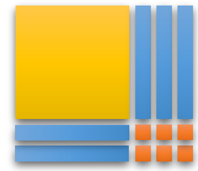
F.2 Factoring Trinomials

In this section, we discuss factoring trinomials. We start with factoring quadratic trinomials of the form $x^2 + bx + c$, then quadratic trinomials of the form $ax^2 + bx + c$, where $a \neq 1$, and finally trinomials reducible to quadratic by means of substitution.



Factorization of Quadratic Trinomials $x^2 + bx + c$

GUESSING METHOD

Factorization of a quadratic trinomial $x^2 + bx + c$ is the reverse process of the FOIL method of multiplying two linear binomials. Observe that

$$(x+p)(x+q) = x^2 + qx + px + pq = x^2 + (p+q)x + pq$$

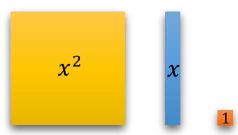
So, to reverse this multiplication, we look for two numbers p and q, such that the product pq equals to the free term c and the sum $p + q$ equals to the middle coefficient b of the trinomial.

$$x^2 + \underbrace{b}_{(p+q)} x + \underbrace{c}_{pq} = (x+p)(x+q)$$

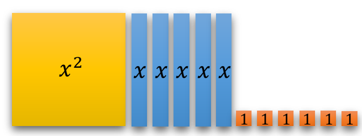
For example, to factor $x^2 + 5x + 6$, we think of two integers that multiply to 6 and add to 5. Such integers are 2 and 3, so $x^2 + 5x + 6 = (x+2)(x+3)$. Since multiplication is commutative, the order of these factors is not important.

VISUALIZATION OF FACTORING

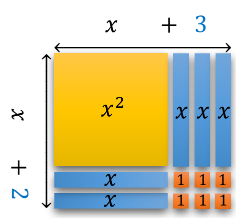
This could also be illustrated geometrically, using algebra tiles.



The area of a square with the side length x is equal to x^2. The area of a rectangle with the dimensions x by 1 is equal to x, and the area of a unit square is equal to 1. So, the trinomial $x^2 + 5x + 6$ can be represented as



To factor this trinomial, we would like to rearrange these tiles to fulfill a rectangle.



The area of such rectangle can be represented as the product of its length, $(x + 3)$, and width, $(x + 2)$ which becomes the factorization of the original trinomial.

In the trinomial examined above, the signs of the middle and the last terms are both positive. To analyse how different signs of these terms influence the signs used in the factors, observe the next three examples.

To factor $x^2 - 5x + 6$, we look for two integers that multiply to 6 and add to -5. Such integers are -2 and -3, so $x^2 - 5x + 6 = (x-2)(x-3)$.

To factor $x^2 + x - 6$, we look for two integers that multiply to -6 and add to 1. Such integers are -2 and 3, so $x^2 + x - 6 = (x-2)(x+3)$.

To factor $x^2 - x - 6$, we look for two integers that multiply to -6 and add to -1. Such integers are 2 and -3, so $x^2 - x - 6 = (x+2)(x-3)$.

Observation: The **positive constant** $\boldsymbol{c}$ in a trinomial $x^2 + bx + c$ tells us that the integers p and q in the factorization $(x+p)(x+q)$ are both of the **same sign** and their **sum** is the middle coefficient b. In addition, if b is positive, both p and q are positive, and if b is negative, both p and q are negative.

The **negative constant** $\boldsymbol{c}$ in a trinomial $x^2 + bx + c$ tells us that the integers p and q in the factorization $(x+p)(x+q)$ are of **different signs** and a **difference** of their absolute values is the middle coefficient b. In addition, the integer whose absolute value is larger takes the sign of the middle coefficient b.

These observations are summarized in the following **Table of Signs**.

Assume that $|p| \geq |q|$.

sum $\boldsymbol{b}$	product $\boldsymbol{c}$	$\boldsymbol{p}$	$\boldsymbol{q}$	**comments**
+	+	+	+	b is the ***sum*** of p and q
−	+	−	−	b is the ***sum*** of p and q
+	−	+	−	b is the ***difference*** $\lvert p\rvert - \lvert q\rvert$
−	−	−	+	b is the ***difference*** $\lvert q\rvert - \lvert p\rvert$

Example 1 ► **Factoring Trinomials with the Leading Coefficient Equal to 1**

Factor each trinomial, if possible.

a. $x^2 - 10x + 24$
b. $x^2 + 9x - 36$
c. $x^2 - 39xy - 40y^2$
d. $x^2 + 7x + 9$

Solution ► **a.** To factor the trinomial $x^2 - 10x + 24$, we look for two integers with a product of 24 and a sum of -10. The two integers are fairly easy to guess, -4 and -6. However, if one wishes to follow a more methodical way of finding these numbers, one can list the possible two-number factorizations of 24 and observe the sums of these numbers.

For simplicity, the table doesn't include signs of the integers. The signs are determined according to the **Table of Signs**.

product = 24 (pairs of factors of 24)	**sum = −10** (sum of factors)
1 · 24	25
2 · 12	14
3 · 8	11
4 · 6	10

Bingo!

Since the product is positive and the sum is negative, both integers must be negative. So, we take -4 and -6.

Thus, $x^2 - 10x + 24 = (\boldsymbol{x} - \boldsymbol{4})(\boldsymbol{x} - \boldsymbol{6})$. The reader is encouraged to check this factorization by multiplying the obtained binomials.

b. To factor the trinomial $x^2 + 9x - 36$, we look for two integers with a product of -36 and a sum of 9. So, let us list the possible factorizations of 36 into two numbers and observe the differences of these numbers.

product $= \mathbf{-36}$ (pairs of factors of 36)	**sum** $= \mathbf{9}$ (difference of factors)
$\mathbf{1 \cdot 36}$	35
$\mathbf{2 \cdot 18}$	16
$\mathbf{3 \cdot 12}$	9
$\mathbf{4 \cdot 9}$	5
$\mathbf{6 \cdot 6}$	0

This row contains the solution, so there is no need to list any of the subsequent rows.

Since the product is negative and the sum is positive, the integers are of different signs and the one with the larger absolute value assumes the sign of the sum, which is positive. So, we take 12 and -3.

Thus, $x^2 + 9x - 36 = (\boldsymbol{x} + \mathbf{12})(\boldsymbol{x} - \mathbf{3})$. Again, the reader is encouraged to check this factorization by mltiplying the obtained binomials.

c. To factor the trinomial $x^2 - 39xy - 40y^2$, we look for two binomials of the form $(x + ?\,y)(x + ?\,y)$ where the question marks are two integers with a product of -40 and a sum of 39. Since the two integers are of different signs and the absolute values of these integers differ by 39, the two integers must be -40 and 1.

Therefore, $x^2 - 39xy - 40y^2 = (\boldsymbol{x} - \mathbf{40}\boldsymbol{y})(\boldsymbol{x} + \boldsymbol{y})$.

Suggestion: Create a table of pairs of factors only if guessing the two integers with the given product and sum becomes too difficult.

d. When attempting to factor the trinomial $x^2 + 7x + 9$, we look for a pair of integers that would multiply to 9 and add to 7. There are only two possible factorizations of 9: $9 \cdot 1$ and $3 \cdot 3$. However, neither of the sums, $9 + 1$ or $3 + 3$, are equal to 7. So, there is no possible way of factoring $x^2 + 7x + 9$ into two linear binomials with integral coefficients. Therefore, if we admit only integral coefficients, this polynomial is **not factorable**.

Factorization of Quadratic Trinomials $ax^2 + bx + c$ with $a \neq 0$

Before discussing factoring quadratic trinomials with a leading coefficient different than 1, let us observe the multiplication process of two linear binomials with integral coefficients.

$$(\boldsymbol{m}x + p)(\boldsymbol{n}x + q) = mnx^2 + mqx + npx + pq = \underbrace{\boldsymbol{a}}_{\boldsymbol{mn}}\, x^2 + \underbrace{\boldsymbol{b}}_{(\boldsymbol{mq}+\boldsymbol{np})}\, x + \underbrace{\boldsymbol{c}}_{\boldsymbol{pq}}$$

To reverse this process, notice that this time, we are looking for four integers m, n, p, and q that satisfy the conditions

$$mn = a, \quad pq = c, \quad mq + np = b,$$

where a, b, c are the coefficients of the quadratic trinomial that needs to be factored. This produces a lot more possibilities to consider than in the guessing method used in the case of the leading coefficient equal to 1. However, if at least one of the outside coefficients, a or c, are prime, the guessing method still works reasonably well.

For example, consider $2x^2 + x - 6$. Since the coefficient $a = 2 = mn$ is a prime number, there is only one factorization of a, which is $1 \cdot 2$. So, we can assume that $m = 2$ and $n = 1$. Therefore,

$$2x^2 + x - 6 = (2x \pm |p|)(x \mp |q|)$$

Since the constant term $c = -6 = pq$ is negative, the binomial factors have different signs in the middle. Also, since pq is negative, we search for such p and q that the inside and outside products **differ** by the middle term $b = x$, up to its sign. The only factorizations of 6 are $1 \cdot 6$ and $2 \cdot 3$. So we try

GUESSING METHOD

$$2x^2 + x - 6 = (2x \pm 1)(x \mp 6)$$

x

$12x$

differs by $11x$ → too much

Observe that these two trials can be disregarded at once as 2 is not a common factor of all the terms of the trinomial, while it is a common factor of the terms of one of the binomials.

$$2x^2 + x - 6 = (2x \pm 6)(x \mp 1)$$

$6x$

$2x$

differs by $4x$ → still too much

$$2x^2 + x - 6 = (2x \pm 2)(x \mp 3)$$

$2x$

$6x$

differs by $4x$ → still too much

$$2x^2 + x - 6 = (2x \pm 3)(x \mp 2)$$

$3x$

$4x$

differs by x → perfect!

Then, since the difference between the inner and outer products should be positive, the larger product must be positive and the smaller product must be negative. So, we distribute the signs as below.

$$2x^2 + x - 6 = (2x - 3)(x + 2)$$

$-3x$

$4x$

In the end, it is a good idea to multiply the product to check if it results in the original polynomial. We leave this task to the reader.

What if the outside coefficients of the quadratic trinomial are both composite? Checking all possible distributions of coefficients m, n, p, and q might be too cumbersome. Luckily, there is another method of factoring, called **decomposition**.

The decomposition method is based on the reverse FOIL process.
Suppose the polynomial $6x^2 + 19x + 15$ factors into $(mx + p)(nx + q)$. Observe that the FOIL multiplication of these two binomials results in the four term polynomial,

$$mnx^2 + mqx + npx + pq,$$

which after combining the two middle terms gives us the original trinomial. So, reversing these steps would lead us to the factored form of $6x^2 + 19x + 15$.

To reverse the FOIL process, we would like to:

This product is often referred to as the **master product** or the ***ac*-product**.

- Express the middle term, $19x$, as a sum of two terms, mqx and npx, such that the product of their coefficients, $mnpq$, is equal to the product of the outside coefficients $ac = 6 \cdot 15 = 90$.
- Then, factor the four-term polynomial by grouping.

Thus, we are looking for two integers with the product of 90 and the sum of 19. One can check that 9 and 10 satisfy these conditions. Therefore,

DECOMPOSITION METHOD

$$\begin{aligned} &6x^2 + 19x + 15 \\ &= 6x^2 + 9x + 10x + 15 \\ &= 3x(2x+3) + 5(2x+3) \\ &= (2x+3)(3x+5) \end{aligned}$$

Example 2 ► **Factoring Trinomials with the Leading Coefficient Different than 1**

Factor completely each trinomial.

a. $6x^3 + 14x^2 + 4x$ **b.** $-6y^2 - 10 + 19y$
c. $18a^2 - 19ab - 12b^2$ **d.** $2(x+3)^2 + 5(x+3) - 12$

Solution ► **a.** First, we factor out the GCF, which is $2x$. This gives us

$$6x^3 + 14x^2 + 4x = 2x(3x^2 + 7x + 2)$$

The outside coefficients of the remaining trinomial are prime, so we can apply the guessing method to factor it further. The first terms of the possible binomial factors must be $3x$ and x while the last terms must be 2 and 1. Since both signs in the trinomial are positive, the signs used in the binomial factors must be both positive as well. So, we are ready to give it a try:

$$2x(3x + 2)(x + 1) \quad \text{or} \quad 2x(3x + 1)(x + 2)$$
$$2x \qquad\qquad\qquad x$$
$$3x \qquad\qquad\qquad 6x$$

The first distribution of coefficients does not work as it would give us $2x + 3x = 5x$ for the middle term. However, the second distribution works as $x + 6x = 7x$, which matches the middle term of the trinomial. So,

$$6x^3 + 14x^2 + 4x = \mathbf{2x(3x+1)(x+2)}$$

b. Notice that the trinomial is not arranged in decreasing order of powers of y. So, first, we rearrange the last two terms to achieve the decreasing order. Also, we factor out the -1, so that the leading term of the remaining trinomial is positive.

$$-6y^2 - 10 + 19y = -6y^2 + 19y - 10 = -(6y^2 - 19y + 10)$$

Then, since the outside coefficients are composite, we will use the decomposition method of factoring. The ac-product equals to 60 and the middle coefficient equals to -19. So, we are looking for two integers that multiply to 60 and add to -19. The integers that satisfy these conditions are -15 and -4. Hence, we factor

$$\begin{aligned} &-(6y^2 - 19y + 10) \\ &= -(6y^2 - 15y - 4y + 10) \\ &= -[3y(2y-5) - 2(2y-5)] \\ &= -(\mathbf{2y-5})(\mathbf{3y-2}) \end{aligned}$$

the **square bracket** is essential because of the negative sign outside

remember to **reverse the sign**!

c. There is no common factor to take out of the polynomial $18a^2 - 19ab - 12b^2$. So, we will attempt to factor it into two binomials of the type $(ma \pm pb)(na \mp qb)$, using the decomposition method. The ac-product equals $-12 \cdot 18 = -2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$ and the middle coefficient equals -19. To find the two integers that multiply to the ac-product and add to -19, it is convenient to group the factors of the product

$$2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$$

in such a way that the products of each group differ by 19. It turns out that grouping all the 2's and all the 3's satisfy this condition, as 8 and 27 differ by 19. Thus, the desired integers are -27 and 8, as the sum of them must be -19. So, we factor

$$\begin{aligned} &18a^2 - 19ab - 12b^2 \\ &= 18a^2 - 27ab + 8ab - 12b^2 \\ &= 9a(2a-3b) + 4b(2a-3b) \\ &= (\mathbf{2a-3b})(\mathbf{9a+4b}) \end{aligned}$$

d. To factor $2(x+3)^2 + 5(x+3) - 12$, first, we notice that treating the group $(x+3)$ as another variable, say a, simplify the problem to factoring the quadratic trinomial

$$2a^2 + 5a - 12$$

This can be done by the guessing method. Since

$$2a^2 + 5a - 12 = (2a-3)(a+4),$$

$$-3a$$

$$8a$$

then

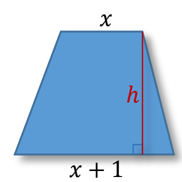
$$\begin{aligned} 2(x+3)^2 + 5(x+3) - 12 &= [2(x+3) - 3][(x+3) + 4] \\ &= (2x + 6 - 3)(x + 3 + 4) \\ &= (\mathbf{2x+3})(\mathbf{x+7}) \end{aligned}$$

Note 1: Polynomials that can be written in the form $\boldsymbol{a}(\quad)^{\mathbf{2}} + \boldsymbol{b}(\quad) + \boldsymbol{c}$, where $a \neq 0$ and $(\quad)$ represents any nonconstant polynomial expression, are referred to as **quadratic in form**. To factor such polynomials, it is convenient to **replace** the expression in the bracket by **a single variable**, different than the original one. This was illustrated in *Example 2d* by substituting a for $(x+3)$. However, when using this **substitution method**, we must **remember to leave the final answer in terms of the original variable**. So, after factoring, we replace a back with $(x+3)$, and then simplify each factor.

Note 2: Some students may feel comfortable factoring polynomials quadratic in form directly, without using substitution.

Example 3 ► **Application of Factoring in Geometry Problems**

If the area of a trapezoid is $2x^2 + 5x + 2$ square meters and the lengths of the two parallel sides are x and $x + 1$ meters, then what polynomial represents the height of the trapezoid?



Solution ► Using the formula for the area of a trapezoid, we write the equation

$$\frac{1}{2}h(a+b) = 2x^2 + 5x + 2$$

Since $a + b = x + (x+1) = 2x + 1$, then we have

$$\frac{1}{2}h(2x+1) = 2x^2 + 5x + 2,$$

which after factoring the right-hand side gives us

$$\frac{1}{2}h(2x+1) = (2x+1)(x+2).$$

To find h, it is enough to divide the above equation by the common factor $(2x+1)$ and then multiply it by 2. So,

$$h = 2(x+2) = \mathbf{2x+4}.$$

F.2 Exercises

Vocabulary Check *Complete each blank with the most appropriate term or phrase from the given list:* ***decomposition, guessing, multiplication, prime, quadratic, sum, variable.***

1. Any factorization can be checked by using __________________.

2. To factor a quadratic trinomial with a leading coefficient equal to 1, we usually use the _______________ method.

3. To factor $x^2 + bx + c$ using the guessing method, write the trinomial as $(x + ?)(x + ?)$, where the question marks are two factors of c whose ________ is b.

4. To factor a quadratic trinomial with a leading coefficient different than 1, we usually use the _______________ method. If one of the outside coefficients is a ___________ number, we can still use the guessing method.

5. To factor polynomials that are ______________ in form, it is convenient to substitute a single variable (different than the original one) for the expression that appears in the first and the second power. However, the final factorization must be expressed back in the original ___________.

Concept Check

6. If $ax^2 + bx + c$ has no monomial factor, can either of the possible binomial factors have a monomial factor?

7. Is $(2x + 5)(2x - 4)$ a complete factorization of the polynomial $4x^2 + 2x - 20$?

8. When factoring the polynomial $-2x^2 - 7x + 15$, students obtained the following answers:
$$(-2x + 3)(x + 5),\ (2x - 3)(-x - 5),\ \text{or}\ -(2x - 3)(x + 5)$$
Which of the above factorizations are correct?

9. Is the polynomial $x^2 - x + 2$ factorable or is it prime?

Concept Check *Fill in the missing factor.*

10. $x^2 - 4x + 3 = (\qquad)(x - 1)$

11. $x^2 + 3x - 10 = (\qquad)(x - 2)$

12. $x^2 - xy - 20y^2 = (x + 4y)(\qquad)$

13. $x^2 + 12xy + 35y^2 = (x + 5y)(\qquad)$

Factor, if possible.

14. $x^2 + 7x + 12$

15. $x^2 - 12x + 35$

16. $y^2 + 2y - 48$

17. $a^2 - a - 42$

18. $x^2 + 2x + 3$

19. $p^2 - 12p - 27$

20. $m^2 - 15m + 56$

21. $y^2 + 3y - 28$

22. $18 - 7n - n^2$

23. $20 + 8p - p^2$

24. $x^2 - 5xy + 6y^2$

25. $p^2 + 9pq + 20q^2$

Factor completely.

26. $-x^2 + 4x + 21$

27. $-y^2 + 14y + 32$

28. $n^4 - 13n^3 - 30n^2$

29. $y^3 - 15y^2 + 54y$

30. $-2x^2 + 28x - 80$

31. $-3x^2 - 33x - 72$

32. $x^4y + 7x^2y - 60y$

33. $24ab^2 + 6a^2b^2 - 3a^3b^2$

34. $40 - 35t^{15} - 5t^{30}$

35. $x^4y^2 + 11x^2y + 30$

36. $64n - 12n^5 - n^9$

37. $24 - 5x^a - x^{2a}$

Discussion Point

38. A polynomial $x^2 + \boxed{}x + 75$ with an unknown coefficient b by the middle term can be factored into two binomials with integral coefficients. What are the possible values of b?

Concept Check *Fill in the missing factor.*

39. $2x^2 + 7x + 3 = (\qquad)(x + 3)$

40. $3x^2 - 10x + 8 = (\qquad)(x - 2)$

41. $4x^2 + 8x - 5 = (2x - 1)(\qquad)$

42. $6x^2 - x - 15 = (2x + 3)(\qquad)$

Factor completely.

43. $2x^2 - 5x - 3$

44. $6y^2 - y - 2$

45. $4m^2 + 17m + 4$

46. $6t^2 - 13t + 6$

47. $10x^2 + 23x - 5$

48. $42n^2 + 5n - 25$

49. $3p^2 - 27p + 24$

50. $-12x^2 - 2x + 30$

51. $6x^2 + 41xy - 7y^2$

52. $18x^2 + 27xy + 10y^2$

53. $8 - 13a + 6a^2$

54. $15 - 14n - 8n^2$

55. $30x^4 + 3x^3 - 9x^2$

56. $10x^3 - 6x^2 + 4x^4$

57. $2y^6 + 7xy^3 + 6x^2$

58. $9x^2y^2 - 4 + 5xy$

59. $16x^2y^3 + 3y - 16xy^2$

60. $4p^4 - 28p^2q + 49q^2$

61. $4(x-1)^2 - 12(x-1) + 9$

62. $2(a+2)^2 + 11(a+2) + 15$

63. $4x^{2a} - 4x^a - 3$

Discussion Point

64. A polynomial $2x^2 + \boxed{}x - 15$ with an unknown coefficient b by the middle term can be factored into two binomials with integral coefficients. What are the possible values of ?

Analytic Skills

65. If the volume of a case of apples is $x^3 + x^2 - 2x$ cubic feet and the height of this box is $(x - 1)$ feet, then what polynomial represents the area of the bottom of the case?

66. A ceremonial red carpet is rectangular in shape and covers $2x^2 + 11x + 12$ square feet. If the width of the carpet is $(x + 4)$ feet, express the length, in feet.