Algebra Handout

Math 063/075

3.6 Linear Equations in One Variable

3.6 OBJECTIVES

1. Combine the addition and multiplication properties to solve an equation
2. Use the order of operations when solving an equation
3. Recognize identities
4. Recognize equations with no solutions

In all our examples thus far, either the addition property or the multiplication property was used in solving an equation. Often, finding a solution will require the use of both properties. If this is the case, we apply the addition property first and then use the multiplication property.

OBJECTIVE 1

Example 1 Solving Equations

(a) Solve.

$$4x - 5 = 7$$

NOTE If we were to begin by dividing by 4, we would need to distribute and divide:

$$\frac{4x}{4} - \frac{5}{4} = \frac{7}{4}$$

This is a valid operation but makes our work more difficult by introducing fractions.

It is easier to begin by adding 5 to both sides.

Here x is *multiplied* by 4. The result, $4x$, then has 5 subtracted from it (or -5 added to it) on the left side of the equation. These two operations mean that both properties must be applied in solving the equation.

Because the variable term is already on the left, we start by adding 5 to both sides:

$$\begin{array}{rcl} 4x - 5 &=& 7 \\ +5 && +5 \\ \hline 4x &=& 12 \end{array}$$

We now divide both sides by 4:

$$\frac{4x}{4} = \frac{12}{4}$$

$$x = 3$$

The solution is 3. To check, replace x with 3 in the original equation. Be careful to follow the rules for the order of operations.

$$4 \cdot 3 - 5 \stackrel{?}{=} 7$$

$$12 - 5 \stackrel{?}{=} 7$$

$$7 = 7 \quad \text{(True)}$$

(b) Solve.

$$\begin{array}{rcl} 3x + 8 &=& -4 \\ -8 && -8 \\ \hline 3x &=& -12 \end{array} \qquad \text{Add } -8 \text{ to both sides.}$$

Now divide both sides by 3 to isolate x on the left.

$$\frac{3x}{3} = \frac{-12}{3}$$

$$x = -4$$

The solution is -4. We'll leave it to you to check this result.

CHECK YOURSELF 1

Solve and check.

(a) $6x + 9 = -15$ **(b)** $5x - 8 = 7$

The variable may appear in any position in an equation. Just apply the rules carefully as you try to write an equivalent equation, and you will find the solution. Example 2 illustrates this property.

Example 2 Solving Equations

CAUTION

The sign in front of a term belongs to that term.

Solve.

$$\begin{array}{rcr} 3 - 2x & = & 9 \\ -3 & & -3 \\ \hline -2x & = & 6 \end{array}$$

First add -3 to both sides.

NOTE $\frac{-2}{-2} = 1$, so we divide by -2 to isolate x on the left.

Now divide both sides by -2. This will leave x alone on the left.

$$\frac{-2x}{-2} = \frac{6}{-2}$$

$$x = -3$$

The solution is -3. We'll leave it to you to check this result.

CHECK YOURSELF 2

Solve and check.

$10 - 3x = 1$

You may also have to combine multiplication with addition or subtraction to solve an equation. Consider Example 3.

Example 3 Solving Equations

(a) Solve.

$$\frac{1}{5}x - 3 = 4$$

To get the x term alone, we first add 3 to both sides.

$$\begin{array}{rcr} \frac{1}{5}x - 3 & = & 4 \\ +3 & & +3 \\ \hline \frac{1}{5}x & = & 7 \end{array}$$

Now, multiply both sides of the equation by 5, which is the reciprocal of $\frac{1}{5}$.

$$5\left(\frac{1}{5}x\right) = 5 \cdot 7$$
$$x = 35$$

The solution is 35. Just return to the original equation to check the result.

$$\frac{1}{5}(35) - 3 \stackrel{?}{=} 4$$
$$7 - 3 \stackrel{?}{=} 4$$
$$4 = 4 \qquad \text{(True)}$$

(b) Solve.

$$5 - \frac{1}{4}x = 2$$

To get the x term alone, we first add -5 to both sides.

$$\begin{array}{rcr} 5 - \frac{1}{4}x & = & 2 \\ \underline{-5 \qquad} & & \underline{-5} \\ \frac{-1}{4}x & = & -3 \end{array}$$

Now multiply both sides by -4, the reciprocal of $\frac{-1}{4}$.

$$(-4)\left(\frac{-1}{4}x\right) = (-4)(-3)$$

or

$$x = 12$$

The solution is 12. We'll leave it to you to check this result.

CHECK YOURSELF 3

Solve and check.

(a) $\frac{1}{6}x + 5 = 3$ **(b)** $-8 - \frac{1}{4}x = 10$

In Section 2.10, you learned how to solve certain equations when the variable appeared on both sides. Example 4 will show you how to extend that work by using the multiplication property of equality.

Example 4 Solving an Equation

Solve.

$6x - 14 = 3x - 2$

First add 14 to both sides. This will undo the subtraction on the left.

$$\begin{array}{rcl} 6x - 14 &=& 3x - 2 \\ +14 && +14 \\ \hline 6x &=& 3x + 12 \end{array}$$

Now add $-3x$ so that the terms in x will be on the left only.

$$\begin{array}{rcl} 6x &=& 3x + 12 \\ -3x && -3x \\ \hline 3x &=& 12 \end{array}$$

Finally divide by 3.

$$\frac{3x}{3} = \frac{12}{3}$$

$$x = 4$$

Check:

$$6(4) - 14 \stackrel{?}{=} 3(4) - 2$$

$$24 - 14 \stackrel{?}{=} 12 - 2$$

$$10 = 10 \quad \text{(True)}$$

As you know, the basic idea is to use our two properties to form an equivalent equation with the x isolated. Here we added 14 and then subtracted $3x$. You can do these steps in either order. Try it for yourself the other way. In either case, the multiplication property is then used as the *last step* in finding the solution.

CHECK YOURSELF 4

Solve and check.

$7x - 5 = 3x + 15$

We will look at two approaches to solving equations in which the coefficient on the right side is greater than the coefficient on the left side.

Example 5 Solving an Equation (Two Methods)

Solve $4x - 8 = 7x + 7$.

Method 1

$$
\begin{array}{rcrl}
4x - 8 & = & 7x + 7 & \\
-7x & & -7x & \text{Adding } -7x \text{ will get the variable terms on the left.} \\
\hline
-3x - 8 & = & 7 & \\
+8 & & +8 & \text{Adding 8 will leave the } x \text{ term alone on the left.} \\
\hline
-3x & = & 15 & \\
\dfrac{-3x}{-3} & = & \dfrac{15}{-3} & \text{Dividing by } -3 \text{ will isolate } x \text{ on the left.} \\
x & = & -5 &
\end{array}
$$

We'll let you check this result.

To avoid a negative coefficient (in this example, -3), some students prefer a different approach.

This time we'll work toward having the number on the *left* and the x term on the *right*, or

$\square = x$.

Method 2

NOTE It is usually easier to isolate the variable term on the side that will result in a positive coefficient.

$$
\begin{array}{rcrl}
4x - 8 & = & 7x + 7 & \\
-4x & & -4x & \text{Add } -4x \text{ to get the variables on the right.} \\
\hline
-8 & = & 3x + 7 & \\
-7 & & -7 & \text{Add } -7 \text{ to both sides.} \\
\hline
-15 & = & 3x & \\
\dfrac{-15}{3} & = & \dfrac{3x}{3} & \text{Divide by 3 to isolate } x \text{ on the right.} \\
-5 & = & x &
\end{array}
$$

Because $-5 = x$ and $x = -5$ are equivalent equations, it really makes no difference; the solution is still -5! You can use whichever approach you prefer.

CHECK YOURSELF 5

Solve $5x + 3 = 9x - 21$ by finding equivalent equations of the form $x = \square$ and $\square = x$ to compare the two methods of finding the solution.

It may also be necessary to remove grouping symbols in solving an equation.

OBJECTIVE 2

Example 6 Solving Equations that Contain Parentheses

Solve and check.

RECALL
$5(x-3)$
$= 5(x+(-3))$
$= 5x + 5(-3)$
$= 5x + (-15)$
$= 5x - 15$

$$
\begin{array}{rcll}
5(x-3) - 2x &=& x + 7 & \text{First, apply the distributive property.} \\
5x - 15 - 2x &=& x + 7 & \text{Combine like terms.} \\
3x - 15 &=& x + 7 & \\
+15 && +15 & \text{Add 15.} \\
\hline
3x &=& x + 22 & \\
-x && -x & \text{Add } -x. \\
\hline
2x &=& 22 & \text{Divide by 2.} \\
x &=& 11 &
\end{array}
$$

The solution is 11. To check, substitute 11 for x in the original equation. Again note the use of our rules for the order of operations.

$$
\begin{array}{rl}
5(11-3) - 2 \cdot 11 \stackrel{?}{=} 11 + 7 & \text{Simplify terms in parentheses.} \\
5 \cdot 8 - 2 \cdot 11 \stackrel{?}{=} 11 + 7 & \text{Multiply.} \\
40 - 22 \stackrel{?}{=} 11 + 7 & \text{Add and subtract.} \\
18 = 18 & \text{A true statement.}
\end{array}
$$

CHECK YOURSELF 6

Solve and check.

$7(x+5) - 3x = x - 7$

An equation that is true for any value of x is called an **identity.**

OBJECTIVE 3

Example 7 Solving an Equation

Solve the equation $2(x-3) = 2x - 6$.

$$
\begin{array}{rcl}
2(x-3) &=& 2x - 6 \\
2x - 6 &=& 2x - 6 \\
-2x && -2x \\
\hline
-6 &=& -6
\end{array}
$$

NOTE We could ask the question "For what values of x does $-6 = -6$?"

The statement $-6 = -6$ is true for any value of x. The original equation is an identity.

CHECK YOURSELF 7

Solve the equation $3(x-4) - 2x = x - 12$.

There are also equations for which there are no solutions.

OBJECTIVE 4

Example 8 Solving an Equation

Solve the equation $3(2x - 5) - 4x = 2x + 1$.

$$\begin{aligned} 3(2x - 5) - 4x &= 2x + 1 \\ 6x - 15 - 4x &= 2x + 1 \\ 2x - 15 &= 2x + 1 \\ -2x \qquad &\quad -2x \\ \hline -15 &= 1 \end{aligned}$$

These two numbers are never equal. The original equation has no solutions.

NOTE We could ask the question "For what values of x does $-15 = 1$?"

CHECK YOURSELF 8

Solve the equation $2(x - 5) + x = 3x - 3$.

Step by Step: Solving Linear Equations

Step 1 Use the distributive property to remove any grouping symbols. Then simplify by combining like terms on each side of the equation.

Step 2 Add or subtract the same term on each side of the equation until the variable term is on one side and a number is on the other.

Step 3 Multiply or divide both sides of the equation by the same nonzero number so that the variable is alone on one side of the equation. If no variable remains, determine whether the original equation is an identity or whether it has no solutions.

Step 4 Check the solution in the original equation.

NOTE Such an outline of steps is sometimes called an **algorithm** for the process.

READING YOUR TEXT

The following fill-in-the-blank exercises are designed to assure that you understand the key vocabulary used in this section. Each sentence usually comes directly from the section. You will find the correct answers in Appendix C.

Section 3.6

(a) Often, in solving an equation, both the __________ property and the multiplication property must be used.

(b) When both addition and multiplication are used in solving an equation, the __________ property is used as the last step.

(c) It is usually easier to isolate the variable term on the side that will result in a __________ coefficient.

(d) An equation that is true for any value of x is called an __________.

CHECK YOURSELF ANSWERS

1. **(a)** -4; **(b)** 3 **2.** 3 **3.** **(a)** -12; **(b)** -72 **4.** 5 **5.** 6 **6.** -14
7. The equation is an identity; x is any real number. **8.** There are no solutions.

3.6 Exercises

Name ______

Section ______ Date ______

Solve for x and check your result.

1. $2x + 1 = 9$ ALEKS
2. $3x - 1 = 17$ ALEKS
3. $3x - 2 = 7$ VIDEO ALEKS
4. $5x + 3 = 23$ ALEKS
5. $4x + 7 = 35$ ALEKS
6. $7x - 8 = 13$ ALEKS
7. $2x + 9 = 5$ ALEKS
8. $6x + 25 = -5$ ALEKS
9. $4 - 7x = 18$
10. $8 - 5x = -7$
11. $3 - 4x = -9$
12. $5 - 4x = 25$
13. $\frac{1}{2}x + 1 = 5$
14. $\frac{1}{3}x - 2 = 3$
15. $\frac{1}{4}x - 5 = 3$
16. $\frac{1}{5}x + 3 = 8$
17. $5x = 2x + 9$
18. $7x = 18 - 2x$
19. $3x = 10 - 2x$
20. $11x = 7x + 20$
21. $9x + 2 = 3x + 38$
22. $8x - 3 = 4x + 17$ VIDEO
23. $4x - 8 = x - 14$
24. $6x - 5 = 3x - 29$
25. $7x - 3 = 9x + 5$
26. $5x - 2 = 8x - 11$
27. $5x + 4 = 7x - 8$
28. $2x + 23 = 6x - 5$
29. $2x - 3 + 5x = 7 + 4x + 2$
30. $8x - 7 - 2x = 2 + 4x - 5$
31. $6x + 7 - 4x = 8 + 7x - 26$ VIDEO
32. $7x - 2 - 3x = 5 + 8x + 13$
33. $9x - 2 + 7x + 13 = 10x - 13$
34. $5x + 3 + 6x - 11 = 8x + 25$
35. $7(2x - 1) - 5x = x + 25$ ALEKS
36. $9(3x + 2) - 10x = 12x - 7$ ALEKS

ANSWERS

1. ______	2. ______
3. ______	4. ______
5. ______	6. ______
7. ______	8. ______
9. ______	10. ______
11. ______	12. ______
13. ______	14. ______
15. ______	16. ______
17. ______	18. ______
19. ______	20. ______
21. ______	22. ______

23. ______
24. ______
25. ______
26. ______
27. ______
28. ______
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35. ______
36. ______

ANSWERS

37. ______
38. ______
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41. ______
42. ______
43. ______
44. ______
45. ______
46. ______
47. ______
48. ______
49. ______
50. ______

37. $4(x + 5) = 4x + 20$

38. $-3(2x - 4) - 12 = -6x$

39. $5(x + 1) - 4x = x - 5$

40. $-4(2x - 3) = -8x + 5$

41. $6x - 4x + 1 = 12 + 2x - 11$

42. $-2x + 5x - 9 = 3(x - 4) - 5$

43. $-4(x + 2) - 11 = 2(-2x - 3) - 13$

44. $4(-x - 2) + 5 = -2(2x + 7)$

45. Create an equation of the form $ax + b = c$ that has 2 as a solution.

46. Create an equation of the form $ax + b = c$ that has 7 as a solution.

47. The equation $3x = 3x + 5$ has no solution, whereas the equation $7x + 8 = 8$ has zero as a solution. Explain the difference between a solution of zero and no solution.

48. Construct an equation for which every real number is a solution.

In exercises 49 and 50, write and solve an equation that models the given situation.

49. Science and Medicine To determine the upper limit for a person's heart rate during aerobic training, we subtract the person's age from 220 and then multiply by $\frac{9}{10}$. How old is a person whose upper limit heart rate is 153?

50. Science and Medicine To determine the lower limit for a person's heart rate during aerobic training, we subtract the person's age from 220 and then multiply by $\frac{3}{5}$. How old is a person whose lower limit heart rate is 111?

Answers

1. 4 **3.** 3 **5.** 7 **7.** -2 **9.** -2 **11.** 3 **13.** 8 **15.** 32
17. 3 **19.** 2 **21.** 6 **23.** -2 **25.** -4 **27.** 6 **29.** 4
31. 5 **33.** -4 **35.** 4 **37.** Identity **39.** No solution
41. Identity **43.** Identity **45.** **47.** **49.** 50

Special types of equations:
Identity, Contradiction (No solution), Conditional

You have practiced solving equations which look like this: $3x + 7 = -2$.
You would proceed as follows

$$\begin{aligned} 3x + 7 &= -2 \\ -7 \quad & -7 \\ 3x \quad &= -9 \\ x \quad &= -3 \end{aligned}$$

This type of equation with one (or several) solutions is called a **conditional equation**. If you substitute the solution back into the original equation and evaluate both sides you will get a true statement, like $-2 = -2$.

There are 2 other types of equations we will look at now.

1. Solve $5(x + 3) + 2x - 4 = 7x + 11$

First we would multiply the bracket by the 5, combine like terms, etc.

$$\begin{aligned} 5(x + 3) + 2x - 4 &= 7x + 11 \\ 5x + 15 + 2x - 4 &= 7x + 11 \\ 7x + 11 &= 7x + 11 \end{aligned}$$

That seems strange to get the identical expression on both sides. Let's keep going.

$$\begin{aligned} 7x + 11 \quad &= 7x + 11 \\ -7x \qquad & \quad - 7x \\ 11 &= 11 \end{aligned}$$

The x has disappeared from the equation and now we are left with the true statement $11 = 11$. When this happens, the original equation $5(x + 3) + 2x - 4 = 7x + 11$ is called an **identity** and the solution is **all real numbers**. This means you can choose any real number, like 3, 0 or -25, to substitute in for x in the original equation, evaluate both sides and you will get a true statement (like $32 = 32$ if you chose $x = 3$).

If you are asked to solve an equation and it turns out to be an identity, you can state for your answer: *The solution is all real numbers. The equation is an identity.*

You have seen that **conditional equations** have one (or several) solutions, and **identity equations** have all real numbers as their solution. We will now look at a type of equation called **contradictory equations** (or contradictions) which have no solutions.

2. Solve $4x + 5 = 2(2x - 8)$.

First we would multiply the bracket by the 2, combine like terms, etc

$$\begin{aligned} 4x + 5 &= 2(2x - 8) \\ 4x + 5 &= 4x - 16 \\ -4x \quad & \quad -4x \\ 5 \quad &= -16 \end{aligned}$$

The x has disappeared from the equation and now we are left with the false statement $5 = -16$. When this happens, the original equation $4x + 5 = 2(2x - 8)$ is called an **contradiction** and the solution is **no real numbers**. There is no value that you can substitute in for x and get a true statement.

If you are asked to solve an equation and it turns out to be a contradiction, you can state for your answer: *There is no solution. The equation is a contradiction.*

Practice:
State if the following is an identity, contradiction or conditional equation. State the solution.

1. $4x - 1 = 4(x + 1)$
2. $-3(2x + 1) - 4x = -10(x + 1) + 7$
3. $2x - 6 + 8x = 10x - 3 + x$
4. $3x + 4 = 6x - 11$
5. $-x + 15 - 3(x + 1) = 5 - 4x$
6. $5(x - 1) = 7x - 2(x + 1) - 3$

Answers: 1. Contradiction, no solution. 2. Identity, all real numbers. 3. Conditional, $x = -3$. 4. Conditional, $x = 5$. 5. Contradiction, no solution. 6. Identity, all real numbers.