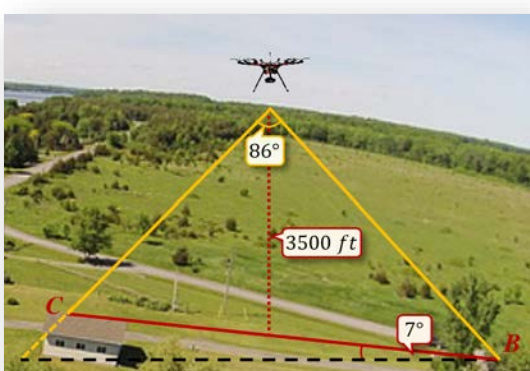
37. Radio direction finders are placed at points A and B, which are 3.46 mi apart on an east-west line, with A west of B. A radio transmitter is found to be at the direction of 47.7° from A and 302.5° from B. Find the distance of the transmitter from A, to the nearest hundredth of a mile.



38. Observers at P and Q are located on the side of a hill that is inclined 32° to the horizontal, as shown in the accompanying figure. The observer at P determines the angle of elevation to a hot-air balloon to be 62°. At the same instant, the observer at Q measures the angle of elevation to the balloon to be 71°. If P is 60 meters down the hill from Q, find the distance from Q to the balloon.

39. What is the length of the chord subtending a central angle of 19° in a circle of radius 30 ft?

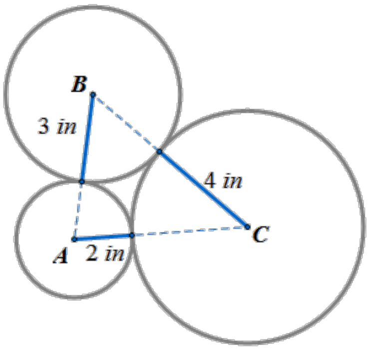
40. A pilot flies her plane on a heading of 35° from point X to point Y, which is 400 mi from X. Then she turns and flies on a heading of 145° to point Z, which is 400 mi from her starting point X. What is the heading of Z from X, and what is the distance YZ?



41. A painter is going to apply a special coating to a triangular metal plate on a new building. Two sides measure 16.1 m and 15.2 m. She knows that the angle between these sides is 125°. What is the area of the surface she plans to cover with the coating?

42. A camera lens with a 6-in. focal length has an angular coverage of 86°. Suppose an aerial photograph is taken vertically with no tilt at an altitude of 3500 ft over ground with an increasing slope of 7°, as shown in the accompanying figure. Calculate the ground distance CB that will appear in the resulting photograph.

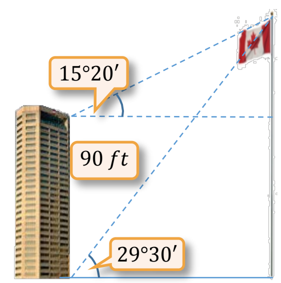
43. A solar panel with a width of 1.2 m is positioned on a flat roof, as shown in the accompanying figure. What is the angle of elevation α of the solar panel?



44. An engineer wants to position three pipes so that they are tangent to each other. A perpendicular cross section of the structure is shown in the accompanying figure. If pipes with centers A, B, and C have radii 2 in., 3 in., and 4 in., respectively, then what are the angles of the triangle ABC?

45. A flagpole 95 ft tall is on the top of a building. From a point on level ground, the angle of elevation of the top of the flagpole is $35°$, and the angle of elevation of the bottom of the flagpole is $26°$. Find the height of the building.

46. The angle of elevation (*see the figure to the right*) from the top of a building 90 ft high to the top of a nearby mast is $15°20'$. From the base of the building, the angle of elevation of the tower is $29°30'$. Find the height of the mast.

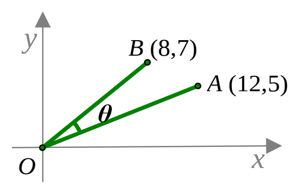


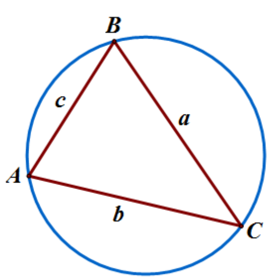
47. A real estate agent wants to find the area of a triangular lot. A surveyor takes measurements and finds that two sides are 52.1 m and 21.3 m, and the angle between them is 42.2°. What is the area of the triangular lot?

48. A painter needs to cover a triangular region with sides of lengths 75 meters, 68 meters, and 85 meters. A can of paint covers 75 square meters of area. How many cans will be needed?

Analytic Skills

49. Find the measure of angle θ enclosed by the segments OA and OB, as on the accompanying diagram.

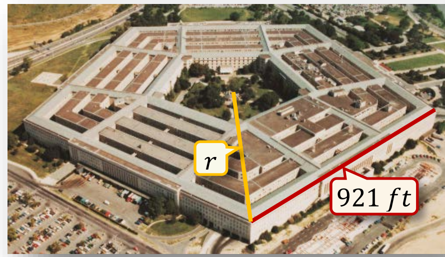




50. Prove that for a triangle inscribed in a circle of radius r (*see the diagram to the left*), the law of sine ratios $\frac{a}{\sin\angle A}$, $\frac{b}{\sin\angle B}$, and $\frac{c}{\sin\angle C}$ have value $2r$. Then confirm that in a circle of diameter 1, the following equations hold: $\sin\angle A = a$, $\sin\angle B = b$, and $\sin\angle C = c$.
(This provides an alternative way to define the sine function for angles between 0° and 180°. It was used nearly 2000 years ago by the mathematician Ptolemy to construct one of the earliest trigonometric tables.)

51. Josie places her lawn sprinklers at the vertices of a triangle that has sides of 9 m, 10 m, and 11 m. The sprinklers water in circular patterns with radii of 4, 5, and 6 m. No area is watered by more than one sprinkler. What amount of area inside the triangle is not watered by any of the three sprinklers? *Round the answer to the nearest hundredth of a square meter.*

52. The Pentagon in Washington D.C. is 921 ft on each side, as shown in the accompanying figure. What is the distance r from a vertex to the center of the Pentagon?



TRIGONOMETRY - ANSWERS

T.1 Exercises

1. Complementary, 180

3. coterminal, 360

5. $20.075°$

7. $274.304°$

9. $15.168°$

11. $18°0'45''$

13. $65°0'5''$

15. $175°23'58''$

17. $83°59'$

19. $33°50'$

21. $28°03'03''$

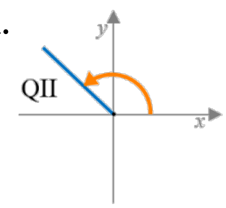
23. $60°, 150°$

25. $45°, 135°$

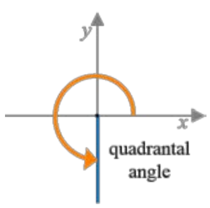
27. $74°30', 164°30'$

29. $180 - \theta°$

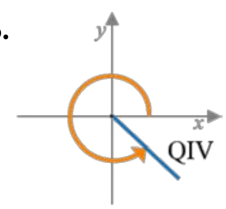
31.



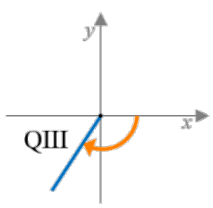
33.



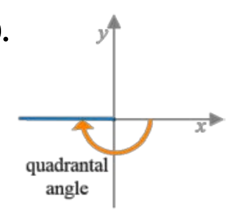
35.



37.



39.



41. $15°$

43. $135°$

45. $30° + k \cdot 360°$

47. $k \cdot 360°$

49. $\alpha° + k \cdot 360°$

51. $7.5°$

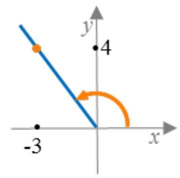
T.2 Exercises

1. $\sin\theta = \frac{3}{5}, \quad \cos\theta = \frac{4}{5}, \quad \tan\theta = \frac{3}{4}$

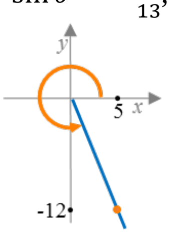
3. $\sin\theta = \frac{\sqrt{3}}{2}, \quad \cos\theta = \frac{1}{2}, \quad \tan\theta = \sqrt{3}$

5. $\sin\theta = \frac{n}{\sqrt{n^2+4}}, \quad \cos\theta = \frac{2}{\sqrt{n^2+4}}, \quad \tan\theta = \frac{n}{2}$

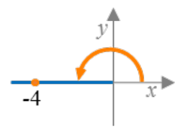
7. $\sin\theta = \frac{4}{5}, \quad \cos\theta = -\frac{3}{5}, \quad \tan\theta = -\frac{4}{3}$



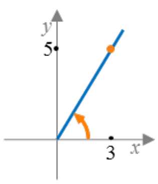
9. $\sin\theta = -\frac{12}{13}, \quad \cos\theta = \frac{5}{13}, \quad \tan\theta = -\frac{12}{5}$



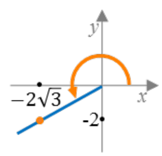
11. $\sin\theta = 0, \quad \cos\theta = -1, \quad \tan\theta = 0$



13. $\sin\theta = \frac{5\sqrt{34}}{34}, \quad \cos\theta = \frac{3\sqrt{34}}{34}, \quad \tan\theta = \frac{5}{3}$



15. $\sin\theta = -\frac{1}{2}, \quad \cos\theta = -\frac{\sqrt{3}}{2}, \quad \tan\theta = \frac{\sqrt{3}}{3}$



17. sine and cosine is negative, tangent is positive

19. negative

21. negative

23. positive

25. positive

27. negative

29. 1

31. -1

33. 0

35. 0

37. $undefined$

39. $\cos\beta = -\frac{\sqrt{5}}{3}$

$\tan\beta = \frac{2\sqrt{5}}{5}$

T.3 Exercises

1. approximated

3. reference, acute, x-axis

5. 0.6000

7. -0.9106

9. $\frac{\sqrt{2}}{2}$

11. $\frac{\sqrt{3}}{2}$

13. $\frac{1}{2}$

15. 1

17. $\cos 67.5°$

19. 82°

21. 13°

23. 6°

25. QIII and QIV

27. QII

29. QIV

31. negative

33. negative

35. positive

37. positive

39. $\frac{\sqrt{3}}{2}$

41. $\frac{1}{2}$

43. $-\frac{\sqrt{3}}{2}$

45. 1

47. $60°, 300°$

49. $60°, 120°$

51. $135°, 225°$

53. $150°, 330°$

55. $\sin\alpha = -\frac{4}{5}$
$\tan\alpha = -\frac{4}{3}$

T.4 Exercises

1. solve, three, sides

3. inverse

5. 25, East, South

7. 37.8°

9. 138.6°

11. 48.6°

13. $a \simeq 19.3,\ \angle B = 51.5°,\ c \simeq 24.3$

15. $a \simeq 15.3,\ \angle B = 48°,\ c \simeq 22.9$

17. $\angle A = 26°48',\ a \simeq 9.6,\ c \simeq 21.4$

19. $a = 4,\ b = 4\sqrt{3},\ d = 4\sqrt{6},\ h = 4\sqrt{3}$

21. $a = 3\sqrt{6},\ b = 3\sqrt{6},\ c = 12,\ d = 6\sqrt{3}$

23. 48 m

25. 88.3 ft

27. 14°

29. 134.7 cm

31. 324.5 km in the direction of 193.3°

33. 2.4 m/s

35. 75°

37. Yes, the car is speeding at 94.8 kph.

T.5 Exercises

1. oblique

3. three, enclosed, cosines

5. longest, angle

7. ambiguous, largest

9. area, sides

11. $y \simeq 13.8$ m, $\angle Z = 78.4°$, $c \simeq 14.5$ m

13. $\angle F \simeq 28.0°$, $\angle G \simeq 31°$, $g \simeq 43.8$ m

15. $\angle R \simeq 24.7°$, $\angle S \simeq 114.3°$, $r \simeq 11.5$ k

17. $\angle B = 10°$, $b \simeq 69.5$, $c \simeq 136.8$

19. $\angle B \simeq 18°13'26''$, $\angle C \simeq 51°31'34''$, $c \simeq 40.1$

21. $\angle A \simeq 17.8°$, $b \simeq 56.6$ ft, $\angle C \simeq 21.2°$

23. $\angle R \simeq 32.2°$, $\angle S \simeq 91.4°$, $\angle T \simeq 56.4.1°$

25. $\angle E \simeq 118.6°$, $\angle F \simeq 25°$, $\angle G \simeq 36.4°$

27. $\angle A \simeq 28.3°$, $b \simeq 45$, $\angle C \simeq 39.7°$

29. $\angle A \simeq 112.8°$, $\angle B \simeq 19°$, $\angle C \simeq 48.2°$

31. $AB \simeq 118$ m

33. ~1687 m

35. ~8.9 mi

37. ~1.93 mi

39. ~9.9 ft

41. ~100.2 m^2

43. ~19.2°

45. ~218.1 ft

47. ~372.7 m^2

49. $\theta \simeq 18.6°$

51. ~3.85 m^2