Additional Functions, Conic Sections, and Nonlinear Systems

Relations and functions are an essential part of mathematics as they allow to describe interactions between two or more variable quantities. In this chapter, we will give a quick overview of the commonly used functions, their properties such as domain and range, and some of their transformations, particularly translations. Some of these functions (i.e., linear, quadratic, square root, and reciprocal functions) were already discussed in detail in *sections G1*, *Q3*, *RD1*, and *RT5*. In *section C1*, we will explore some additional functions, such as absolute value or greatest integer functions, as well as functions of the form $\frac{1}{f(x)}$ or $|f(x)|$.

Aside from new functions, we will discuss equations and graphs of commonly used relations such as circles, ellipses, and hyperbolas. These relations are known as conic sections as their graphs have the shape of a curve formed by the intersection of a cone and a plane. Conic sections are geometric representations of quadratic equations in two variables and as such, they include parabolas. Thus, studying conic sections is an extension of studying parabolas. When working with conic sections, we are often in need of finding intersection points of given curves. Thus, at the end of this chapter, we will discuss solving systems of nonlinear equations as well as nonlinear inequalities.

C.1 Properties and Graphs of Additional Functions

The graphs of some **basic functions**, such as

$$f(x) = x^2,\ f(x) = |x|, f(x) = \sqrt{x}, \text{ or } f(x) = \frac{1}{x},$$

were already presented throughout this text. Knowing the shapes of the graphs of these functions is very useful for graphing related functions, such as $g(x) = |x| - 2$ or $f(x) = \sqrt{x+1}$. In *section Q3*, we observed that the graph of function $g(x) = (x-p)^2 + q$ could be obtained by translating a graph of the basic parabola p units horizontally and q units vertically. This observation applies to any function $f(x)$.

To graph a function $\boldsymbol{f(x-a)+b}$, it is enough to translate the graph of $\boldsymbol{f(x)}$ by ***a* units horizontally** and ***b* units vertically**.

Examine the relations between the defining formula of a function and its graph in the following examples.

Basic Functions and Their Translations

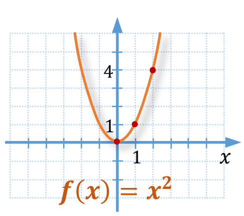


Figure 1.1a

Parabola $f(x) = x^2$

Recall the shape of the graph of the basic parabola $f(x) = x^2$, as in *Figure 1.1a*. The domain of this function is $\mathbb{R}$, and the range is the interval $[0, \infty)$.

The graph of the basic parabola can be used to graph other quadratic functions such as $\boldsymbol{g(x) = (x-3)^2 - 1}$. Function g

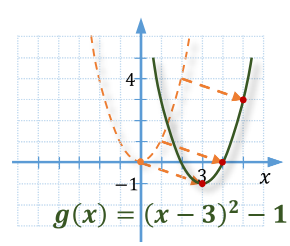


Figure 1.1b

can be graphed by translating the basic parabola **3 units to the right** and **1 unit down**, as in *Figure 1.1b*.

Observe that under this translation,

- the vertex $(0,0)$ of the basic parabola is moved to the **vertex** $(\mathbf{3},-\mathbf{1})$ of function g;
- the **domain** of function g remains unchanged, and it is still $\mathbb{R}$;
- the **range** of function g is the interval $[-\mathbf{1},\infty)$ as a result of the translation of the range $[0,\infty)$ of the basic parabola by 1 unit down.

Absolute Value $f(x) = |x|$

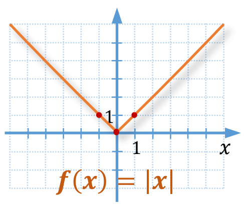


Figure 1.2a

x	$f(x)$
-2	2
-1	1
0	0
1	1
2	2

Using a table of values, we can graph the basic absolute value function, $f(x) = |x|$, as in *Figure 1.2a*. The domain of this function is $\mathbb{R}$, and the range is the interval $[0,\infty)$. Similarly as in the case of the basic parabola, the lowest point, called the vertex, is at $(0,0)$.

The graph of the basic absolute value function can be used to graph other absolute value functions such as $\boldsymbol{g(x) = |x+1| + 2}$. Function g can be graphed by translating function $f(x) = |x|$ by **1 unit to the left** and **2 units up**, as in *Figure 1.2b*.

x	$g(x)$
-3	4
-2	3
-1	2
0	3
1	4

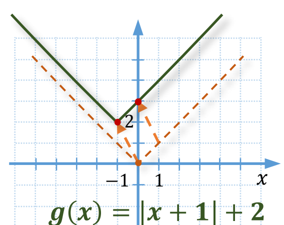


Figure 1.2b

Observe that under this translation,

- the vertex $(0,0)$ of function f is moved to the **vertex** $(-\mathbf{1},\mathbf{2})$ of function g;
- the **domain** of function g remains unchanged and it is still $\mathbb{R}$;
- the **range** of function g is the interval $[\mathbf{2},\infty)$, as a result of the translation of the range $[0,\infty)$ of function f by 2 units up.

Square Root $f(x) = \sqrt{x}$

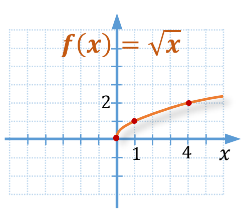


Figure 1.3a

x	$f(x)$
-2	2
-1	1
0	0
1	1
2	2

Using a table of values, we can graph the basic square root function, $f(x) = \sqrt{x}$, as in *Figure 1.3a*. The domain of this function is the interval $[0,\infty)$, and the range is also $[0,\infty)$. The curve starts at the origin $(0,0)$.

The graph of the basic square root function can be used to graph other square root functions such as $\boldsymbol{g(x) = \sqrt{x+1} - 2}$. Function g can be graphed by translating function $f(x) = \sqrt{x}$ by **1 unit to the left** and **2 units down**, as in *Figure 1.3b*.

x	$g(x)$
-2	2
-1	1
0	0
1	1
2	2

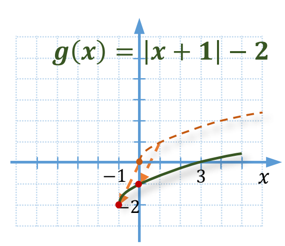


Figure 1.3b

Observe that under this translation,

- the initial point $(0,0)$ of function f is moved to the **initial point** $(-\mathbf{1},\mathbf{2})$ of function g;
- the **domain** of function g is moved to $[-\mathbf{1},\infty)$, by subtracting 1 from all domain values $[0,\infty)$ of function f;
- the **range** of function g is moved to $[-\mathbf{2},\infty)$, by subtracting 2 from all range values $[0,\infty)$ of function f.

Cubic $f(x) = x^3$

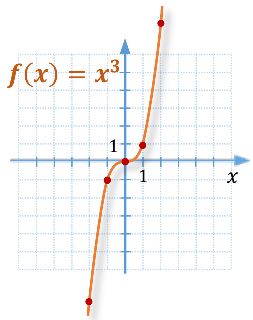


Figure 1.4a

x	$f(x)$
-2	-8
-1	-1
$-\frac{1}{2}$	$-\frac{1}{8}$
0	1
$\frac{1}{2}$	$\frac{1}{8}$
1	1
2	8

x	$g(x)$
-2	2
-1	1
0	0
1	1
2	2

Using a table of values, we can graph the basic cubic function, $f(x) = x^3$, as in *Figure 1.4a*. The domain and range of this function are both $\mathbb{R}$. The curve is symmetric about the origin $(0,0)$.

The graph of the basic cubic function can be used to graph other cubic functions such as $\boldsymbol{g(x) = (x-3)^3 - 2}$. Function g can be graphed by translating function $f(x) = x^3$ by **3 units to the right** and **2 units down**, as in *Figure 1.4b*.

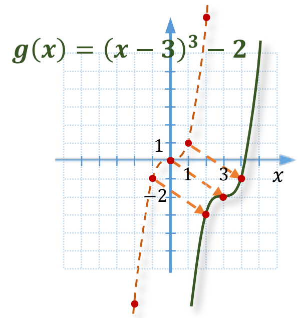


Figure 1.4b

Observe that under this translation,

- the central point $(0,0)$ of function f is moved to the **central point** $(\mathbf{3}, \mathbf{-2})$ of function g;
- the **domain** of function g remains unchanged, and it is still $\mathbb{R}$;
- the **range** of function g remains unchanged, and it is still $\mathbb{R}$.

Reciprocal $f(x) = \frac{1}{x}$

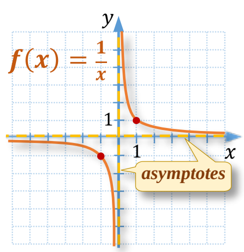


Figure 1.5a

x	$f(x)$
-4	$-\frac{1}{4}$
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
0	*undefined*
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
4	$\frac{1}{4}$

x	$g(x)$
0	$\frac{1}{2}$
1	0
$\frac{3}{2}$	-1
2	*undefined*
$\frac{5}{2}$	3
3	2
4	$\frac{3}{2}$

Using a table of values, we can graph the basic reciprocal function, $f(x) = \frac{1}{x}$, as in *Figure 1.5a*. The domain and range of this function is the set of all real numbers except for zero, $\mathbb{R} \setminus \{\mathbf{0}\}$. The graph consists of two curves that are approaching two asymptotes, the horizontal asymptote $y = 0$ and the vertical asymptote $x = 0$.

The graph of the basic reciprocal function can be used to graph other reciprocal functions such as $g(x) = \frac{1}{x-2} + 1$. Function g can be graphed by translating function $f(x) = \frac{1}{x}$ by 2 units to the right and 1 unit up, as in *Figure 1.5b*.

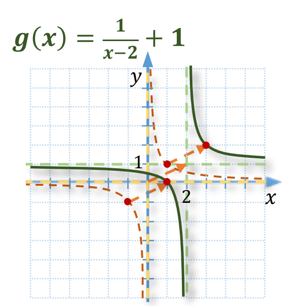


Figure 1.5b

Observe that under this translation,

- the **horizontal asymptote** of function f is moved **1 unit up**, and the **vertical asymptote** of function f is moved **2 units to the right**;
- the **domain** of function g is moved to $\mathbb{R} \setminus \{\mathbf{2}\}$, by adding 2 to all domain values $\mathbb{R} \setminus \{0\}$ of function f;
- the **range** of function g is moved to $\mathbb{R} \setminus \{\mathbf{1}\}$, by adding 1 to all range values $\mathbb{R} \setminus \{0\}$ of function f.

Greatest Integer $f(x) = [\![x]\!]$

Definition 1.1 ► The greatest integer, denoted $[\![x]\!]$, of a real number x is the **greatest integer that does not exceed this number** x. For example,

$$[\![0.9]\!] = 0, \quad [\![1]\!] = 1, \quad [\![1.1]\!] = 1, \quad [\![1.9]\!] = 1$$

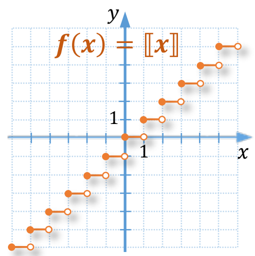


Figure 1.6a

x	$f(x)$
−1.5	−2
−1.1	−2
−1	−1
−0.5	−1
−0.1	−1
0	0
0.5	0
0.9	0
1	1
1.2	1
2	2

x	$g(x)$
−0.5	1
0	2
0.5	2
1	3
1.5	3

Using a table of values, we can graph the basic greatest integer function, $f(x) = [\![x]\!]$, as in *Figure 1.6a*. The domain of this function is the set of real numbers $\mathbb{R}$ while the range is the set of integers $\mathbb{Z}$. The graph consists of infinitely many half-open segments that line up along the diagonal, $y = x$.

The graph of the basic greatest integer function can be used to graph other greatest integer functions such as $\boldsymbol{g(x) = [\![x]\!] + 2}$. Function g can be graphed by translating function $f(x) = [\![x]\!]$ by 1 unit up, as in *Figure 1.6b*.

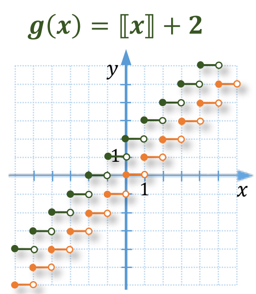


Figure 1.6b

Observe that under this translation,

- the segments of the graph g line up along the line $y = x + 2$;
- the **domain** of function g remains unchanged, and it is still $\mathbb{R}$;
- the **range** of function g remains unchanged, and it is still the set of all integers $\mathbb{Z}$;

Other Transformations of Basic Functions

Aside from translating, graphs can be transformed by flipping them along x- or y-axis, or stretching or shrinking (dilating) in different directions. In the next two examples, observe the relation between the defining formula of a function and the graph transformation of the corresponding basic function.

Example 1 ► **Graphing Functions and Identifying Transformations**

Graph each function. State the transformation(s) of the corresponding basic function that would result in the obtained graph. Then, describe the main properties of the function, such as domain, range, vertex, asymptotes, and symmetry, if applicable.

a. $f(x) = -\frac{1}{x}$ **b.** $f(x) = 2[\![x]\!]$ **d.** $f(x) = \frac{1}{2}|x| - 3$

Solution ► **a.** To graph $f(x) = -\frac{1}{x}$, we first observe that this is a modified reciprocal function. So, we expect that the graph might have some asymptotes.

Since we cannot divide by zero, $x = 0$ does not belong to the domain of this function. This suggests that the graph may have a vertical asymptote, $x = 0$. Also, since the numerator of the fraction $-\frac{1}{x}$ is never equal to zero, then function $f(x) = -\frac{1}{x}$ would never assume the value of zero. So, zero is out of the range of this function. This suggests that the graph may have a horizontal asymptote, $y = 0$.

After graphing the two asymptotes and plotting a few points of the graph, we obtain the final graph, as in *Figure 1.7*.

x	$f(x)$
-2	$\frac{1}{2}$
-1	1
$-\frac{1}{2}$	2
0	*undefined*
$\frac{1}{2}$	-2
1	-1
2	$-\frac{1}{2}$

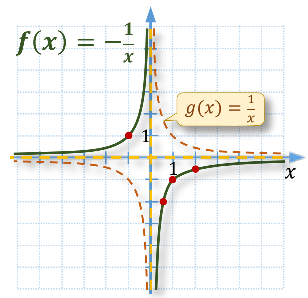


Figure 1.7

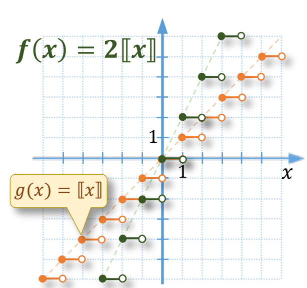
Notice that the graph of function $f(x) = -\frac{1}{x}$ could be obtained by **reflecting** the graph of the basic reciprocal function, $g(x) = \frac{1}{x}$, in the ***x*-axis**.

Function f has the following properties:
Domain: $\mathbb{R} \setminus \{\mathbf{0}\}$
Range: $\mathbb{R} \setminus \{\mathbf{0}\}$
Equations of asymptotes: $\boldsymbol{x = 0},\ \boldsymbol{y = 0}$
Symmetry: The graph is **symmetrical with respect to the origin**.

b. To graph $f(x) = 2[\![x]\!]$, first, we observe that this is a modified greatest integer function. So, we expect that the graph will consist of half-open segments that line up along a certain line.

Notice that for every x, the value of function f is obtained by multiplying the corresponding value of function $g(x) = [\![x]\!]$ by the factor of 2. Since the segments of the graph of the basic greatest integer function line up along the line $y = x$, we may predict that the segments of the graph of function $f(x) = 2[\![x]\!]$ would line up along the line $y = 2x$. This can be confirmed by calculating and plotting a sufficient number of points, as below.

x	$f(x)$
-0.5	-2
0	0
0.5	0
1	2
1.9	2
2	4



Notice that the graph of function $f(x) = 2[\![x]\!]$ could be obtained by **stretching** the graph of the basic greatest integer function, $g(x) = [\![x]\!]$, in $\boldsymbol{y}$**-axis** by a factor of **2**.

Function f has the following properties:
Domain: $\mathbb{R}$
Range: $\mathbb{Z}$
The segments of the graph line up along the line $y = 2x$.

c. To graph $f(x) = \frac{1}{2}|x| - 3$, first, we observe that this is a modified absolute value function. So, we expect a "V" shape for its graph.

Notice that for every x, the value of function f is obtained by multiplying the corresponding value of the basic absolute value function $g(x) = |x|$ by a factor of $\frac{1}{2}$, and then subtracting 3. Observe how these operations impact the vertex $(0,0)$ of the basic "V" shape. Since the y-value of the vertex is zero, multiplying it by a factor $\frac{1}{2}$ does not change its position. However, subtracting 3 from the y-value of zero causes the vertex to move to $(0, -3)$.

After plotting the vertex and a few more points, as computed in the table below, we obtain the final graph, as illustrated in *Figure 1.8*.

$\boldsymbol{x}$	$\boldsymbol{f(x)}$
$\mathbf{-2}$	$\frac{1}{2}$
$\mathbf{-1}$	1
$\mathbf{-\frac{1}{2}}$	2
$\mathbf{0}$	*undefined*
$\mathbf{\frac{1}{2}}$	-2
$\mathbf{1}$	-1
$\mathbf{2}$	$-\frac{1}{2}$

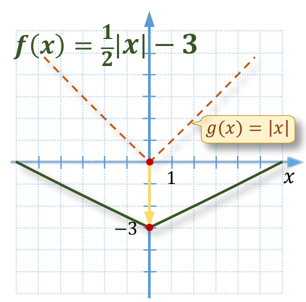


Figure 1.8

Notice that the obtained shape is wider than the shape of the basic absolute value graph. This is because the slopes of the linear sections of the graph are half as steep. So, the function $f(x) = \frac{1}{2}|x| - 3$ could be obtained by

- **compressing** the graph of the basic absolute value function, $g(x) = |x|$, in $\boldsymbol{y}$**-axis** by a factor of $\frac{1}{2}$, and then
- **translating** the resulting graph by **3 units down**.

Function f has the following properties:
Domain: $\mathbb{R}$
Range: $[\mathbf{-3}, \infty)$
Vertex: $(\mathbf{0}, \mathbf{-3})$
Symmetry: The graph is **symmetrical with respect to the** $\boldsymbol{y}$**-axis**.

Generally, to graph a function $\boldsymbol{kf(x)}$, it is enough to **dilate** the graph of $\boldsymbol{f(x)}$, $\boldsymbol{k}$ times in **y-axis**. This dilation is a

- **stretching**, if $|\boldsymbol{k}| > \mathbf{1}$
- **compressing**, if $\mathbf{0} < |\boldsymbol{k}| < \mathbf{1}$
- **flipping** over the $\boldsymbol{x}$**-axis**, if $\boldsymbol{k} = \mathbf{-1}$

Functions of the form $\frac{1}{f(x)}$ or $|f(x)|$

Consider the graphs of a linear function, $f(x) = x + 2$, and its reciprocal, $g(x) = \frac{1}{x+2}$, as illustrated in *Figure 1.9*.

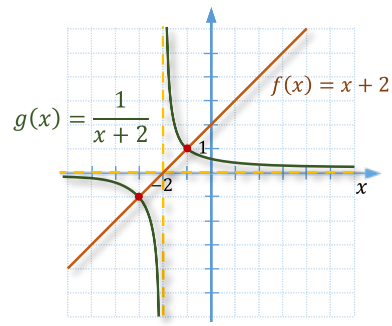


Figure 1.9

Notice that:

- the reciprocal function (in green) has its vertical asymptote at the x-intercept of the linear function (in orange);
- the horizontal asymptote of the reciprocal function $g(x) = \frac{1}{x+2}$ is the x-axis, $y = 0$;
- the points with the y-coordinate equal to 1 or -1 are common for both functions;
- the reciprocal of values close to zero are far away from zero while the reciprocals of values that are far away from zero are close to zero;
- the values of the reciprocal function are of the same sign as the corresponding values of the linear function.

$\boldsymbol{f(x) \to \frac{1}{f(x)}}$

Generally, the graph of the reciprocal of a linear function, $\boldsymbol{g(x) = \frac{1}{ax+b}}$, has the $\boldsymbol{x}$**-axis** as its **horizontal asymptote** and $\boldsymbol{y = -\frac{b}{a}}$ as its **vertical asymptote**.

Now, consider the graphs of the quadratic function $f(x) = x^2 + x - 2$ and the absolute value of this function $g(x) = |x^2 + x - 2|$, as illustrated in *Figure 1.10*.

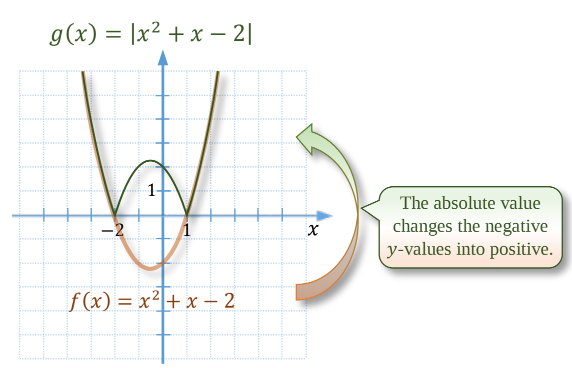


Figure 1.10

Notice that the absolute value function g (in green) follows the original function f (in orange) wherever function f assumes positive values. Otherwise, function g assumes opposite values to function f. So, the graph of the absolute value function g can be obtained by flipping the negative section (section below the x-axis) of the graph of the quadratic function f over the x-axis and leaving the positive sections (above the x-axis) unchanged.

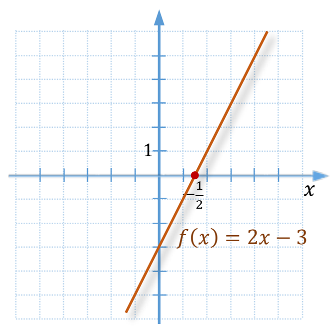
$f(x) \to |f(x)|$

Generally, the graph of the absolute value of any given function $f(x)$, $\boldsymbol{g(x) = |f(x)|}$, can be obtained by **flipping the section(s)** of the graph of the original function f **below the x-axis over the x-axis** and leaving the section(s) above the x-axis unchanged.

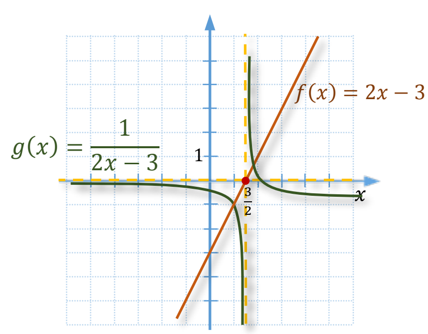
Example 2 ► **Graphing the Reciprocal of a Linear Function**

Using the graph of function $f(x) = 2x - 3$, graph function $g(x) = \frac{1}{f(x)} = \frac{1}{2x+3}$. Determine the x-intercept of function f and the equation of the vertical asymptote of function g.

Solution ► First, we graph function $f(x) = 2x - 3$ as below.



Then, we plot a few 'reciprocal' points. For example, since point $(0, -3)$ belongs to function f, then point $\left(0, -\frac{1}{3}\right)$ must belong to function g. Notice that points $(1, -1)$ and $(2,1)$ are common to both functions, as the reciprocals of -1 and 1 are the same numbers -1 and 1. The graph of function g arises by joining the obtained 'reciprocal' points, as illustrated below.



The equation of the vertical asymptote of the graph of function g is $x = \frac{3}{2}$, and it crosses the x-axis at the x-intercept of function f, which is $\left(\frac{3}{2}, 0\right)$.

Example 3 ► **Graphing the Absolute Value of a Given Function**

Using the graph of function $f(x) = 2x - x^2$, graph function $g(x) = |f(x)| = |2x - x^2|$.

Solution ► To graph function $g(x) = |2x - x^2|$, we may graph function $f(x) = 2x - x^2$ first. Since $2x - x^2 = x(2 - x)$ then the x-intercepts of this parabola are at $x = 0$ and $x = 2$. The first coordinate of the vertex is the average of the two intercepts, so it is 1. Since $f(1) = 1$, then the parabola has its vertex at the point (1,1). So, the graph of function f can be obtained by connecting the two intercepts and the vertex with a parabolic curve. See the orange graph in *Figure 1.11*.

Since $|f(x)| = \begin{cases} f(x) & if \quad f(x) > 0 \\ -f(x) & if \quad f(x) < 0 \end{cases}$, then the graph of function $g(x) = |f(x)|$ is obtained by

- following the (orange) graph of f for the parts where this graph is above the x-axis and
- flipping the parts of the orange graph that lie below the x-axis over the x-axis, as illustrated in green in *Figure 1.11*.

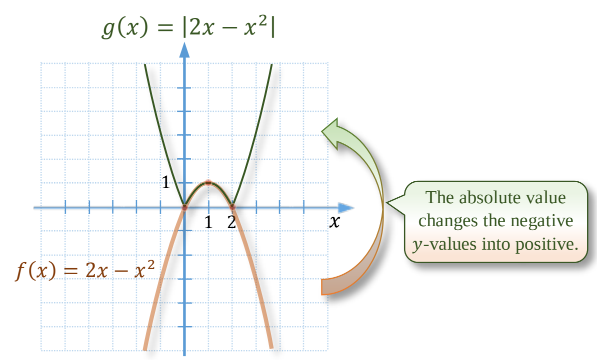


Figure 1.11

Step Function in Applications

The greatest integer function, $[\![x]\!]$, is an example of a larger class of functions, called **step functions**.

Definition 1.2 ► A **step function** is a function whose graph consists of a series of horizontal line segments with jumps in-between them. The line segments can be half-open, open, or closed.

A step function is a constant function on given intervals. However, the value of this function is different for each interval. For example, the function defined as follows:

$f(x) = 1$ for all x-values from the interval $[-2, 1)$,

$f(x) = 2$ for all x-values from the interval $[1, 3)$,

$f(x) = 4$ for all x-values from the interval $[3, 4]$, is a step function with a staircase-like graph, as illustrated in *Figure 1.12*. Such function can be defined with the use of a **piecewise notation**, as below.

$$f(x) = \begin{cases} 0, & if \quad -2 \leq x < 1 \\ 2, & if \quad 1 \leq x < 3 \\ 4, & if \quad 3 \leq x \leq 4 \end{cases}$$

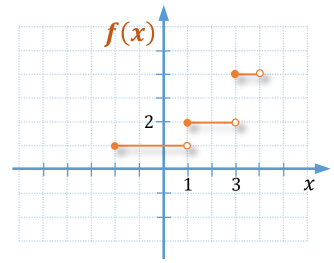


Figure 1.12

Step functions are used in many areas of life, particularly in business. For example, utilities or taxes are often billed according to a step function.

Example 4 ► **Finding a Step Function that Models a Parking Charge**

The cost of parking a car at an airport hourly parking lot is \$5 for the first hour or its portion and \$3 for each additional hour or its portion. Let $C(t)$ represent the cost of parking a car for t hours. Graph $C(t)$ for t in the interval $(0, 4]$. Then, using piecewise notation, state the formula for the graphed function.

Solution ► To graph $C(t)$, we may create a table of values first. Observe that

t	$C(t)$
0.5	5
1	5
1.5	$5 + 3 = 8$
2	$5 + 3 = 8$
2.5	$5 + 2 \cdot 3 = 11$
3	$5 + 2 \cdot 3 = 11$
3.5	$5 + 3 \cdot 3 = 14$
4	$5 + 3 \cdot 3 = 14$

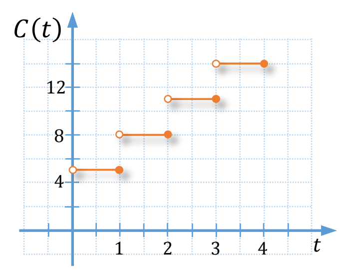
$C(t) = 5$ for all t-values from the interval $(0, 1]$,

$C(t) = 8$ for all t-values from the interval $(1, 2]$,

$C(t) = 11$ for all t-values from the interval $(2, 3]$, and

$C(t) = 14$ for all t-values from the interval $(3, 4]$.

So, we graph $C(t)$ as below.



Using piecewise notation, function C can be written as

$$C(t) = \begin{cases} 5, & if \quad 0 < t \leq 1 \\ 8, & if \quad 1 < t \leq 2 \\ 11, & if \quad 2 < t \leq 3 \\ 14, & if \quad 3 < t \leq 4 \end{cases}$$

C.1 Exercises

Vocabulary Check *Complete each blank with the most appropriate term or phrase from the given list:* ***translating, vertically, dilating, below, above, zeros, step.***

1. The graph of function $f(x-a)$ can be obtained by ______________ the graph of $f(x)$ by a units horizontally.

2. The graph of function $f(x)+b$ can be obtained by translating the graph of $f(x)$ by b units ______________.

3. The graph of function $kf(x)$ can be obtained by _____________ the graph of $f(x)$, k times vertically.

4. The graph of function $|f(x)|$ can be obtained by flipping the section(s) of the graph of $f(x)$ that are ________ the x-axis over the x-axis and leaving unchanged the section(s) that are _________ the x-axis.

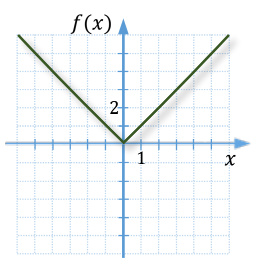
5. The vertical asymptotes of the graph of function $\frac{1}{f(x)}$ are the lines that pass through the ________ of the graph of $f(x)$.

6. The greatest integer function $f(x) = [\![x]\!]$ is an example of a _______ function.

Concept Check

7. Match the name of the basic function provided in **a.-f.** with the corresponding graph in **I-VI**. Then, give the equation of this function and state its domain and range.

a. quadratic **b.** cubic **c.** square root
d. absolute value **e.** greatest integer **f.** reciprocal

I


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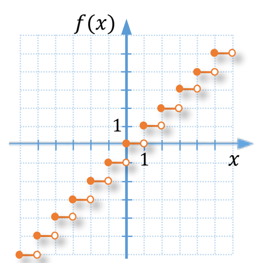

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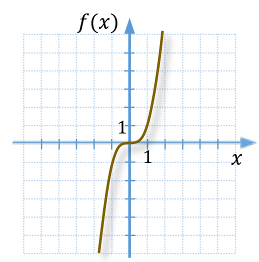

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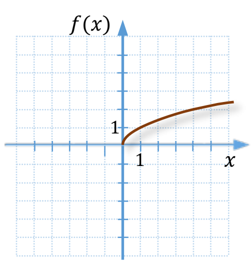

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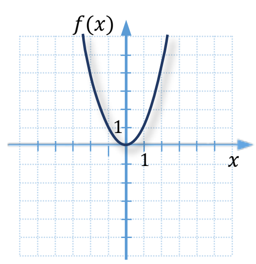

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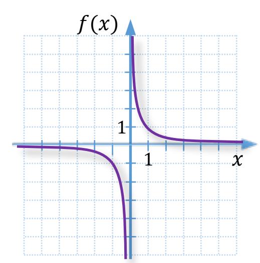
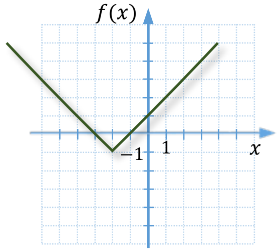


Concept Check

8. Match each absolute value function given in **a.-d.** with its graph in **I-IV**.

a. $f(x) = -|x-1|+2$ **b.** $f(x) = 2|x|-1$ **c.** $f(x) = -|x-2|+1$
d. $f(x) = |x+2|-1$ **e.** $f(x) = -|x|+2$ **f.** $f(x) = 2|x+1|-2$

I



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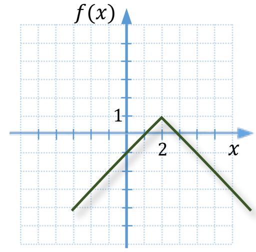

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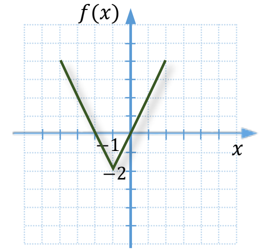

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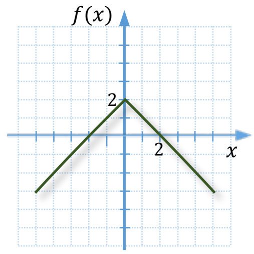

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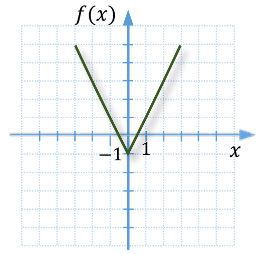
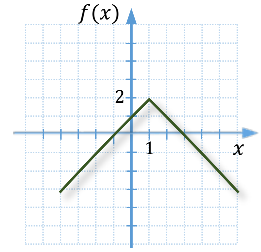


VI



Concept Check

9. How is the graph of $f(x) = \frac{1}{x-5} + 3$ obtained from the graph of $g(x) = \frac{1}{x}$?

10. How is the graph of $f(x) = \sqrt{x+4} - 1$ obtained from the graph of $g(x) = \sqrt{x}$?

Graph each function. Give the domain and range. For rational functions, give the equations of their asymptotes.

11. $f(x) = |x+2|$ **12.** $f(x) = |x-3|$ **13.** $f(x) = \sqrt{x} + 2$

14. $f(x) = \sqrt{x} - 3$ **15.** $f(x) = \frac{1}{x} - 2$ **16.** $f(x) = \frac{1}{x} + 1$

17. $f(x) = -\frac{2}{x-1}$ **18.** $f(x) = \frac{1}{x+3} - 2$ **19.** $f(x) = -\sqrt{x+3}$

20. $f(x) = -(x+2)^3 + 1$ **21.** $f(x) = 2(x+3)^2 - 4$ **22.** $f(x) = 2|x+1| - 3$

Concept Check *Evaluate each expression.*

23. $[\![2.1]\!]$ **24.** $[\![-2.1]\!]$ **25.** $-[\![2.1]\!]$ **26.** $-[\![-1.9]\!]$

Graph each function.

27. $f(x) = -[\![x]\!]$ **28.** $f(x) = [\![x]\!] - 2$ **29.** $f(x) = [\![x+3]\!]$

For each function $f(x)$, graph its reciprocal $g(x) = \frac{1}{f(x)}$. Determine the x-intercept of function f and the equation of the vertical asymptote of function g.

30. $f(x) = -x$ **31.** $f(x) = \frac{1}{2}x - 2$ **32.** $f(x) = -2x + 1$

33. $f(x) = x + 3$ **34.** $f(x) = -x + 2$ **35.** $f(x) = 4x - 3$

For each function $f(x)$, graph its absolute value $g(x) = |f(x)|$.

36. $f(x) = x^2 - 4$

37. $f(x) = (2 - x)(x + 3)$

38. $f(x) = 2x^2 + 3x$

39. $f(x) = (2x + 1)(x - 3)$

40. $f(x) = -2x^2 - 5x$

41. $f(x) = x^2 + 3x - 4$

Analytic Skills *Solve each problem.*

42. A certain long-distance carrier provides service between Podunk and Nowhereville. If x represents the number of minutes for the call, where $x > 0$, then the function f defined by

$$f(x) = 0.40[\![x]\!] + 0.75$$

gives the total cost of the call in dollars.

a. Find the cost of a 5.5-minute call

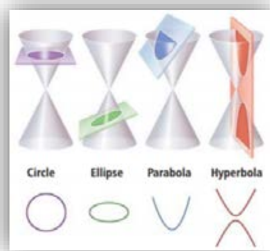
b. Find the cost of a 20.75-minute call.

43. The cost of parking a car at an airport hourly parking lot is $3 for the first half-hour or its portion and $2 for each additional half-hour or its portion. Let $C(t)$ represent the cost of parking a car for t hours. Graph $C(t)$ for t in the interval $(0, 2]$. Then, using piecewise notation, state the formula for the graphed function.

44. An overnight delivery service charges $25 for a package weighing up to 2 kg. For each additional kilogram or its portion, an additional $3 is charged. Let $D(x)$ represent the cost to send a package weighing x kilograms. Graph $D(x)$ for x in the interval $(0, 5]$. Then, using piecewise notation, state the formula for the graphed function.

45. A furniture store pays employees a bonus based on their monthly sales. For sales of $5,000 up to $15,000, the bonus is $500. For sales of $15,000 up to $20,000, the bonus is $800. For sales of $20,000 or more, the bonus is $1,000. Let $B(m)$ represent the amount of bonus received for the monthly sales m. Graph $B(m)$ for m in the interval $(0, 30000]$. Then, using piecewise notation, state the formula for the graphed function.

C.2 Equations and Graphs of Conic Sections



In this section, we give an overview of the main properties of the curves called **conic sections**. Geometrically, these curves can be defined as intersections of a plane with a double cone. These intersections can take the shape of a point, a line, two intersecting lines, a circle, an ellipse, a parabola, or a hyperbola, depending on the position of the plane with respect to the cone.

Conic sections play an important role in mathematics, physics, astronomy, and other sciences, including medicine. For instance, planets, comets, and satellites move along conic pathways. Radio telescopes are built with the use of parabolic dishes while reflecting telescopes often contain hyperbolic mirrors. Conic sections are present in both analyzing and constructing many important structures in our world.

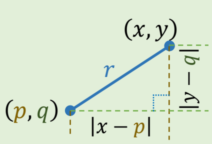
Since lines and parabolas were already discussed in previous chapters, this section will focus on circles, ellipses, and hyperbolas.

Circles

A circle is a conic section formed by the intersection of a cone and a plane parallel to the base of the cone. In coordinate geometry, a circle is defined as follows.

Definition 2.1 ► A **circle** with a fixed **centre** and the **radius** of length $\boldsymbol{r}$ is the set of all points in a plane that lie at the **constant distance** $\boldsymbol{r}$ from this centre.

Equation of a Circle in Standard Form



A **circle** with **centre** $(\boldsymbol{p}, \boldsymbol{q})$ and **radius** $\boldsymbol{r}$ is given by the equation:

$$(\boldsymbol{x}-\boldsymbol{p})^2+(\boldsymbol{y}-\boldsymbol{q})^2=\boldsymbol{r}^2$$

In particular, the equation of a circle centered at the origin and with radius $\boldsymbol{r}$ takes the form

$$\boldsymbol{x}^2+\boldsymbol{y}^2=\boldsymbol{r}^2$$

Proof: ► Suppose a point (x, y) belongs to the circle with centre (p, q) and radius r. By *definition 2.1*, the distance between this point and the centre is equal to r. Using the distance formula that was developed in *section RD.3*, we have

$$r=\sqrt{(x-p)^2+(y-q)^2}$$

Hence, after squaring both sides of this equation, we obtain the equation of the circle:

$$r^2=(x-p)^2+(y-q)^2$$

Example 1 ► **Finding an Equation of a Circle and Graphing It**

Find an equation of the circle with radius 2 and center at $(0,1)$ and graph it.

Solution ► By substituting $p = 0$, $q = 1$, and $r = 2$ into the standard form of the equation of a circle, we obtain

$$x^2 + (y - 1)^2 = 4$$

To graph this circle, we plot the centre $(0,1)$ first, and then plot points that are 2 units apart in the four main directions, East, West, North, and South. The circle passes through these four points, as in *Figure 2.1*.

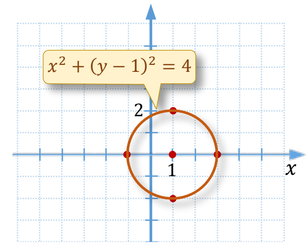


Figure 2.1

Example 2 ► **Graphing a Circle Given Its Equation**

Identify the center and radius of each circle. Then graph it and state the domain and range of the relation.

a. $x^2 + y^2 = 7$ **b.** $(x - 3)^2 + (y + 2)^2 = 6.25$

c. $x^2 + 4x + y^2 - 2y = 4$

Solution ►

a. The equation can be written as $(x - 0)^2 + (y - 0)^2 = \left(\sqrt{7}\right)^2$. So, the **centre** of this circle is at $(\mathbf{0}, \mathbf{0})$, and the length of the **radius** is $\sqrt{\mathbf{7}}$. The graph is shown in *Figure 2.2a*.
By projecting the graph onto the x-axis, we observe that the **domain** of this relation is $\left[-\sqrt{\mathbf{7}}, \sqrt{\mathbf{7}}\right]$. Similarly, by projecting the graph onto the y-axis, we obtain the **range**, which is also $\left[-\sqrt{\mathbf{7}}, \sqrt{\mathbf{7}}\right]$.

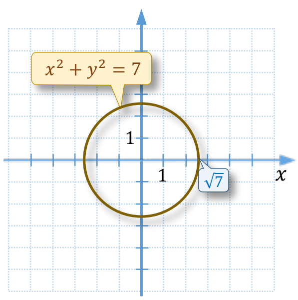


Figure 2.2a

b. The **centre** of this circle is at $(\mathbf{3}, \mathbf{-2})$ and the length of the **radius** is $\sqrt{6.25} = \mathbf{2.5}$. The graph is shown in *Figure 2.2b*. The **domain** of the relation is $[\mathbf{0.5}, \mathbf{5.5}]$, and the **range** is $[\mathbf{-4.5}, \mathbf{0.5}]$.

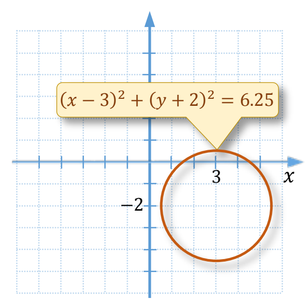


Figure 2.2b

c. The given equation is not in standard form. To rewrite it in standard form, we apply the completing the square procedure to the x-terms and to the y-terms.

$$x^2 + 4x + y^2 - 2y = 4$$
$$(x + 2)^2 - 4 + (y - 1)^2 - 1 = 4$$
$$(x + 2)^2 + (y - 1)^2 = 9$$
$$(x + 2)^2 + (y - 1)^2 = 3^2$$

So, the **centre** of this circle is at $(\mathbf{-2}, \mathbf{1})$ and the length of the **radius** is $\mathbf{3}$. The graph is shown in *Figure 2.2c*. The **domain** of the relation is $[\mathbf{-5}, \mathbf{1}]$ and the **range** is $[\mathbf{-2}, \mathbf{4}]$.

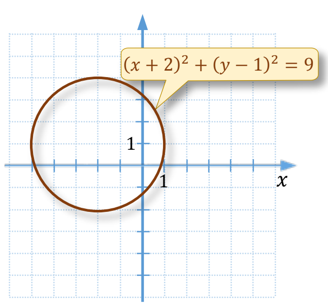
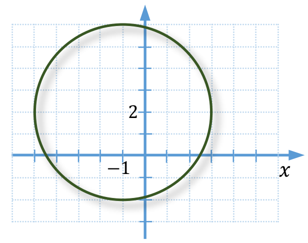


Figure 2.2c

Example 3 ► **Finding Equation of a Circle Given Its Graph**

Determine the equation of the circle shown in the graph.



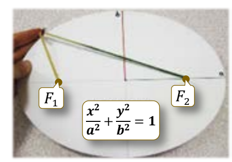
Solution ► Reading from the graph, the centre of the circle is at $(-1,2)$ and the radius is 4. So the equation of this circle is

$$(x+1)^2 + (y-2)^2 = 4^2$$

Ellipses

A conic section formed by the intersection of a cone and a plane slanted to the base but not parallel to the side of the cone is called an ellipse. In coordinate geometry, an ellipse is defined as follows.

Definition 2.2 ► An **ellipse** is the set of points in a plane with a constant *sum* of distances from two fixed points. These fixed points are called **foci** (*singular:* **focus**). The point halfway between the two foci is called the **center** of the ellipse.

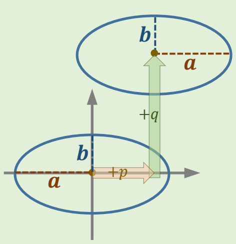


An ellipse has an interesting property of reflection.

Reflecting Property of an Ellipse

When a ray of light or sound emanating from one focus of an ellipse bounces off the ellipse, it passes through the other focus.

Equation of an Ellipse in Standard Form



An **ellipse** with its **centre** at the origin, **radius along the x-axis (r_x)** of length a, and **radius along the y-axis (r_y)** of length b is given by the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

An **ellipse** with its **centre** at the point (p, q), **radius along the x-axis (r_x)** of length a, and **radius along the y-axis (r_y)** of length b is given by the equation:

$$\frac{(x-p)^2}{a^2} + \frac{(y-q)^2}{b^2} = 1$$

Note: A circle is a special case of an ellipse, where $a = b = r$.

Example 4 ► **Graphing an Ellipse Given Its Equation**

Identify the center and the two radii of each ellipse. Then graph it and state the domain and range of the relation.

a. $9x^2 + y^2 = 9$ **b.** $\frac{(x-1)^2}{16} + \frac{(y+2)^2}{4} = 1$

Solution ►

a. First, we may want to change the equation to its standard form. This can be done by dividing both sides of the given equation by 9, to make the right side equal to 1. So, we obtain

$$x^2 + \frac{y^2}{9} = 1$$

or equivalently,

$$x^2 + \frac{y^2}{3^2} = 1$$

Hence, the **centre** of this ellipse is at $(\mathbf{0}, \mathbf{0})$, and the two **radii** are $\boldsymbol{r_x = 1}$ and $\boldsymbol{r_y = 3}$. Thus, we graph this ellipse as in *Figure 2.3a*. The **domain** of the relation is $[\mathbf{-1}, \mathbf{1}]$ and the **range** is $[\mathbf{-3}, \mathbf{3}]$.

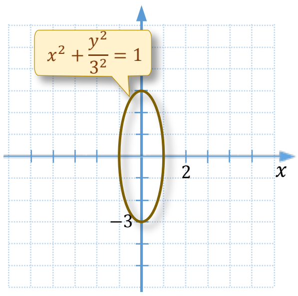


Figure 2.3a

b. The given equation can be written as

$$\frac{(x-1)^2}{4^2} + \frac{(y+2)^2}{2^2} = 1$$

So, the **centre** of this ellipse is at $(\mathbf{1}, \mathbf{-2})$ and the two **radii** are $\boldsymbol{r_x = 4}$ and $\boldsymbol{r_y = 2}$. The graph is shown in *Figure 2.3b*. The **domain** of the relation is $[\mathbf{-3}, \mathbf{5}]$, and the **range** is $[\mathbf{-4}, \mathbf{0}]$.

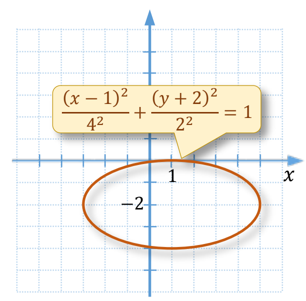
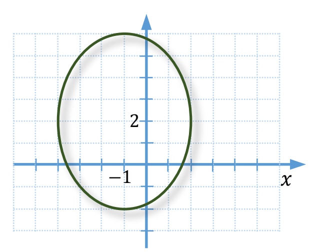


Figure 2.3b

Example 5 ► **Finding Equation of an Ellipse Given Its Graph**

Give the equation of the ellipse shown in the accompanying graph.

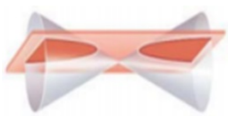


Solution ►

Reading from the graph, the centre of the ellipse is at $(-1,2)$, the radius r_x equals 3, and the radius r_y equals 4. So, the equation of this ellipse is

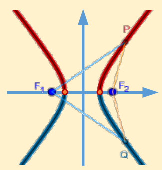
$$\frac{(x+1)^2}{3^2} + \frac{(y-2)^2}{4^2} = 1$$

Hyperbolas



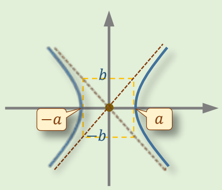
A conic section formed by the intersection of a cone and a plane perpendicular to the base of the cone is called a hyperbola. In coordinate geometry, a hyperbola is defined as follows.

Definition 2.2 ►



A **hyperbola** is the set of points in a plane with a constant absolute value of the *difference* of distances from two fixed points. These fixed points are called **foci** (*singular:* **focus**). The point halfway between the two foci is the **center** of the hyperbola. The graph of a hyperbola consists of two branches and has two axes of symmetry. The axis of symmetry that passes through the foci is called the **transverse** axis. The intercepts of the hyperbola and its transverse are the **vertices** of the hyperbola. The line passing through the centre of the hyperbola and perpendicular to the transverse is the other axis of symmetry, called the **conjugate** axis.

Equation of a Hyperbola in Standard Form

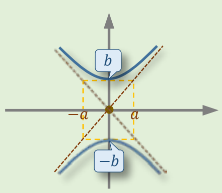


A **hyperbola** with its **centre** at the origin, **transverse axis** on the $\boldsymbol{x}$**-axis**, and **vertices** at $(-a, 0)$ and $(a, 0)$ is given by the equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

A **hyperbola** with its **centre** at (p, q), **horizontal transverse axis**, and **vertices** at $(-a, 0)$ and $(a, 0)$ is given by the equation:

$$\frac{(x-p)^2}{a^2} - \frac{(y-q)^2}{b^2} = 1$$



A **hyperbola** with its **centre** at the origin, **transverse axis** on the $\boldsymbol{y}$**-axis**, and **vertices** at $(0, -b)$ and $(0, b)$ is given by the equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

A **hyperbola** with its **centre** at (p, q), **vertical transverse axis**, and **vertices** at $(0, -b)$ and $(0, b)$ is given by the equation:

$$\frac{(x-p)^2}{a^2} - \frac{(y-q)^2}{b^2} = -1$$

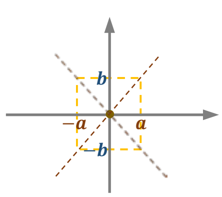


Figure 2.4a

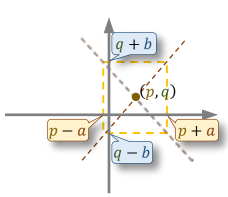


Figure 2.4b

Fundamental Rectangle and Asymptotes of a Hyperbola

The graph of a hyperbola given by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$ is based on a rectangle formed by the lines $x = \pm a$ and $y = \pm b$. This rectangle is called the **fundamental rectangle** (see *Figure 2.4*). The extensions of the diagonals of the fundamental rectangle are the **asymptotes** of the hyperbola. Their equations are $y = \pm \frac{b}{a} x$.

Generally, the **fundamental rectangle** of a hyperbola given by the equation

$$\frac{(x-p)^2}{a^2} - \frac{(y-q)^2}{b^2} = \pm 1$$

is formed by the lines $x = p \pm a$ and $y = q \pm b$. The extensions of the diagonals of this rectangle are the **asymptotes** of the hyperbola.

Example 6 ► Graphing a Hyperbola Given Its Equation

Determine the center, transverse axis, and vertices of each hyperbola. Graph the fundamental rectangle and asymptotes of the hyperbola. Then, graph the hyperbola and state its domain and range.

a. $9x^2 - 4y^2 = 36$ **b.** $(x-2)^2 - \frac{(y+1)^2}{4} = -1$

Solution ►

a. First, we may want to change the equation to its standard form. This can be done by dividing both sides of the given equation by 36, to make the right side equal to 1. So, we obtain

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

or equivalently

$$\frac{x^2}{2^2} - \frac{y^2}{3^2} = 1$$

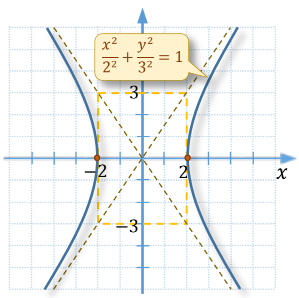


Figure 2.5

Hence, the **centre** of this hyperbola is at $(\mathbf{0},\mathbf{0})$, and the transverse axis is on the ***x*-axis**. Thus, the vertices of the hyperbola are $(\mathbf{-2},\mathbf{0})$ and $(\mathbf{2},\mathbf{0})$.

The **fundamental rectangle** is centered at the origin, and it spans 2 units horizontally apart from the centre and 3 units vertically apart from the centre, as in *Figure 2.5*. The **asymptotes** pass through the opposite vertices of the fundamental rectangle. The final graph consists of two branches. Each of them passes through the corresponding vertex and is shaped by the asymptotes, as shown in *Figure 2.5*.

The **domain** of the relation is $(-\infty,\mathbf{-2}] \cup [\mathbf{2},\infty)$ and the **range** is $\mathbb{R}$.

b. The equation can be written as

$$\frac{(x-2)^2}{1^2} - \frac{(y+1)^2}{2^2} = -1$$

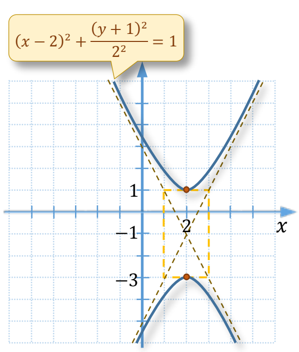


Figure 2.6

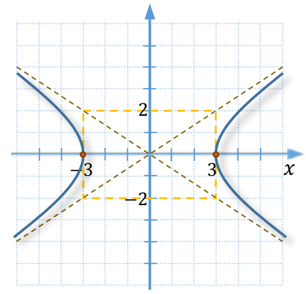
The **centre** of this hyperbola is at $(\mathbf{2},\mathbf{-1})$. The -1 on the right side of this equation indicates that the **transverse axis is vertical**. Thus, the vertices of the hyperbola are 2 units vertically apart from the centre. So, they are $(\mathbf{2},\mathbf{-3})$ and $(\mathbf{2},\mathbf{1})$.

The **fundamental rectangle** is centered at $(\mathbf{2},\mathbf{-1})$ and it spans 1 unit horizontally apart from the centre and 2 units vertically apart from the centre, as in *Figure 2.6*. The **asymptotes** pass through the opposite vertices of the fundamental box. The final graph consists of two branches. Each of them passes through the corresponding vertex and is shaped by the asymptotes, as shown in *Figure 2.6*.

The **domain** of the relation is $\mathbb{R}$, and the **range** is $(-\infty,\mathbf{-3}] \cup [\mathbf{1},\infty)$.

Example 7 ► **Finding the Equation of a Hyperbola Given Its Graph**

Give the equation of a hyperbola shown in the accompanying graph.



Solution ► Reading from the graph, the centre of the hyperbola is at $(0,0)$, the transverse axis is the x-axis, and the vertices are $(-3,0)$ and $(3,0)$. The fundamental rectangle spans 2 units vertically apart from the centre. So, we substitute $p = 0$, $q = 0$, $a = 3$, and $b = 2$ to the standard equation of a hyperbola. Thus the equation is

$$\frac{x^2}{3^2} - \frac{y^2}{2^2} = 1$$

Generalized Square Root Functions $f(x) = \sqrt{g(x)}$ for Quadratic Functions $g(x)$

Conic sections are relations but usually not functions. However, we could consider parts of conic sections that are already functions. For example, when solving the equation of a circle

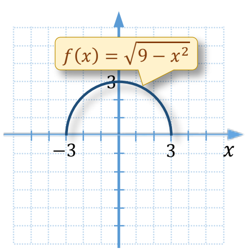
$$x^2 + y^2 = 9$$

for y, we obtain

$$y^2 = 9 - x^2$$

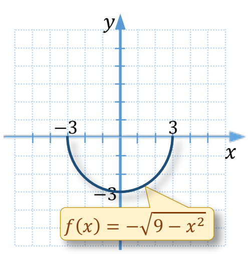
$$|y| = \sqrt{9 - x^2}$$

$$y = \pm\sqrt{9 - x^2}.$$



So, the graph of this circle can be obtained by graphing the two functions: $y = \sqrt{9 - x^2}$ and $y = -\sqrt{9 - x^2}$.

Since the equation $\boldsymbol{y = \sqrt{9 - x^2}}$ describes all the points of the circle with a nonnegative y-coordinate, its graph must be the **top half of the circle** centered at the origin and with the radius of length 3. So, the domain of this function is $[-3,3]$ and the range is $[0,3]$.



Likewise, since the equation $\boldsymbol{y = -\sqrt{9 - x^2}}$ describes all the points of the circle with a nonpositive y-coordinate, its graph must be the **bottom half of the circle** centered at the origin and with the radius of length 3. Thus, the domain of this function is $[-3,3]$ and the range is $[-3,0]$.

Note: Notice that the function $f(x) = \sqrt{9 - x^2}$ is a composition of the square root function and the quadratic function $g(x) = 9 - x^2$. One could prove that the graph of the square root of any quadratic function is the top half of one of the conic sections. Similarly, the graph of the negative square root of any quadratic function is the bottom half of one of the conic sections.

Example 8 ► **Graphing Generalized Square Root Functions**

Graph each function. Give its domain and range.

a. $f(x) = 1 - \sqrt{16 - x^2}$ **b.** $f(x) = 2\sqrt{1 + \frac{x^2}{9}}$

Solution ►

a. To recognize the shape of the graph of function *f*, let us rearrange its equation first.

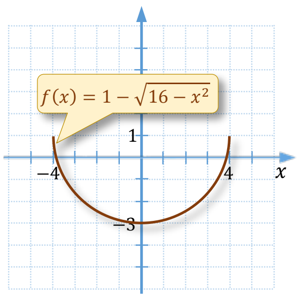


Figure 2.7

$$y = 1 - \sqrt{16 - x^2}$$

$$y - 1 = -\sqrt{16 - x^2}$$

$$(y-1)^2 = \left(-\sqrt{16 - x^2}\right)^2$$

$$(y-1)^2 = 16 - x^2$$

$$x^2 + (y-1)^2 = 16$$

The resulting equation represents a circle with its centre at $(0,1)$ and a radius of 4. So, the graph of $f(x) = 1 - \sqrt{16 - x^2}$ must be part of this circle. Since $y - 1 = -\sqrt{16 - x^2} \le 0$, then $y \le 1$. Thus the graph of function f is the **bottom half** of this **circle**, as shown in *Figure 2.7*.
So, the **domain** of function f is $[\mathbf{-4}, \mathbf{4}]$ and the **range** is $[\mathbf{-3}, \mathbf{1}]$.

b. To recognize the shape of the graph of function *f*, let us rearrange its equation first.

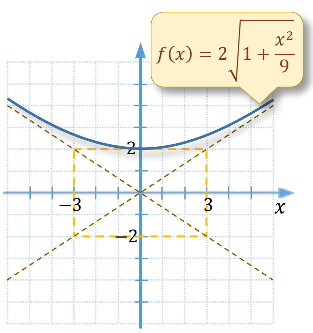


Figure 2.8

$$y = 2\sqrt{1 + \frac{x^2}{9}}$$

$$\frac{y}{2} = \sqrt{1 + \frac{x^2}{9}}$$

$$\left(\frac{y}{2}\right)^2 = 1 + \frac{x^2}{9}$$

$$-1 = \frac{x^2}{9} - \left(\frac{y}{2}\right)^2$$

$$\frac{x^2}{3^2} - \frac{y^2}{2^2} = -1$$

The resulting equation represents a hyperbola centered at the origin, with a vertical transverse axis. Its fundamental rectangle spans horizontally 3 units and vertically 2 units from the centre. Since the graph of $f(x) = 2\sqrt{1 + \frac{x^2}{9}}$ must be a part of this hyperbola and the values $f(x)$ are nonnegative, then its graph is the **top half** of this **hyperbola**, as shown in *Figure 2.8*.
So, the **domain** of function f is $\mathbb{R}$, and the **range** is $[\mathbf{2}, \infty]$.

C.2 Exercises

Vocabulary Check *Complete each blank with the most appropriate term or phrase from the given list:* ***ellipse, circle, hyperbola, center, fundamental rectangle, transverse, focus, conic.***

1. The set of all points in a plane that are equidistant from a fixed point is a ___________ .

2. The set of all points in a plane with a constant sum of their distances from two fixed points is an _____________ .

3. The set of all points in a plane with a constant difference between the distances from two fixed points is a _______________.

4. The _________ of a hyperbola is the point that lies halfway between the vertices of this hyperbola.

5. The asymptotes of a hyperbola pass through the opposite vertices of the _____________________________ of this hyperbola.

6. The _____________ axis of a hyperbola passes through the two vertices of the hyperbola.

7. A ray of light emanated from one focus of an ellipse passes through the other _________.

8. The graph of a square root of a quadratic function is the top or the bottom half of one of the __________ sections.

Concept Check *True or false.*

9. A circle is a set of points, where the center is one of these points.

10. If the foci of an ellipse coincide, then the ellipse is a circle.

11. The x-intercepts of $\frac{x^2}{9} + \frac{y^2}{4} = 1$ are $(-9,0)$ and $(9,0)$.

12. The graph of $2x^2 + y^2 = 1$ is an ellipse.

13. The y-intercepts of $x^2 + \frac{y^2}{3} = 1$ are $(-\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$.

14. The graph of $y^2 = 1 - x^2$ is a hyperbola centered at the origin.

15. The transverse axis of the hyperbola $-y^2 = 1 - x^2$ is the x-axis.

Find the equation of a circle satisfying the given conditions.

16. centre at $(-1, -2)$; radius 1

17. centre at $(3,1)$; radius $\sqrt{3}$

18. centre at $(2, -1)$; diameter 6

19. centre at $(-2,2)$; diameter 5

*Find the **center** and **radius** of each circle.*

20. $x^2 + y^2 + 4x + 6y + 9 = 0$

21. $x^2 + y^2 - 8x - 10y + 5 = 0$

22. $x^2 + y^2 + 6x - 16 = 0$

23. $x^2 + y^2 - 12x + 12 = 0$

24. $2x^2 + 2y^2 + 20y + 10 = 0$

25. $3x^2 + 3y^2 - 12y - 24 = 0$

*Identify the **center** and **radius** of each circle. Then graph the relation and state its **domain** and **range**.*

26. $x^2 + (y - 1)^2 = 16$

27. $(x + 1)^2 + y^2 = 2.25$

28. $(x - 2)^2 + (y + 3)^2 = 4$

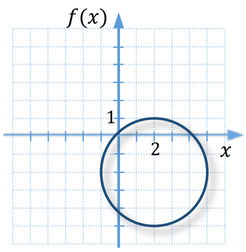
29. $(x + 3)^2 + (y - 2)^2 = 9$

30. $x^2 + y^2 + 2x + 2y - 23 = 0$

31. $x^2 + y^2 + 4x + 2y + 1 = 0$

Concept Check *Use the given graph to determine the equation of the **circle**.*

32.



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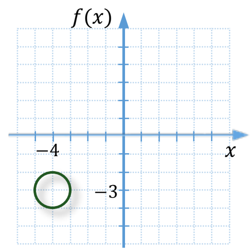

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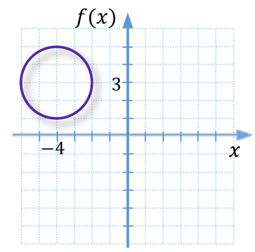
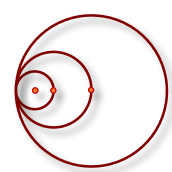


Discussion Point

35. The equation of the smallest circle shown is $x^2 + y^2 = r^2$. What is the equation of the largest circle?



Concept Check

*Identify the **center** and the horizontal (r_x) and vertical (r_y) **radii** of each ellipse. Then graph the relation and state its **domain** and **range**.*

36. $\frac{x^2}{4} + (y - 1)^2 = 1$

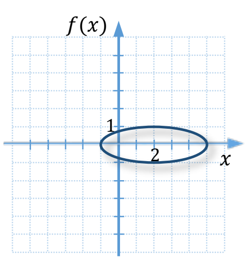
37. $(x + 1)^2 + \frac{y^2}{9} = 1$

38. $\frac{(x-2)^2}{16} + \frac{(y+3)^2}{4} = 1$

39. $\frac{(x-4)^2}{4} + \frac{(y-2)^2}{9} = 1$

Concept Check *Use the given graph to determine the equation of the **ellipse**.*

40.



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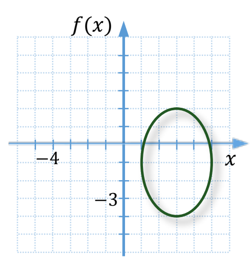

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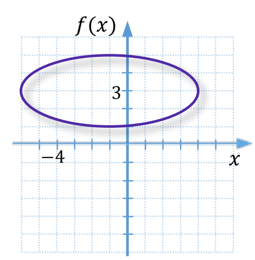


Concept Check *Identify the* ***center*** *and the* ***transverse axis*** *of each hyperbola. Then graph the* ***fundamental box*** *and the* ***hyperbola****. State the* ***domain*** *and* ***range*** *of the relation.*

43. $\frac{x^2}{4} - (y-1)^2 = 1$

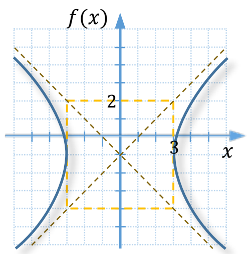
44. $(x+1)^2 + \frac{y^2}{9} = -1$

45. $\frac{(x-3)^2}{4} - \frac{(y+2)^2}{4} = -1$

46. $\frac{(x+2)^2}{9} - \frac{(y-1)^2}{9} = 1$

Concept Check *Use the given graph to determine the equation of the* ***hyperbola****.*

47.



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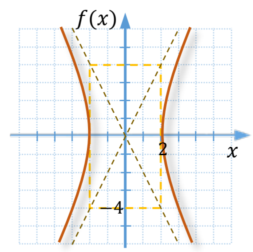

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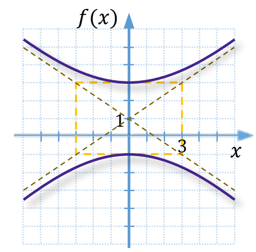


Concept Check

50. Match each equation with its graph.

a. $\frac{x^2}{9} + \frac{y^2}{16} = 1$

b. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

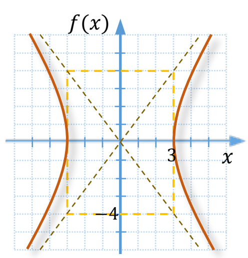
c. $\frac{x^2}{9} - \frac{y^2}{16} = -1$

d. $\frac{x^2}{16} - \frac{y^2}{9} = 1$

e. $\frac{x^2}{9} + \frac{y^2}{9} = 1$

f. $\frac{x^2}{16} + \frac{y^2}{9} = 1$

I



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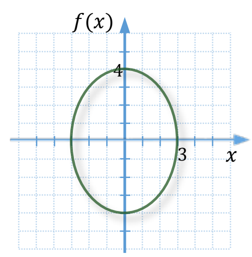
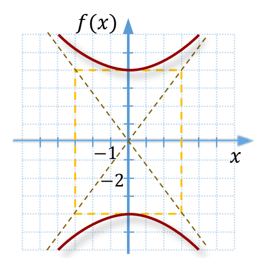
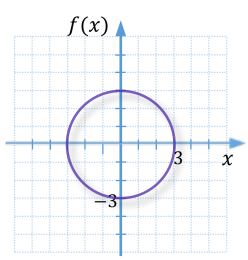


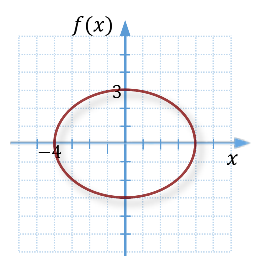
III



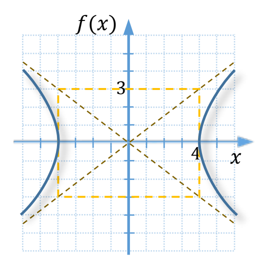
IV



V



VI



Graph each ***generalized square root*** *function. Give the* ***domain*** *and* ***range****.*

51. $f(x) = \sqrt{4 - x^2}$

52. $f(x) = -\sqrt{25 - x^2}$

53. $f(x) = -2\sqrt{1 - \frac{x^2}{9}}$

54. $f(x) = 3\sqrt{1 - \frac{x^2}{4}}$

55. $\frac{y}{3} = \sqrt{x^2 - 1}$

56. $\frac{y}{2} = -\sqrt{1 + \frac{x^2}{9}}$

Analytic Skills *Solve each problem.*

57. The arch under a bridge is designed as the upper half of an ellipse as illustrated in the accompanying figure. Assuming that the ellipse is modeled by $25x^2 + 144y^2 = 3600$, where x and y are in meters, find the width and height of the arch (*above the yellow line*).

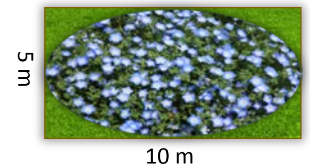
58. Suppose a power outage affects all homes and businesses within a 5-km radius of the power station.

a. If the power station is located 2 km east and 6 km south of the center of town, find an equation of the circle that represents the boundary of the power outage.

b. Will a mall located 4 km east and 4 km north of the power station be affected by the outage?

59. Two buildings in a sports complex are shaped and positioned like a portion of the branches of the hyperbola with equation $400x^2 - 625y^2 = 250{,}000$, where x and y are in meters. How far apart are the buildings at their closest point?

60. The area of an ellipse is given by the formula $A = \pi ab$, where a and b are the two radii of the ellipse.



a. To the nearest tenth of a square meter, find the area of the largest elliptic flower bed that fits in a rectangular space that is 5 meters wide and 10 meters long, as shown in the accompanying figure.

b. Assuming that each square meter of this flower bed is filled with 25 plants, approximate the number of plants in the entire flower bed.

C.3 Nonlinear Systems of Equations and Inequalities

In *section E1*, we discussed methods of solving systems of two linear equations. Recall that solutions to such systems are the intercepts of the two lines. So, we could have either zero, or one, or infinitely many solutions.

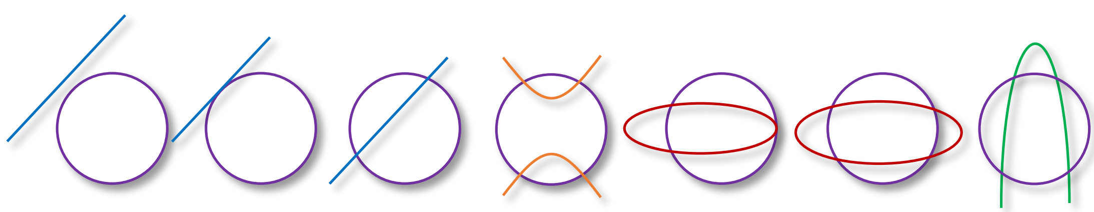
In this section, we will consider systems of two equations that are not necessarily linear. In particular, we will focus on solving systems composed of equations of conic sections. Since the solutions to such systems can be seen as the intercepts of the curves represented by the equations, we may expect a different number of solutions. For example, a circle may intercept an ellipse in 0, 1, 2, 3, or 4 points. We encourage the reader to visualise these situations by drawing a circle and an ellipse in various positions.

Aside from discussing methods of solving nonlinear systems of equations, we will also graph solutions of nonlinear systems of inequalities, using similar techniques as presented in *section G4*, where solutions to linear inequalities were graphed.

Nonlinear Systems of Equations

Definition 3.1 ► A **nonlinear system of equations** is a system of equations containing at least one equation that is not linear.

When solving a nonlinear system of two equations, it is useful to predict the possible number of solutions by visualising the shapes and position of the graphs of these equations. For example, the number of solutions to a system of two equations representing conic sections can be determined by observing the number of intercept points of the two curves. Here are some possible situations.



0 solutions 1 solution 2 solutions 4 solutions 3 solutions 4 solutions 4 solutions

To solve a nonlinear system of two equations, we may use any of the algebraic methods discussed in *section E1*, the substitution or the elimination method, whichever makes the calculations easier.

Example 1 ► **Solving Nonlinear Systems of Two Equations by Substitution**

Solve each system of equations.

a. $\begin{cases} xy = 4 \\ 4y + x = 8 \end{cases}$ **b.** $\begin{cases} x^2 + y^2 = 9 \\ x - y = 1 \end{cases}$

Solution ►

a. The system $\begin{cases} xy = 4 \\ 4y + x = 8 \end{cases}$ consists of a reciprocal function (which is a hyperbola) and a line. So, we may expect 0, 1, or 2 solutions. To solve this system, we may want to solve the second equation for x,

$$x = -4y + 8,$$

and then substituting the resulting expression into the first equation. So, we have

$$(8 - 4y)y = 4 \qquad / -4$$
$$8y - 4y^2 - 4 = 0 \qquad / \cdot(-1)$$
$$4y^2 - 8y + 4 = 0 \qquad / \div 4$$
$$y^2 - 2y + 1 = 0$$
$$(y - 1)^2 = 0$$
$$y = 1$$

Then, using the substitution equation, we calculate

$$x = -4 \cdot 1 + 8 = 4.$$

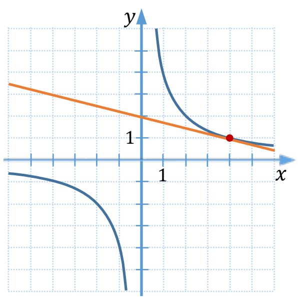


Figure 3.1

So, the solution set consists of one point, $(\mathbf{4}, \mathbf{1})$, as illustrated in *Figure 3.1*.

b. The system $\begin{cases} x^2 + y^2 = 9 \\ x - y = 1 \end{cases}$ consists of a circle and a line. So, we may expect 0, 1, or 2 solutions. To solve this system, we may want to solve the second equation, for example for x,

$$x = y + 1,$$

and then substitute the resulting expression into the first equation. So, we have

$$(y + 1)^2 + y^2 = 9$$
$$y^2 + 2y + 1 + y^2 = 9$$
$$2y^2 + 2y - 8 = 0 \qquad / \div 2$$
$$y^2 + y - 4 = 0$$
$$y = \frac{-1 \pm \sqrt{1 + 4 \cdot 4}}{2} = \frac{-1 \pm \sqrt{17}}{2}$$

Then, using substitution, we calculate

$$x = y + 1 = \frac{-1 \pm \sqrt{17}}{2} + 1 = \frac{-1 \pm \sqrt{17} + 2}{2} = \frac{1 \pm \sqrt{17}}{2}.$$

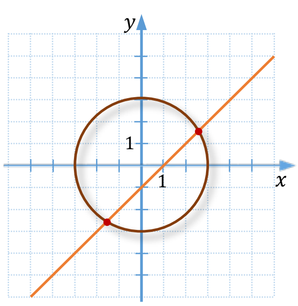


Figure 3.2

So, the solution set consists of two points, $\left(\mathbf{\frac{1-\sqrt{17}}{2}}, \mathbf{\frac{-1-\sqrt{17}}{2}}\right)$ and $\left(\mathbf{\frac{1+\sqrt{17}}{2}}, \mathbf{\frac{-1+\sqrt{17}}{2}}\right)$, as illustrated in *Figure 3.2*.
Their approximations are $(-1.56, -2.56)$ and $(2.56, 1.56)$.

Example 2 ► Solving Nonlinear Systems of Two Equations by Elimination

Solve the system of equations $\begin{cases} x^2 + y^2 = 9 \\ 2x^2 - y^2 = -6 \end{cases}$ using elimination.

Solution ►

The system consists of a circle and a hyperbola, so we may expect up to four solutions. To solve it, we can start by eliminating the y-variable by adding the two equations, side by side.

$$+\begin{cases} x^2 + y^2 = 9 \\ 2x^2 - y^2 = -6 \end{cases} \qquad / \div 3$$

$$3x^2 = 3$$

$$x^2 = 1$$

$$x = \pm 1$$

Then, by substituting the obtained x-values into the first equation, we can find the corresponding y-values. So, if $x = 1$, we have

$$1^2 + y^2 = 9 \qquad / -1$$

$$y^2 = 8$$

$$y = \pm\sqrt{8} = \pm 2\sqrt{2}$$

Similarly, if $x = -1$, we have

$$(-1)^2 + y^2 = 9 \qquad / -1$$

$$y^2 = 8$$

$$y = \pm\sqrt{8} = \pm 2\sqrt{2}$$

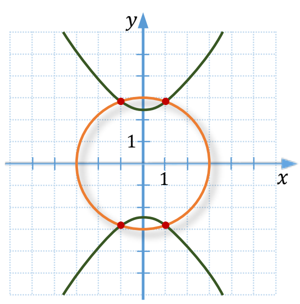


Figure 3.3

So, the solution set is $\{(\mathbf{1}, \mathbf{2\sqrt{2}}), (\mathbf{1}, -\mathbf{2\sqrt{2}}), (-\mathbf{1}, \mathbf{2\sqrt{2}}), (\mathbf{1}, -\mathbf{2\sqrt{2}})\}$. These are the four intersection points of the two curves, circle and hyperbola, as illustrated in *Figure 3.3*.

Nonlinear systems of equations appear in many application problems, especially in the field of geometry, physics, astronomy, astrophysics, engineering, etc.

Example 3 ► Nonlinear Systems of Two Equations in Applied Problems

Find the dimensions of a can having a volume V of 250 cubic centimeters and a side area A of 200 square centimeters.

Solution ►

Using the formulas for the volume, $V = \pi r^2 h$, and side area, $A = 2\pi r h$, of a cylinder with radius r and height h, we can set up the system of two equations,

$$\begin{cases} \pi r^2 h = 250 \\ 2\pi r h = 200 \quad / \div 2 \end{cases}$$

To solve this system, first, we may want to divide the second equation by 2 and then divide the two equations, side by side. So, we obtain

$$\begin{cases} \pi r^2 h = 250 \\ \pi r h = 100 \end{cases}$$

$$r = 2.5$$

After substituting this value into the equation $2\pi r h = 200$, we can find the corresponding h-value:

$$2\pi(2.5)h = 200 \quad / \div 5\pi$$

$$h = \frac{200}{5\pi} \approx 12.7$$

So, the can should have a **radius** of **2.5 cm** and a **height** of about **12.7 cm**.

Nonlinear Systems of Inequalities

In *Section G.4* we discussed graphical solutions to linear inequalities and systems of linear inequalities in two variables. Nonlinear inequalities in two variables and systems of such inequalities can be solved using similar graphic techniques.

Example 4 ► **Graphing Solutions to a Nonlinear Inequality**

Graph the solution set of each inequality.

a. $y \geq (x-2)^2 - 3$ **b.** $9x^2 - 4y^2 < 36$

Solution ► **a.** First, we graph the related equation of the parabola $y = (x-2)^2 - 3$, using a solid curve. So, we plot the vertex $(2, -3)$ and follow the shape of the basic parabola $y = x^2$, with arms directed upwards.

This parabola separates the plane into two regions, the one above the parabola and the one below the parabola. The inequality $y \geq (x-2)^2 - 3$ indicates that the y-values of the solution points are **above** the parabola $y = (x-2)^2 - 3$. To confirm this observation, we may want to pick a **test point** outside of the parabola and check whether or not it satisfies the inequality. For example, the point $(2,0)$ makes the inequality

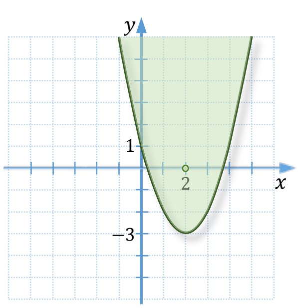


Figure 3.4

$$0 \geq (2-2)^2 - 3 = -3$$

a true statement. Thus, the point $(2,0)$ is one of the solutions of the inequality $y \geq (x-2)^2 - 3$ and so are the points of the whole region containing $(2,0)$. To illustrate the solution set of the given inequality, we shade this region, as in *Figure 3.4*. Thus, the solution set consists of all points above the parabola, including the parabola itself.

b. The related equation, $9x^2 - 4y^2 = 36$ represents a hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$ centered at the origin, with a horizontal transverse axis, and with a fundamental rectangle that

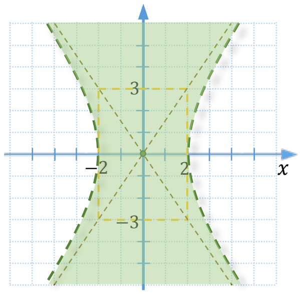


Figure 3.5

stretches 2 units horizontally and 3 units vertically apart from the centre. Since the inequality is strong (<), we graph this hyperbola using a dashed line. This indicates that the points on the hyperbola are not among the solutions of the inequality.

To decide which region should be shaded as the set of solutions for the given inequality, we can pick a test point that is easy to calculate, for instance (0,0). Since

$$9 \cdot 0^2 - 4 \cdot 0^2 = 0 < 36$$

is a true statement, the point (0,0) is one of the solutions of the inequality $9x^2 - 4y^2 < 36$, and so are the points of the whole region containing (0,0). Thus, the solution set consists of all the points shaded in green (see *Figure 3.5*), but not the points of the hyperbola itself.

Note: If we chose a test point that does not satisfy the inequality, then the solution set is the region that does not contain this test point.

Example 5 ► **Graphing Solutions to a System of Nonlinear Inequalities**

Graph the solution set of each system of inequalities.

a. $\begin{cases} x^2 + y < 4 \\ y - x \geq 2 \end{cases}$ **b.** $\begin{cases} \frac{x^2}{9} + \frac{y^2}{16} \leq 1 \\ x^2 - y^2 \geq -1 \end{cases}$

Solution ►

a. Observe that the first inequality, $y < -x^2 + 4$, represents the sets of points **below the parabola** $y = -x^2 + 4$. We will shade it in blue, as in *Figure 3.6*.

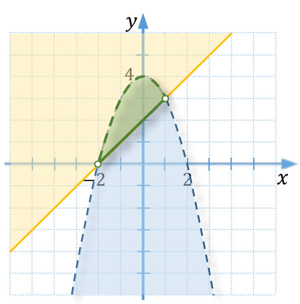


Figure 3.6

The second inequality, $y \geq x + 2$, represents the sets of points **above the line** $y = x + 2$, including the points on the line. We will shade it in yellow, as in *Figure 3.6*.

Thus, the solution set of the system of these inequalities is the **intersection of the blue and yellow region**, as illustrated in green in *Figure 3.6*. The top boundary of the green region is marked by a **dashed line** as these points do not belong to the solution set, and the bottom boundary is marked by a **solid line**, indicating that these points are among the solutions to the system. Also, since the intersection points of the two curves do not satisfy the first inequality, they are not solutions to the system. So, we mark them with **hollow circles**.

b. Observe that the first inequality, $\frac{x^2}{9} + \frac{y^2}{16} \leq 1$, represents the sets of points **inside the ellipse** $\frac{x^2}{9} + \frac{y^2}{16} = 1$, including the points on the ellipse. This can be confirmed by testing a point, for instance (0,0). Since $\frac{0^2}{9} + \frac{0^2}{16} = 0 \leq 1$ is a true statement, then the region containing the origin is the solution set to this inequality. We will shade it in blue, as in *Figure 3.7*.

The second inequality, $x^2 - y^2 \geq -1$, represents the sets of points **outside the hyperbola** $x^2 - y^2 \geq -1$, including the points on the curve. Again, we can confirm

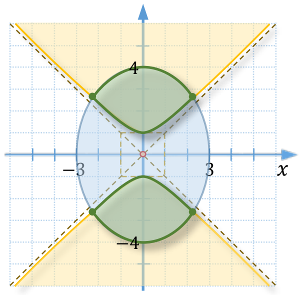


Figure 3.6

this by testing the $(0,0)$ point. Since $0^2 - 0^2 = 0 \geq -1$ is a false statement, then the solution set to this inequality is outside the region containing the origin. We will shade it in yellow, as in *Figure 3.7*.

Thus, the solution set of the system of these inequalities is the **intersection of the blue and yellow region**, as illustrated in green in *Figure 3.7*. The boundary of the green region is marked by a **solid line** as these points satisfy both inequalities and therefore are among the solutions of the system. Since the intersection points of the two curves also satisfy both inequalities, they are solutions of the system as well. So, we mark them using **filled-in circles**.

C.3 Exercises

Concept Check *Suppose that a nonlinear system is composed of equations whose graphs are those described, and the number of points of intersection of the two graphs is as given. Make a sketch satisfying these conditions. (There may be more than one way to do this.)*

1. a line and a circle; one intercept

2. a line and a hyperbola; no intercepts

3. a line and a hyperbola; two intercepts

4. a circle and an ellipse; four intercepts

5. a circle and an ellipse; three intercepts

6. a parabola and a hyperbola; one intercept

7. a parabola and a hyperbola; two intercepts

8. an ellipse and an ellipse; two intercepts

9. an ellipse and a hyperbola; no intercepts

10. an ellipse and a parabola; four intercepts

Concept Check

11. Give the maximum number of points at which the following pairs of graphs can intersect.

- **a.** a line and an ellipse
- **b.** a line and a parabola
- **c.** two different ellipses
- **d.** two different circles with centers at the origin
- **e.** two hyperbolas with centers at the origin

Solve each system.

12. $\begin{cases} y = x^2 + 6x \\ y = 4x \end{cases}$

13. $\begin{cases} y = x^2 + 8x + 16 \\ x - y = -4 \end{cases}$

14. $\begin{cases} xy = 12 \\ x + y = 8 \end{cases}$

15. $\begin{cases} xy = -5 \\ 2x + y = 3 \end{cases}$

16. $\begin{cases} x^2 + y^2 = 2 \\ 2x + y = 1 \end{cases}$

17. $\begin{cases} 2x^2 + 4y^2 = 4 \\ x = 4y \end{cases}$

18. $\begin{cases} x^2 + y^2 = 4 \\ y = x^2 - 2 \end{cases}$

19. $\begin{cases} x^2 + y^2 = 9 \\ y = 3 - x^2 \end{cases}$

20. $\begin{cases} x^2 + y^2 = 4 \\ x + y = 3 \end{cases}$

21. $\begin{cases} x^2 - 2y^2 = 1 \\ x = 2y \end{cases}$

22. $\begin{cases} 3x^2 + 2y^2 = 12 \\ x^2 + 3y^2 = 4 \end{cases}$

23. $\begin{cases} 2x^2 + 3y^2 = 6 \\ x^2 + 3y^2 = 3 \end{cases}$

24. $\begin{cases} (x-4)^2 + y^2 = 4 \\ (x+2)^2 + y^2 = 16 \end{cases}$

25. $\begin{cases} 4x^2 + y^2 = 30 \\ 5x^2 - y^2 = 15 \end{cases}$

26. $\begin{cases} \frac{x^2}{9} - y^2 = -1 \\ \frac{x^2}{16} - \frac{y^2}{4} = 1 \end{cases}$

Analytic Skills *Solve each problem by using a nonlinear system.*

27. Find the length and width of a rectangular room whose perimeter is 50 m and whose area is 100 m^2.

28. A calculator company has determined that the cost y (in thousands) to make x (thousand) calculators is

$$y = 4x^2 + 36x + 20,$$

while the revenue y (in thousands) from the sale of x (thousand) calculators is

$$36x^2 - 3y = 0.$$

Find the **break-even point**, where cost equals revenue.

Concept Check *True or False.*

29. A nonlinear system of equations can have up to four solutions.

30. The solution set of the inequality $x^2 + \frac{y^2}{25} \ge 1$ consists of points outside of the ellipse $x^2 + \frac{y^2}{25} = 1$, including the points of the ellipse.

31. The solution set of the inequality $x^2 + \frac{y^2}{25} < 1$ consists of points inside the ellipse $x^2 + \frac{y^2}{25} = 1$.

32. The intersection points of the curves $x^2 + y^2 = 5$ and $x - y = 3$ belong to the solution set of the system $\begin{cases} x^2 + y^2 \ge 5 \\ x - y > 3 \end{cases}$

33. The solution set of the inequality $y \ge x^2 + 3$ consists of points above or on the parabola $y = x^2 + 3$.

34. The solution set of the inequality $y < x^2 + 3$ consists of points below or on the parabola $y = x^2 + 3$.

Concept Check

35. Fill in each blank with the appropriate response.

The graph of the system $\begin{cases} x^2 + y^2 < 16 \\ y > -x \end{cases}$ consists of all points $\underset{(outside/inside)}{\underline{\qquad\qquad}}$ the circle $x^2 + y^2 = 16$ and $\underset{(above/below)}{\underline{\qquad\qquad}}$ the line $y = -x$.

Concept Check

36. Match each inequality with the graph of its solution set.

a. $y \geq x^2 - 2$ **b.** $y \leq x^2 - 2$ **c.** $y > x^2 - 2$ **d.** $y < x^2 - 2$

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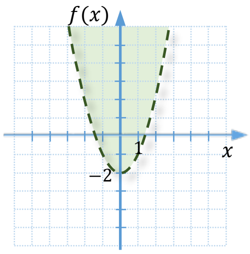

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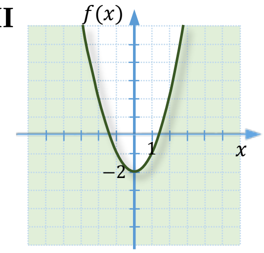

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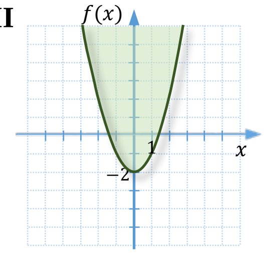

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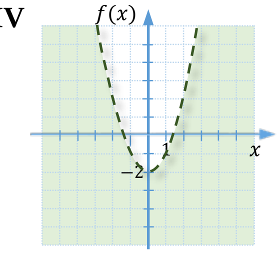

37. Match each inequality with the graph of its solution set.

a. $\frac{x^2}{16} + \frac{y^2}{4} < 1$ **b.** $\frac{x^2}{16} + \frac{y^2}{4} \leq 1$ **c.** $\frac{x^2}{16} + \frac{y^2}{4} \geq 1$ **d.** $\frac{x^2}{16} + \frac{y^2}{4} > 1$

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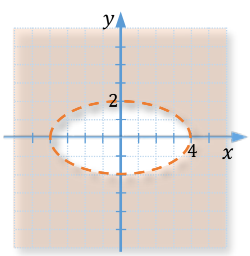

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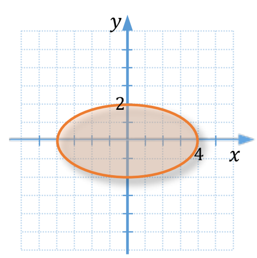

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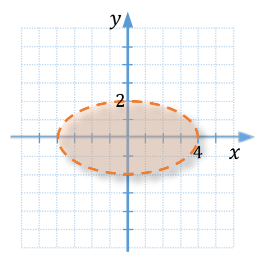

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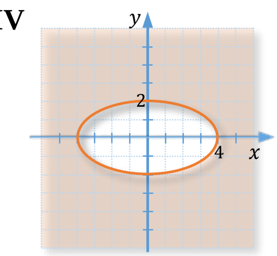

Graph each nonlinear inequality.

38. $x^2 + y^2 > 9$

39. $(x-1)^2 + (y+2)^2 \leq 16$

40. $y < 2x^2 - 6x$

41. $9x^2 + 4y^2 \geq 36$

42. $4x^2 - y^2 > 16$

43. $y \leq \frac{1}{2}(x+3)^2$

44. $x^2 + 9y^2 < 36$

45. $x^2 - 4 \geq -4y^2$

46. $y \geq x^2 - 8x + 12$

Graph the solution set to each nonlinear system of inequalities.

47. $\begin{cases} x^2 + y^2 < 16 \\ y > -2x \end{cases}$

48. $\begin{cases} y > x^2 - 4 \\ y < -x^2 + 3 \end{cases}$

49. $\begin{cases} x^2 + y^2 \geq 4 \\ x \geq 0 \end{cases}$

50. $\begin{cases} x^2 + y^2 \geq 1 \\ x^2 - 4y^2 \leq 16 \end{cases}$

51. $\begin{cases} x^2 + y^2 < 4 \\ y \geq x^2 + 3 \end{cases}$

52. $\begin{cases} x^2 + 16y^2 > 16 \\ 4x^2 + 9y^2 < 36 \end{cases}$

53. $\begin{cases} x^2 + y^2 \leq 4 \\ (x-2)^2 + y^2 \leq 4 \end{cases}$

54. $\begin{cases} x^2 - y^2 \leq 4 \\ x^2 + y^2 \leq 9 \end{cases}$

55. $\begin{cases} x^2 - y^2 \geq -4 \\ y < 1 - x^2 \end{cases}$

Discussion Point

56. Is it possible for a single point to be the only solution of a system of nonlinear inequalities? If so, give an example of such a system. If not, explain why.