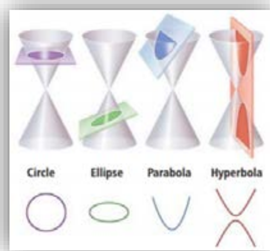
C.2 Equations and Graphs of Conic Sections



In this section, we give an overview of the main properties of the curves called **conic sections**. Geometrically, these curves can be defined as intersections of a plane with a double cone. These intersections can take the shape of a point, a line, two intersecting lines, a circle, an ellipse, a parabola, or a hyperbola, depending on the position of the plane with respect to the cone.

Conic sections play an important role in mathematics, physics, astronomy, and other sciences, including medicine. For instance, planets, comets, and satellites move along conic pathways. Radio telescopes are built with the use of parabolic dishes while reflecting telescopes often contain hyperbolic mirrors. Conic sections are present in both analyzing and constructing many important structures in our world.

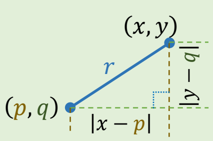
Since lines and parabolas were already discussed in previous chapters, this section will focus on circles, ellipses, and hyperbolas.

Circles

A circle is a conic section formed by the intersection of a cone and a plane parallel to the base of the cone. In coordinate geometry, a circle is defined as follows.

Definition 2.1 ► A **circle** with a fixed **centre** and the **radius** of length $\boldsymbol{r}$ is the set of all points in a plane that lie at the **constant distance** $\boldsymbol{r}$ from this centre.

Equation of a Circle in Standard Form



A **circle** with **centre** $(\boldsymbol{p}, \boldsymbol{q})$ and **radius** $\boldsymbol{r}$ is given by the equation:

$$(\boldsymbol{x} - \boldsymbol{p})^2 + (\boldsymbol{y} - \boldsymbol{q})^2 = \boldsymbol{r}^2$$

In particular, the equation of a circle centered at the origin and with radius $\boldsymbol{r}$ takes the form

$$\boldsymbol{x}^2 + \boldsymbol{y}^2 = \boldsymbol{r}^2$$

Proof: ► Suppose a point (x, y) belongs to the circle with centre (p, q) and radius r. By *definition 2.1*, the distance between this point and the centre is equal to r. Using the distance formula that was developed in *section RD.3*, we have

$$r = \sqrt{(x-p)^2 + (y-q)^2}$$

Hence, after squaring both sides of this equation, we obtain the equation of the circle:

$$r^2 = (x-p)^2 + (y-q)^2$$

Example 1 ► **Finding an Equation of a Circle and Graphing It**

Find an equation of the circle with radius 2 and center at $(0,1)$ and graph it.

Solution ► By substituting $p = 0$, $q = 1$, and $r = 2$ into the standard form of the equation of a circle, we obtain

$$x^2 + (y-1)^2 = 4$$

To graph this circle, we plot the centre $(0,1)$ first, and then plot points that are 2 units apart in the four main directions, East, West, North, and South. The circle passes through these four points, as in *Figure 2.1*.

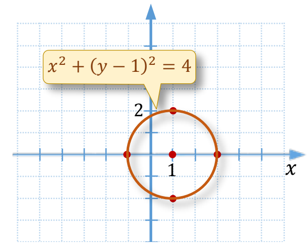


Figure 2.1

Example 2 ► **Graphing a Circle Given Its Equation**

Identify the center and radius of each circle. Then graph it and state the domain and range of the relation.

a. $x^2 + y^2 = 7$ **b.** $(x-3)^2 + (y+2)^2 = 6.25$

c. $x^2 + 4x + y^2 - 2y = 4$

Solution ►

a. The equation can be written as $(x-0)^2 + (y-0)^2 = \left(\sqrt{7}\right)^2$. So, the **centre** of this circle is at $(\mathbf{0}, \mathbf{0})$, and the length of the **radius** is $\sqrt{\mathbf{7}}$. The graph is shown in *Figure 2.2a*.
By projecting the graph onto the x-axis, we observe that the **domain** of this relation is $\left[-\sqrt{\mathbf{7}}, \sqrt{\mathbf{7}}\,\right]$. Similarly, by projecting the graph onto the y-axis, we obtain the **range**, which is also $\left[-\sqrt{\mathbf{7}}, \sqrt{\mathbf{7}}\,\right]$.

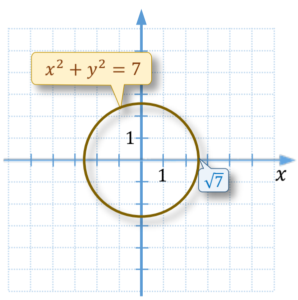


Figure 2.2a

b. The **centre** of this circle is at $(\mathbf{3}, -\mathbf{2})$ and the length of the **radius** is $\sqrt{6.25} = \mathbf{2.5}$. The graph is shown in *Figure 2.2b*. The **domain** of the relation is $[\mathbf{0.5}, \mathbf{5.5}]$, and the **range** is $[-\mathbf{4.5}, \mathbf{0.5}]$.

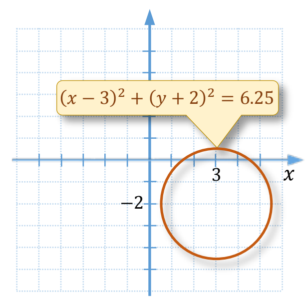


Figure 2.2b

c. The given equation is not in standard form. To rewrite it in standard form, we apply the completing the square procedure to the x-terms and to the y-terms.

$$x^2 + 4x + y^2 - 2y = 4$$
$$(x+2)^2 - 4 + (y-1)^2 - 1 = 4$$
$$(x+2)^2 + (y-1)^2 = 9$$
$$(x+2)^2 + (y-1)^2 = 3^2$$

So, the **centre** of this circle is at $(-\mathbf{2}, \mathbf{1})$ and the length of the **radius** is $\mathbf{3}$. The graph is shown in *Figure 2.2c*. The **domain** of the relation is $[-\mathbf{5}, \mathbf{1}]$ and the **range** is $[-\mathbf{2}, \mathbf{4}]$.

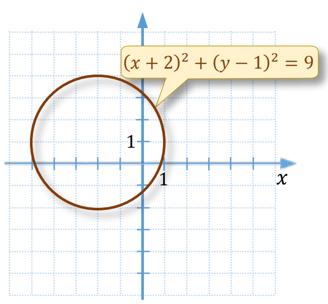
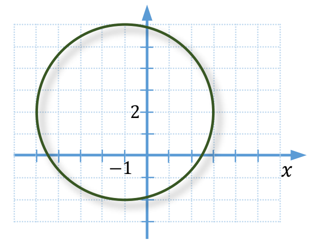


Figure 2.2c

Example 3 ► **Finding Equation of a Circle Given Its Graph**

Determine the equation of the circle shown in the graph.



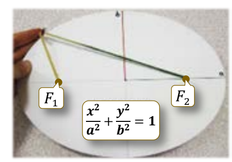
Solution ► Reading from the graph, the centre of the circle is at $(-1,2)$ and the radius is 4. So the equation of this circle is

$$(x+1)^2 + (y-2)^2 = 4^2$$

Ellipses

A conic section formed by the intersection of a cone and a plane slanted to the base but not parallel to the side of the cone is called an ellipse. In coordinate geometry, an ellipse is defined as follows.

Definition 2.2 ► An **ellipse** is the set of points in a plane with a constant *sum* of distances from two fixed points. These fixed points are called **foci** (*singular:* **focus**). The point halfway between the two foci is called the **center** of the ellipse.

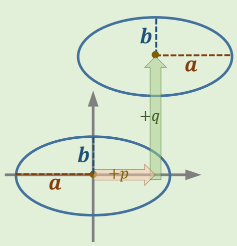


An ellipse has an interesting property of reflection.

Reflecting Property of an Ellipse

When a ray of light or sound emanating from one focus of an ellipse bounces off the ellipse, it passes through the other focus.

Equation of an Ellipse in Standard Form



An **ellipse** with its **centre** at the origin, **radius along the x-axis** (r_x) of length a, and **radius along the y-axis** (r_y) of length b is given by the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

An **ellipse** with its **centre** at the point (p, q), **radius along the x-axis** (r_x) of length a, and **radius along the y-axis** (r_y) of length b is given by the equation:

$$\frac{(x-p)^2}{a^2} + \frac{(y-q)^2}{b^2} = 1$$

Note: A circle is a special case of an ellipse, where $a = b = r$.

Example 4 ► Graphing an Ellipse Given Its Equation

Identify the center and the two radii of each ellipse. Then graph it and state the domain and range of the relation.

a. $9x^2 + y^2 = 9$ **b.** $\frac{(x-1)^2}{16} + \frac{(y+2)^2}{4} = 1$

Solution ►

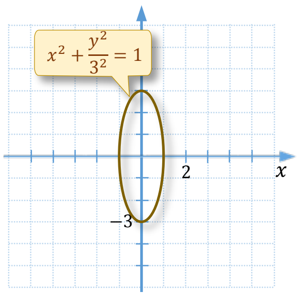


Figure 2.3a

a. First, we may want to change the equation to its standard form. This can be done by dividing both sides of the given equation by 9, to make the right side equal to 1. So, we obtain

$$x^2 + \frac{y^2}{9} = 1$$

or equivalently,

$$x^2 + \frac{y^2}{3^2} = 1$$

Hence, the **centre** of this ellipse is at $(\mathbf{0}, \mathbf{0})$, and the two **radii** are $\boldsymbol{r_x = 1}$ and $\boldsymbol{r_y = 3}$. Thus, we graph this ellipse as in *Figure 2.3a*. The **domain** of the relation is $[\mathbf{-1}, \mathbf{1}]$ and the **range** is $[\mathbf{-3}, \mathbf{3}]$.

b. The given equation can be written as

$$\frac{(x-1)^2}{4^2} + \frac{(y+2)^2}{2^2} = 1$$

So, the **centre** of this ellipse is at $(\mathbf{1}, \mathbf{-2})$ and the two **radii** are $\boldsymbol{r_x = 4}$ and $\boldsymbol{r_y = 2}$. The graph is shown in *Figure 2.3b*. The **domain** of the relation is $[\mathbf{-3}, \mathbf{5}]$, and the **range** is $[\mathbf{-4}, \mathbf{0}]$.

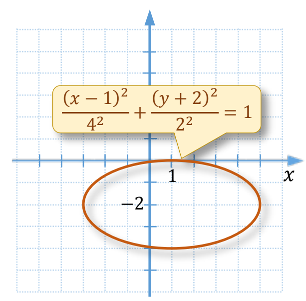
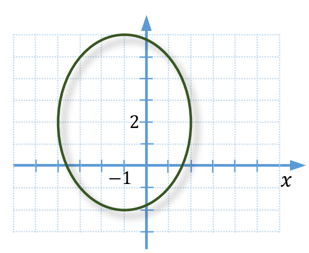


Figure 2.3b

Example 5 ► Finding Equation of an Ellipse Given Its Graph

Give the equation of the ellipse shown in the accompanying graph.



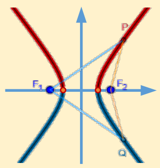
Solution ► Reading from the graph, the centre of the ellipse is at $(-1,2)$, the radius r_x equals 3, and the radius r_y equals 4. So, the equation of this ellipse is

$$\frac{(x+1)^2}{3^2} + \frac{(y-2)^2}{4^2} = 1$$

Hyperbolas

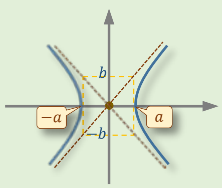
A conic section formed by the intersection of a cone and a plane perpendicular to the base of the cone is called a hyperbola. In coordinate geometry, a hyperbola is defined as follows.

Definition 2.2 ►



A **hyperbola** is the set of points in a plane with a constant absolute value of the *difference* of distances from two fixed points. These fixed points are called **foci** (*singular:* **focus**). The point halfway between the two foci is the **center** of the hyperbola. The graph of a hyperbola consists of two branches and has two axes of symmetry. The axis of symmetry that passes through the foci is called the **transverse** axis. The intercepts of the hyperbola and its transverse are the **vertices** of the hyperbola. The line passing through the centre of the hyperbola and perpendicular to the transverse is the other axis of symmetry, called the **conjugate** axis.

Equation of a Hyperbola in Standard Form

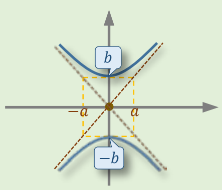


A **hyperbola** with its **centre** at the origin, **transverse axis** on the $\boldsymbol{x}$**-axis**, and **vertices** at $(-\boldsymbol{a}, \mathbf{0})$ and $(\boldsymbol{a}, \mathbf{0})$ is given by the equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

A **hyperbola** with its **centre** at $(\boldsymbol{p}, \boldsymbol{q})$, **horizontal transverse axis**, and **vertices** at $(-\boldsymbol{a}, \mathbf{0})$ and $(\boldsymbol{a}, \mathbf{0})$ is given by the equation:

$$\frac{(x-p)^2}{a^2} - \frac{(y-q)^2}{b^2} = 1$$



A **hyperbola** with its **centre** at the origin, **transverse axis** on the $\boldsymbol{y}$**-axis**, and **vertices** at $(\mathbf{0}, -\boldsymbol{b})$ and $(\mathbf{0}, \boldsymbol{b})$ is given by the equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

A **hyperbola** with its **centre** at $(\boldsymbol{p}, \boldsymbol{q})$, **vertical transverse axis**, and **vertices** at $(\mathbf{0}, -\boldsymbol{b})$ and $(\mathbf{0}, \boldsymbol{b})$ is given by the equation:

$$\frac{(x-p)^2}{a^2} - \frac{(y-q)^2}{b^2} = -1$$

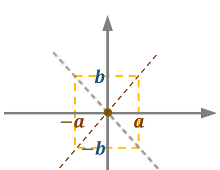


Figure 2.4a

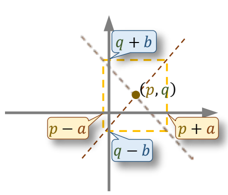


Figure 2.4b

Fundamental Rectangle and Asymptotes of a Hyperbola

The graph of a hyperbola given by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$ is based on a rectangle formed by the lines $x = \pm a$ and $y = \pm b$. This rectangle is called the **fundamental rectangle** (see *Figure 2.4*). The extensions of the diagonals of the fundamental rectangle are the **asymptotes** of the hyperbola. Their equations are $y = \pm \frac{b}{a} x$.

Generally, the **fundamental rectangle** of a hyperbola given by the equation

$$\frac{(x-p)^2}{a^2} - \frac{(y-q)^2}{b^2} = \pm 1$$

is formed by the lines $x = p \pm a$ and $y = q \pm b$. The extensions of the diagonals of this rectangle are the **asymptotes** of the hyperbola.

Example 6 ► **Graphing a Hyperbola Given Its Equation**

Determine the center, transverse axis, and vertices of each hyperbola. Graph the fundamental rectangle and asymptotes of the hyperbola. Then, graph the hyperbola and state its domain and range.

a. $9x^2 - 4y^2 = 36$ **b.** $(x-2)^2 - \frac{(y+1)^2}{4} = -1$

Solution ►

a. First, we may want to change the equation to its standard form. This can be done by dividing both sides of the given equation by 36, to make the right side equal to 1. So, we obtain

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

or equivalently

$$\frac{x^2}{2^2} - \frac{y^2}{3^2} = 1$$

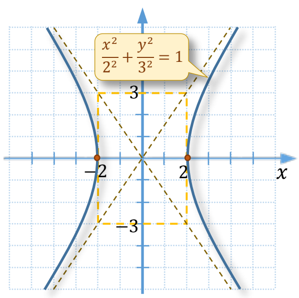


Figure 2.5

Hence, the **centre** of this hyperbola is at $(\mathbf{0}, \mathbf{0})$, and the transverse axis is on the ***x*-axis**. Thus, the vertices of the hyperbola are $(-\mathbf{2}, \mathbf{0})$ and $(\mathbf{2}, \mathbf{0})$.

The **fundamental rectangle** is centered at the origin, and it spans 2 units horizontally apart from the centre and 3 units vertically apart from the centre, as in *Figure 2.5*. The **asymptotes** pass through the opposite vertices of the fundamental rectangle. The final graph consists of two branches. Each of them passes through the corresponding vertex and is shaped by the asymptotes, as shown in *Figure 2.5*.

The **domain** of the relation is $(-\infty, -\mathbf{2}] \cup [\mathbf{2}, \infty)$ and the **range** is $\mathbb{R}$.

b. The equation can be written as

$$\frac{(x-2)^2}{1^2} - \frac{(y+1)^2}{2^2} = -1$$

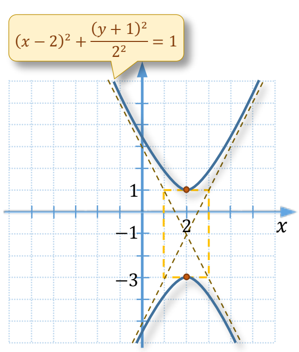


Figure 2.6

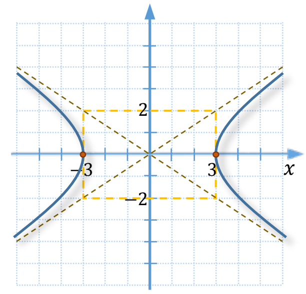
The **centre** of this hyperbola is at $(\mathbf{2}, -\mathbf{1})$. The -1 on the right side of this equation indicates that the **transverse axis is vertical**. Thus, the vertices of the hyperbola are 2 units vertically apart from the centre. So, they are $(\mathbf{2}, -\mathbf{3})$ and $(\mathbf{2}, \mathbf{1})$.

The **fundamental rectangle** is centered at $(\mathbf{2}, -\mathbf{1})$ and it spans 1 unit horizontally apart from the centre and 2 units vertically apart from the centre, as in *Figure 2.6*. The **asymptotes** pass through the opposite vertices of the fundamental box. The final graph consists of two branches. Each of them passes through the corresponding vertex and is shaped by the asymptotes, as shown in *Figure 2.6*.

The **domain** of the relation is $\mathbb{R}$, and the **range** is $(-\infty, -\mathbf{3}] \cup [\mathbf{1}, \infty)$.

Example 7 ► **Finding the Equation of a Hyperbola Given Its Graph**

Give the equation of a hyperbola shown in the accompanying graph.



Solution ► Reading from the graph, the centre of the hyperbola is at (0,0), the transverse axis is the x-axis, and the vertices are $(-3,0)$ and $(3,0)$. The fundamental rectangle spans 2 units vertically apart from the centre. So, we substitute $p = 0$, $q = 0$, $a = 3$, and $b = 2$ to the standard equation of a hyperbola. Thus the equation is

$$\frac{x^2}{3^2} - \frac{y^2}{2^2} = 1$$

Generalized Square Root Functions $f(x) = \sqrt{g(x)}$ for Quadratic Functions $g(x)$

Conic sections are relations but usually not functions. However, we could consider parts of conic sections that are already functions. For example, when solving the equation of a circle

$$x^2 + y^2 = 9$$

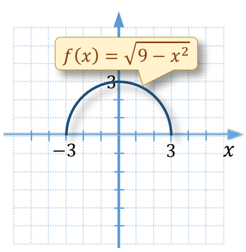
for y, we obtain

$$y^2 = 9 - x^2$$

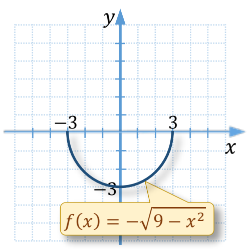
$$|y| = \sqrt{9 - x^2}$$

$$y = \pm\sqrt{9 - x^2}.$$

So, the graph of this circle can be obtained by graphing the two functions: $y = \sqrt{9 - x^2}$ and $y = -\sqrt{9 - x^2}$.



Since the equation $\boldsymbol{y = \sqrt{9 - x^2}}$ describes all the points of the circle with a nonnegative y-coordinate, its graph must be the **top half of the circle** centered at the origin and with the radius of length 3. So, the domain of this function is $[-3,3]$ and the range is $[0,3]$.



Likewise, since the equation $\boldsymbol{y = -\sqrt{9 - x^2}}$ describes all the points of the circle with a nonpositive y-coordinate, its graph must be the **bottom half of the circle** centered at the origin and with the radius of length 3. Thus, the domain of this function is $[-3,3]$ and the range is $[-3,0]$.

Note: Notice that the function $f(x) = \sqrt{9 - x^2}$ is a composition of the square root function and the quadratic function $g(x) = 9 - x^2$. One could prove that the graph of the square root of any quadratic function is the top half of one of the conic sections. Similarly, the graph of the negative square root of any quadratic function is the bottom half of one of the conic sections.

Example 8 ► **Graphing Generalized Square Root Functions**

Graph each function. Give its domain and range.

a. $f(x) = 1 - \sqrt{16 - x^2}$ **b.** $f(x) = 2\sqrt{1 + \frac{x^2}{9}}$

Solution ►

a. To recognize the shape of the graph of function f, let us rearrange its equation first.

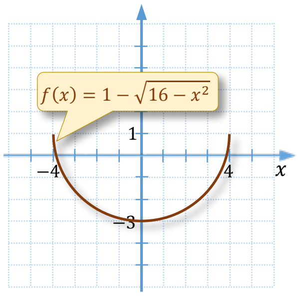


Figure 2.7

$$y = 1 - \sqrt{16 - x^2}$$

$$y - 1 = -\sqrt{16 - x^2}$$

$$(y - 1)^2 = \left(-\sqrt{16 - x^2}\right)^2$$

$$(y - 1)^2 = 16 - x^2$$

$$x^2 + (y - 1)^2 = 16$$

The resulting equation represents a circle with its centre at $(0,1)$ and a radius of 4. So, the graph of $f(x) = 1 - \sqrt{16 - x^2}$ must be part of this circle. Since $y - 1 = -\sqrt{16 - x^2} \le 0$, then $y \le 1$. Thus the graph of function f is the **bottom half** of this **circle**, as shown in *Figure 2.7*.
So, the **domain** of function f is $[\mathbf{-4}, \mathbf{4}]$ and the **range** is $[\mathbf{-3}, \mathbf{1}]$.

b. To recognize the shape of the graph of function f, let us rearrange its equation first.

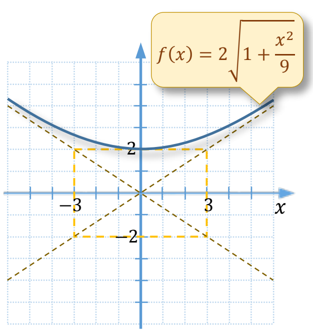


Figure 2.8

$$y = 2\sqrt{1 + \frac{x^2}{9}}$$

$$\frac{y}{2} = \sqrt{1 + \frac{x^2}{9}}$$

$$\left(\frac{y}{2}\right)^2 = 1 + \frac{x^2}{9}$$

$$-1 = \frac{x^2}{9} - \left(\frac{y}{2}\right)^2$$

$$\frac{x^2}{3^2} - \frac{y^2}{2^2} = -1$$

The resulting equation represents a hyperbola centered at the origin, with a vertical transverse axis. Its fundamental rectangle spans horizontally 3 units and vertically 2 units from the centre. Since the graph of $f(x) = 2\sqrt{1 + \frac{x^2}{9}}$ must be a part of this hyperbola and the values $f(x)$ are nonnegative, then its graph is the **top half** of this **hyperbola**, as shown in *Figure 2.8*.
So, the **domain** of function f is $\mathbb{R}$, and the **range** is $[\mathbf{2}, \infty]$.

C.2 Exercises

Vocabulary Check *Complete each blank with the most appropriate term or phrase from the given list:* ***ellipse, circle, hyperbola, center, fundamental rectangle, transverse, focus, conic.***

1. The set of all points in a plane that are equidistant from a fixed point is a ___________ .

2. The set of all points in a plane with a constant sum of their distances from two fixed points is an _____________ .

3. The set of all points in a plane with a constant difference between the distances from two fixed points is a _______________.

4. The _________ of a hyperbola is the point that lies halfway between the vertices of this hyperbola.

5. The asymptotes of a hyperbola pass through the opposite vertices of the _____________________________ of this hyperbola.

6. The _____________ axis of a hyperbola passes through the two vertices of the hyperbola.

7. A ray of light emanated from one focus of an ellipse passes through the other _________.

8. The graph of a square root of a quadratic function is the top or the bottom half of one of the __________ sections.

Concept Check *True or false.*

9. A circle is a set of points, where the center is one of these points.

10. If the foci of an ellipse coincide, then the ellipse is a circle.

11. The x-intercepts of $\frac{x^2}{9}+\frac{y^2}{4}=1$ are $(-9,0)$ and $(9,0)$.

12. The graph of $2x^2+y^2=1$ is an ellipse.

13. The y-intercepts of $x^2+\frac{y^2}{3}=1$ are $(-\sqrt{3},0)$ and $(\sqrt{3},0)$.

14. The graph of $y^2=1-x^2$ is a hyperbola centered at the origin.

15. The transverse axis of the hyperbola $-y^2=1-x^2$ is the x-axis.

Find the equation of a circle satisfying the given conditions.

16. centre at $(-1,-2)$; radius 1

17. centre at $(3,1)$; radius $\sqrt{3}$

18. centre at $(2,-1)$; diameter 6

19. centre at $(-2,2)$; diameter 5

*Find the **center** and **radius** of each circle.*

20. $x^2 + y^2 + 4x + 6y + 9 = 0$

21. $x^2 + y^2 - 8x - 10y + 5 = 0$

22. $x^2 + y^2 + 6x - 16 = 0$

23. $x^2 + y^2 - 12x + 12 = 0$

24. $2x^2 + 2y^2 + 20y + 10 = 0$

25. $3x^2 + 3y^2 - 12y - 24 = 0$

*Identify the **center** and **radius** of each circle. Then graph the relation and state its **domain** and **range**.*

26. $x^2 + (y - 1)^2 = 16$

27. $(x + 1)^2 + y^2 = 2.25$

28. $(x - 2)^2 + (y + 3)^2 = 4$

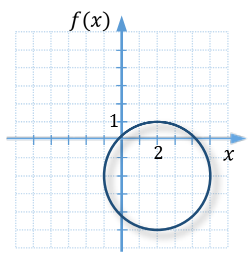
29. $(x + 3)^2 + (y - 2)^2 = 9$

30. $x^2 + y^2 + 2x + 2y - 23 = 0$

31. $x^2 + y^2 + 4x + 2y + 1 = 0$

Concept Check *Use the given graph to determine the equation of the **circle**.*

32.



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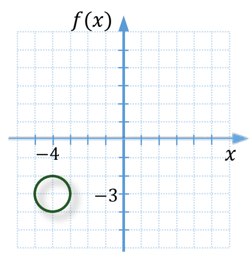

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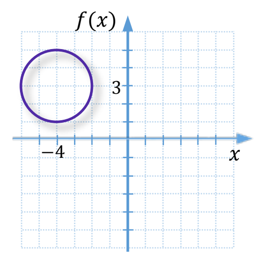
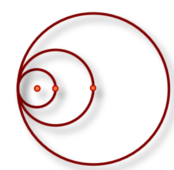


Discussion Point

35. The equation of the smallest circle shown is $x^2 + y^2 = r^2$. What is the equation of the largest circle?



Concept Check

*Identify the **center** and the horizontal ($\boldsymbol{r_x}$) and vertical ($\boldsymbol{r_y}$) **radii** of each ellipse. Then graph the relation and state its **domain** and **range**.*

36. $\frac{x^2}{4} + (y - 1)^2 = 1$

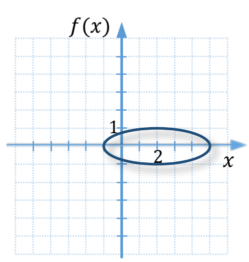
37. $(x + 1)^2 + \frac{y^2}{9} = 1$

38. $\frac{(x-2)^2}{16} + \frac{(y+3)^2}{4} = 1$

39. $\frac{(x-4)^2}{4} + \frac{(y-2)^2}{9} = 1$

Concept Check *Use the given graph to determine the equation of the **ellipse**.*

40.



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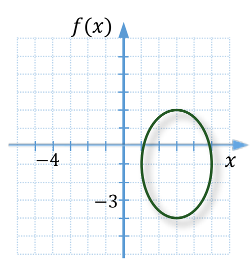

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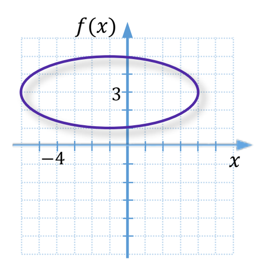


Concept Check *Identify the* ***center*** *and the* ***transverse axis*** *of each hyperbola. Then graph the* ***fundamental box*** *and the* ***hyperbola****. State the* ***domain*** *and* ***range*** *of the relation.*

43. $\frac{x^2}{4} - (y-1)^2 = 1$

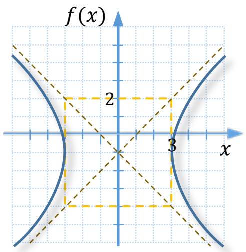
44. $(x+1)^2 + \frac{y^2}{9} = -1$

45. $\frac{(x-3)^2}{4} - \frac{(y+2)^2}{4} = -1$

46. $\frac{(x+2)^2}{9} - \frac{(y-1)^2}{9} = 1$

Concept Check *Use the given graph to determine the equation of the* ***hyperbola****.*

47.



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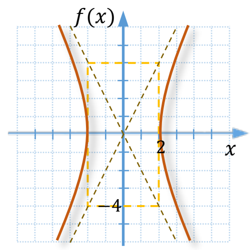

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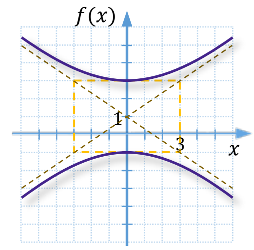


Concept Check

50. Match each equation with its graph.

a. $\frac{x^2}{9} + \frac{y^2}{16} = 1$

b. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

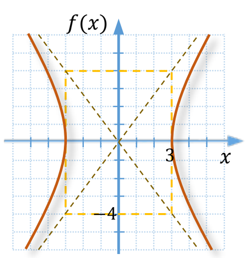
c. $\frac{x^2}{9} - \frac{y^2}{16} = -1$

d. $\frac{x^2}{16} - \frac{y^2}{9} = 1$

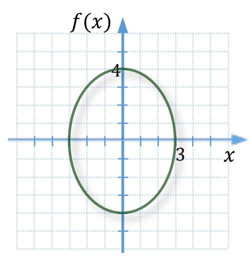
e. $\frac{x^2}{9} + \frac{y^2}{9} = 1$

f. $\frac{x^2}{16} + \frac{y^2}{9} = 1$

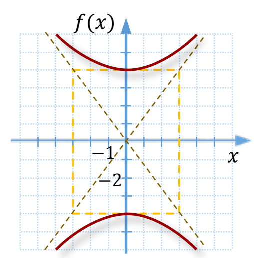
I



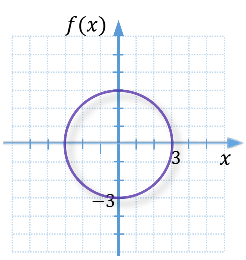
II



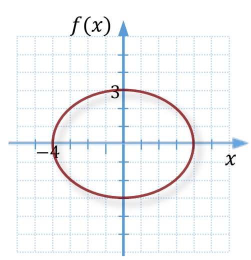
III



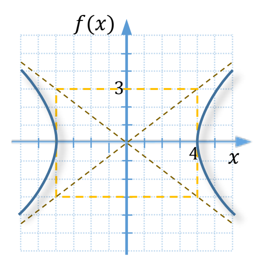
IV



V



VI



Graph each ***generalized square root*** *function. Give the* ***domain*** *and* ***range****.*

51. $f(x) = \sqrt{4 - x^2}$

52. $f(x) = -\sqrt{25 - x^2}$

53. $f(x) = -2\sqrt{1 - \frac{x^2}{9}}$

54. $f(x) = 3\sqrt{1 - \frac{x^2}{4}}$

55. $\frac{y}{3} = \sqrt{x^2 - 1}$

56. $\frac{y}{2} = -\sqrt{1 + \frac{x^2}{9}}$

Analytic Skills *Solve each problem.*

57. The arch under a bridge is designed as the upper half of an ellipse as illustrated in the accompanying figure. Assuming that the ellipse is modeled by $25x^2 + 144y^2 = 3600$, where x and y are in meters, find the width and height of the arch (*above the yellow line*).

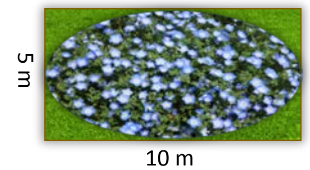
58. Suppose a power outage affects all homes and businesses within a 5-km radius of the power station.

a. If the power station is located 2 km east and 6 km south of the center of town, find an equation of the circle that represents the boundary of the power outage.

b. Will a mall located 4 km east and 4 km north of the power station be affected by the outage?

59. Two buildings in a sports complex are shaped and positioned like a portion of the branches of the hyperbola with equation $400x^2 - 625y^2 = 250{,}000$, where x and y are in meters. How far apart are the buildings at their closest point?

60. The area of an ellipse is given by the formula $A = \pi ab$, where a and b are the two radii of the ellipse.



a. To the nearest tenth of a square meter, find the area of the largest elliptic flower bed that fits in a rectangular space that is 5 meters wide and 10 meters long, as shown in the accompanying figure.

b. Assuming that each square meter of this flower bed is filled with 25 plants, approximate the number of plants in the entire flower bed.