C.3 Nonlinear Systems of Equations and Inequalities

In *section E1*, we discussed methods of solving systems of two linear equations. Recall that solutions to such systems are the intercepts of the two lines. So, we could have either zero, or one, or infinitely many solutions.

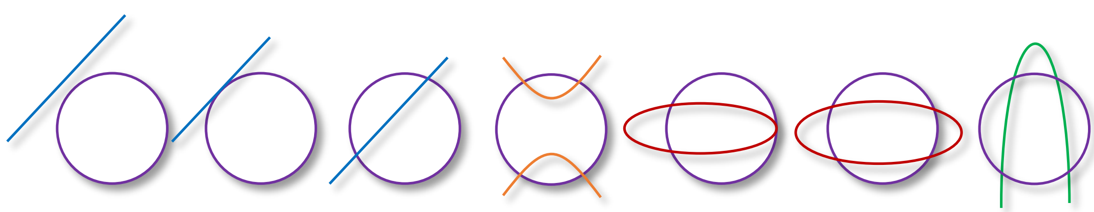
In this section, we will consider systems of two equations that are not necessarily linear. In particular, we will focus on solving systems composed of equations of conic sections. Since the solutions to such systems can be seen as the intercepts of the curves represented by the equations, we may expect a different number of solutions. For example, a circle may intercept an ellipse in 0, 1, 2, 3, or 4 points. We encourage the reader to visualise these situations by drawing a circle and an ellipse in various positions.

Aside from discussing methods of solving nonlinear systems of equations, we will also graph solutions of nonlinear systems of inequalities, using similar techniques as presented in *section G4*, where solutions to linear inequalities were graphed.

Nonlinear Systems of Equations

Definition 3.1 ► A **nonlinear system of equations** is a system of equations containing at least one equation that is not linear.

When solving a nonlinear system of two equations, it is useful to predict the possible number of solutions by visualising the shapes and position of the graphs of these equations. For example, the number of solutions to a system of two equations representing conic sections can be determined by observing the number of intercept points of the two curves. Here are some possible situations.



0 solutions   1 solution   2 solutions   4 solutions   3 solutions   4 solutions   4 solutions

To solve a nonlinear system of two equations, we may use any of the algebraic methods discussed in *section E1*, the substitution or the elimination method, whichever makes the calculations easier.

Example 1 ► **Solving Nonlinear Systems of Two Equations by Substitution**

Solve each system of equations.

a. $\begin{cases} xy = 4 \\ 4y + x = 8 \end{cases}$ **b.** $\begin{cases} x^2 + y^2 = 9 \\ x - y = 1 \end{cases}$

Solution

a. The system $\begin{cases} xy = 4 \\ 4y + x = 8 \end{cases}$ consists of a reciprocal function (which is a hyperbola) and a line. So, we may expect 0, 1, or 2 solutions. To solve this system, we may want to solve the second equation for x,

$$x = -4y + 8,$$

and then substituting the resulting expression into the first equation. So, we have

$$\begin{aligned} (8 - 4y)y &= 4 && / \ -4 \\ 8y - 4y^2 - 4 &= 0 && / \cdot (-1) \\ 4y^2 - 8y + 4 &= 0 && / \ \div 4 \\ y^2 - 2y + 1 &= 0 \\ (y - 1)^2 &= 0 \\ y &= 1 \end{aligned}$$

Then, using the substitution equation, we calculate

$$x = -4 \cdot 1 + 8 = 4.$$

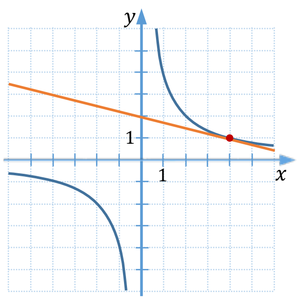


Figure 3.1

So, the solution set consists of one point, $(\mathbf{4}, \mathbf{1})$, as illustrated in *Figure 3.1*.

b. The system $\begin{cases} x^2 + y^2 = 9 \\ x - y = 1 \end{cases}$ consists of a circle and a line. So, we may expect 0, 1, or 2 solutions. To solve this system, we may want to solve the second equation, for example for x,

$$x = y + 1,$$

and then substitute the resulting expression into the first equation. So, we have

$$\begin{aligned} (y + 1)^2 + y^2 &= 9 \\ y^2 + 2y + 1 + y^2 &= 9 \\ 2y^2 + 2y - 8 &= 0 && / \ \div 2 \\ y^2 + y - 4 &= 0 \\ y = \frac{-1 \pm \sqrt{1 + 4 \cdot 4}}{2} &= \frac{-1 \pm \sqrt{17}}{2} \end{aligned}$$

Then, using substitution, we calculate

$$x = y + 1 = \frac{-1 \pm \sqrt{17}}{2} + 1 = \frac{-1 \pm \sqrt{17} + 2}{2} = \frac{1 \pm \sqrt{17}}{2}.$$

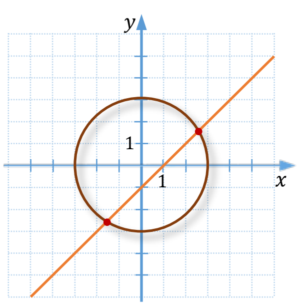


Figure 3.2

So, the solution set consists of two points, $\left(\mathbf{\frac{1-\sqrt{17}}{2}}, \mathbf{\frac{-1-\sqrt{17}}{2}}\right)$ and $\left(\mathbf{\frac{1+\sqrt{17}}{2}}, \mathbf{\frac{-1+\sqrt{17}}{2}}\right)$, as illustrated in *Figure 3.2*.
Their approximations are $(-1.56, -2.56)$ and $(2.56, 1.56)$.

Example 2 ► **Solving Nonlinear Systems of Two Equations by Elimination**

Solve the system of equations $\begin{cases} x^2 + y^2 = 9 \\ 2x^2 - y^2 = -6 \end{cases}$ using elimination.

Solution ► The system consists of a circle and a hyperbola, so we may expect up to four solutions. To solve it, we can start by eliminating the y-variable by adding the two equations, side by side.

$$+\begin{cases} x^2 + y^2 = 9 \\ 2x^2 - y^2 = -6 \end{cases}$$

$$3x^2 = 3 \qquad / \div 3$$

$$x^2 = 1$$

$$x = \pm 1$$

Then, by substituting the obtained x-values into the first equation, we can find the corresponding y-values. So, if $x = 1$, we have

$$1^2 + y^2 = 9 \qquad / -1$$

$$y^2 = 8$$

$$y = \pm\sqrt{8} = \pm 2\sqrt{2}$$

Similarly, if $x = -1$, we have

$$(-1)^2 + y^2 = 9 \qquad / -1$$

$$y^2 = 8$$

$$y = \pm\sqrt{8} = \pm 2\sqrt{2}$$

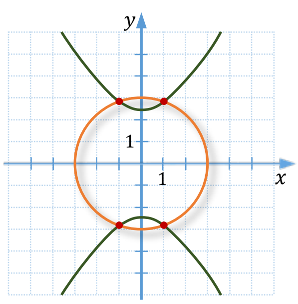


Figure 3.3

So, the solution set is $\{(\mathbf{1}, \mathbf{2\sqrt{2}}), (\mathbf{1}, \mathbf{-2\sqrt{2}}), (\mathbf{-1}, \mathbf{2\sqrt{2}}), (\mathbf{1}, \mathbf{-2\sqrt{2}})\}$. These are the four intersection points of the two curves, circle and hyperbola, as illustrated in *Figure 3.3*.

Nonlinear systems of equations appear in many application problems, especially in the field of geometry, physics, astronomy, astrophysics, engineering, etc.

Example 3 ► **Nonlinear Systems of Two Equations in Applied Problems**

Find the dimensions of a can having a volume V of 250 cubic centimeters and a side area A of 200 square centimeters.

Solution ► Using the formulas for the volume, $V = \pi r^2 h$, and side area, $A = 2\pi r h$, of a cylinder with radius r and height h, we can set up the system of two equations,

$$\begin{cases} \pi r^2 h = 250 \\ 2\pi r h = 200 \end{cases} \qquad / \div 2$$

To solve this system, first, we may want to divide the second equation by 2 and then divide the two equations, side by side. So, we obtain

$$\begin{cases} \underline{\cancel{\pi} r^{\cancel{2}} \cancel{h}} = \underline{250} \\ \cancel{\pi}\cancel{r}\cancel{h} = 100 \end{cases}$$

$$r = 2.5$$

After substituting this value into the equation $2\pi r h = 200$, we can find the corresponding h-value:

$$2\pi(2.5)h = 200 \qquad / \div 5\pi$$

$$h = \frac{200}{5\pi} \approx 12.7$$

So, the can should have a **radius** of **2.5 cm** and a **height** of about **12.7 cm**.

Nonlinear Systems of Inequalities

In *Section G.4* we discussed graphical solutions to linear inequalities and systems of linear inequalities in two variables. Nonlinear inequalities in two variables and systems of such inequalities can be solved using similar graphic techniques.

Example 4 ► **Graphing Solutions to a Nonlinear Inequality**

Graph the solution set of each inequality.

a. $y \geq (x-2)^2 - 3$ **b.** $9x^2 - 4y^2 < 36$

Solution ►

a. First, we graph the related equation of the parabola $y = (x-2)^2 - 3$, using a solid curve. So, we plot the vertex $(2, -3)$ and follow the shape of the basic parabola $y = x^2$, with arms directed upwards.

This parabola separates the plane into two regions, the one above the parabola and the one below the parabola. The inequality $\boldsymbol{y} \geq (x-2)^2 - 3$ indicates that the y-values of the solution points are **above** the parabola $y = (x-2)^2 - 3$. To confirm this observation, we may want to pick a **test point** outside of the parabola and check whether or not it satisfies the inequality. For example, the point (2,0) makes the inequality

$$0 \geq (2-2)^2 - 3 = -3$$

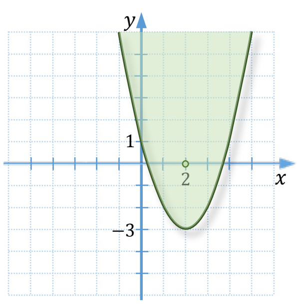


Figure 3.4

a true statement. Thus, the point (2,0) is one of the solutions of the inequality $y \geq (x-2)^2 - 3$ and so are the points of the whole region containing (2,0). To illustrate the solution set of the given inequality, we shade this region, as in *Figure 3.4*. Thus, the solution set consists of all points above the parabola, including the parabola itself.

b. The related equation, $9x^2 - 4y^2 = 36$ represents a hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$ centered at the origin, with a horizontal transverse axis, and with a fundamental rectangle that stretches 2 units horizontally and 3 units vertically apart from the centre. Since the inequality is strong (<), we graph this hyperbola using a dashed line. This indicates that the points on the hyperbola are not among the solutions of the inequality.

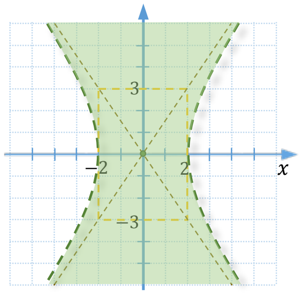


Figure 3.5

To decide which region should be shaded as the set of solutions for the given inequality, we can pick a test point that is easy to calculate, for instance $(0,0)$. Since

$$9 \cdot 0^2 - 4 \cdot 0^2 = 0 < 36$$

is a true statement, the point $(0,0)$ is one of the solutions of the inequality $9x^2 - 4y^2 < 36$, and so are the points of the whole region containing $(0,0)$. Thus, the solution set consists of all the points shaded in green (see *Figure 3.5*), but not the points of the hyperbola itself.

Note: If we chose a test point that does not satisfy the inequality, then the solution set is the region that does not contain this test point.

Example 5 ► Graphing Solutions to a System of Nonlinear Inequalities

Graph the solution set of each system of inequalities.

a. $\begin{cases} x^2 + y < 4 \\ y - x \ge 2 \end{cases}$ **b.** $\begin{cases} \frac{x^2}{9} + \frac{y^2}{16} \le 1 \\ x^2 - y^2 \ge -1 \end{cases}$

Solution ►

a. Observe that the first inequality, $y < -x^2 + 4$, represents the sets of points **below the parabola** $y = -x^2 + 4$. We will shade it in blue, as in *Figure 3.6.*

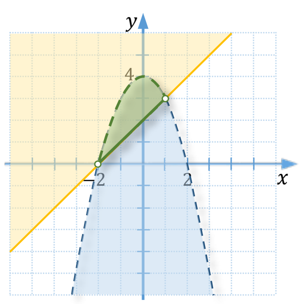


Figure 3.6

The second inequality, $y \ge x + 2$, represents the sets of points **above the line** $y = x + 2$, including the points on the line. We will shade it in yellow, as in *Figure 3.6.*

Thus, the solution set of the system of these inequalities is the **intersection of the blue and yellow region**, as illustrated in green in *Figure 3.6.* The top boundary of the green region is marked by a **dashed line** as these points do not belong to the solution set, and the bottom boundary is marked by a **solid line**, indicating that these points are among the solutions to the system. Also, since the intersection points of the two curves do not satisfy the first inequality, they are not solutions to the system. So, we mark them with **hollow circles**.

b. Observe that the first inequality, $\frac{x^2}{9} + \frac{y^2}{16} \le 1$, represents the sets of points **inside the ellipse** $\frac{x^2}{9} + \frac{y^2}{16} = 1$, including the points on the ellipse. This can be confirmed by testing a point, for instance $(0,0)$. Since $\frac{0^2}{9} + \frac{0^2}{16} = 0 \le 1$ is a true statement, then the region containing the origin is the solution set to this inequality. We will shade it in blue, as in *Figure 3.7.*

The second inequality, $x^2 - y^2 \ge -1$, represents the sets of points **outside the hyperbola** $x^2 - y^2 \ge -1$, including the points on the curve. Again, we can confirm

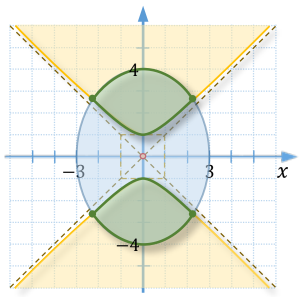


Figure 3.6

this by testing the $(0,0)$ point. Since $0^2 - 0^2 = 0 \geq -1$ is a false statement, then the solution set to this inequality is outside the region containing the origin. We will shade it in yellow, as in *Figure 3.7.*

Thus, the solution set of the system of these inequalities is the **intersection of the blue and yellow region**, as illustrated in green in *Figure 3.7.* The boundary of the green region is marked by a **solid line** as these points satisfy both inequalities and therefore are among the solutions of the system. Since the intersection points of the two curves also satisfy both inequalities, they are solutions of the system as well. So, we mark them using **filled-in circles**.

C.3 Exercises

Concept Check *Suppose that a nonlinear system is composed of equations whose graphs are those described, and the number of points of intersection of the two graphs is as given. Make a sketch satisfying these conditions. (There may be more than one way to do this.)*

1. a line and a circle; one intercept

2. a line and a hyperbola; no intercepts

3. a line and a hyperbola; two intercepts

4. a circle and an ellipse; four intercepts

5. a circle and an ellipse; three intercepts

6. a parabola and a hyperbola; one intercept

7. a parabola and a hyperbola; two intercepts

8. an ellipse and an ellipse; two intercepts

9. an ellipse and a hyperbola; no intercepts

10. an ellipse and a parabola; four intercepts

Concept Check

11. Give the maximum number of points at which the following pairs of graphs can intersect.

a. a line and an ellipse
b. a line and a parabola
c. two different ellipses
d. two different circles with centers at the origin
e. two hyperbolas with centers at the origin

Solve each system.

12. $\begin{cases} y = x^2 + 6x \\ y = 4x \end{cases}$

13. $\begin{cases} y = x^2 + 8x + 16 \\ x - y = -4 \end{cases}$

14. $\begin{cases} xy = 12 \\ x + y = 8 \end{cases}$

15. $\begin{cases} xy = -5 \\ 2x + y = 3 \end{cases}$

16. $\begin{cases} x^2 + y^2 = 2 \\ 2x + y = 1 \end{cases}$

17. $\begin{cases} 2x^2 + 4y^2 = 4 \\ x = 4y \end{cases}$

18. $\begin{cases} x^2 + y^2 = 4 \\ y = x^2 - 2 \end{cases}$

19. $\begin{cases} x^2 + y^2 = 9 \\ y = 3 - x^2 \end{cases}$

20. $\begin{cases} x^2 + y^2 = 4 \\ x + y = 3 \end{cases}$

21. $\begin{cases} x^2 - 2y^2 = 1 \\ x = 2y \end{cases}$

22. $\begin{cases} 3x^2 + 2y^2 = 12 \\ x^2 + 3y^2 = 4 \end{cases}$

23. $\begin{cases} 2x^2 + 3y^2 = 6 \\ x^2 + 3y^2 = 3 \end{cases}$

24. $\begin{cases} (x-4)^2 + y^2 = 4 \\ (x+2)^2 + y^2 = 16 \end{cases}$

25. $\begin{cases} 4x^2 + y^2 = 30 \\ 5x^2 - y^2 = 15 \end{cases}$

26. $\begin{cases} \frac{x^2}{9} - y^2 = -1 \\ \frac{x^2}{16} - \frac{y^2}{4} = 1 \end{cases}$

Analytic Skills *Solve each problem by using a nonlinear system.*

27. Find the length and width of a rectangular room whose perimeter is 50 m and whose area is 100 m^2.

28. A calculator company has determined that the cost y (in thousands) to make x (thousand) calculators is

$$y = 4x^2 + 36x + 20,$$

while the revenue y (in thousands) from the sale of x (thousand) calculators is

$$36x^2 - 3y = 0.$$

Find the **break-even point**, where cost equals revenue.

Concept Check *True or False.*

29. A nonlinear system of equations can have up to four solutions.

30. The solution set of the inequality $x^2 + \frac{y^2}{25} \geq 1$ consists of points outside of the ellipse $x^2 + \frac{y^2}{25} = 1$, including the points of the ellipse.

31. The solution set of the inequality $x^2 + \frac{y^2}{25} < 1$ consists of points inside the ellipse $x^2 + \frac{y^2}{25} = 1$.

32. The intersection points of the curves $x^2 + y^2 = 5$ and $x - y = 3$ belong to the solution set of the system $\begin{cases} x^2 + y^2 \geq 5 \\ x - y > 3 \end{cases}$

33. The solution set of the inequality $y \geq x^2 + 3$ consists of points above or on the parabola $y = x^2 + 3$.

34. The solution set of the inequality $y < x^2 + 3$ consists of points below or on the parabola $y = x^2 + 3$.

Concept Check

35. Fill in each blank with the appropriate response.

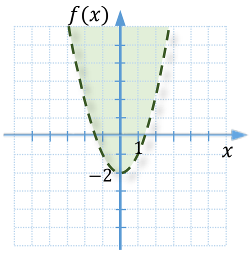
The graph of the system $\begin{cases} x^2 + y^2 < 16 \\ y > -x \end{cases}$ consists of all points $\underset{(outside/inside)}{\underline{\hspace{4cm}}}$ the circle $x^2 + y^2 = 16$ and $\underset{(above/below)}{\underline{\hspace{3cm}}}$ the line $y = -x$.

Concept Check

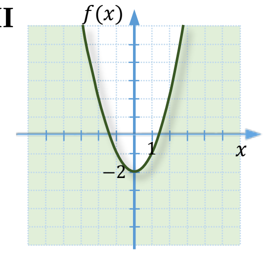
36. Match each inequality with the graph of its solution set.

a. $y \geq x^2 - 2$ **b.** $y \leq x^2 - 2$ **c.** $y > x^2 - 2$ **d.** $y < x^2 - 2$

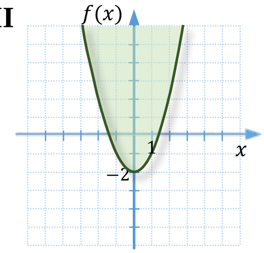
I



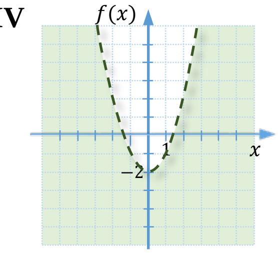
II



III



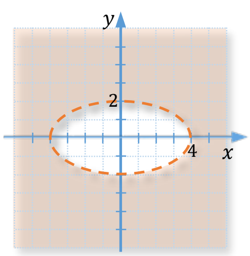
IV



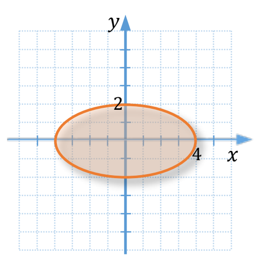
37. Match each inequality with the graph of its solution set.

a. $\frac{x^2}{16} + \frac{y^2}{4} < 1$ **b.** $\frac{x^2}{16} + \frac{y^2}{4} \leq 1$ **c.** $\frac{x^2}{16} + \frac{y^2}{4} \geq 1$ **d.** $\frac{x^2}{16} + \frac{y^2}{4} > 1$

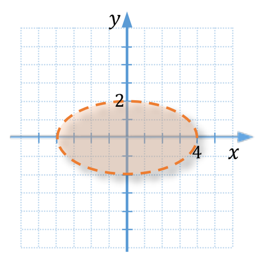
I



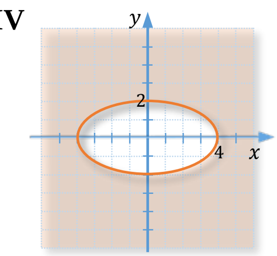
II



III



IV



Graph each nonlinear inequality.

38. $x^2 + y^2 > 9$

39. $(x-1)^2 + (y+2)^2 \leq 16$

40. $y < 2x^2 - 6x$

41. $9x^2 + 4y^2 \geq 36$

42. $4x^2 - y^2 > 16$

43. $y \leq \frac{1}{2}(x+3)^2$

44. $x^2 + 9y^2 < 36$

45. $x^2 - 4 \geq -4y^2$

46. $y \geq x^2 - 8x + 12$

Graph the solution set to each nonlinear system of inequalities.

47. $\begin{cases} x^2 + y^2 < 16 \\ y > -2x \end{cases}$

48. $\begin{cases} y > x^2 - 4 \\ y < -x^2 + 3 \end{cases}$

49. $\begin{cases} x^2 + y^2 \geq 4 \\ x \geq 0 \end{cases}$

50. $\begin{cases} x^2 + y^2 \geq 1 \\ x^2 - 4y^2 \leq 16 \end{cases}$

51. $\begin{cases} x^2 + y^2 < 4 \\ y \geq x^2 + 3 \end{cases}$

52. $\begin{cases} x^2 + 16y^2 > 16 \\ 4x^2 + 9y^2 < 36 \end{cases}$

53. $\begin{cases} x^2 + y^2 \leq 4 \\ (x-2)^2 + y^2 \leq 4 \end{cases}$

54. $\begin{cases} x^2 - y^2 \leq 4 \\ x^2 + y^2 \leq 9 \end{cases}$

55. $\begin{cases} x^2 - y^2 \geq -4 \\ y < 1 - x^2 \end{cases}$

Discussion Point

56. Is it possible for a single point to be the only solution of a system of nonlinear inequalities? If so, give an example of such a system. If not, explain why.