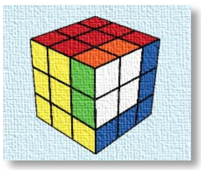
F.3 Special Factoring and a General Strategy of Factoring



Recall that in *Section P2*, we considered formulas that provide a shortcut for finding special products, such as a product of two **conjugate** binomials,

$$(a+b)(a-b)=\boldsymbol{a^2-b^2},$$

or the **perfect square** of a binomial,

$$(a\pm b)^2=\boldsymbol{a^2\pm 2ab+b^2}.$$

Since factoring reverses the multiplication process, these formulas can be used as shortcuts in factoring binomials of the form $\boldsymbol{a^2-b^2}$ (**difference of squares)**, and trinomials of the form $\boldsymbol{a^2\pm 2ab+b^2}$ **(perfect square)**. In this section, we will also introduce a formula for factoring binomials of the form $\boldsymbol{a^3\pm b^3}$ (**sum or difference of cubes)**. These special product factoring techniques are very useful in simplifying expressions or solving equations, as they allow for more efficient algebraic manipulations.

At the end of this section, we give a summary of all the factoring strategies shown in this chapter.

Difference of Squares

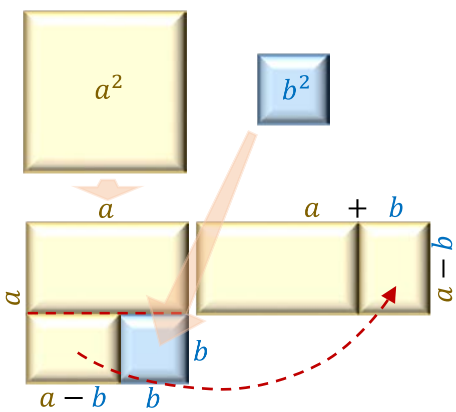


Figure 3.1

Out of the special factoring formulas, the easiest one to use is the difference of squares,

$$\boldsymbol{a^2-b^2=(a+b)(a-b)}$$

Figure 3.1 shows a geometric interpretation of this formula. The area of the yellow square, a^2, diminished by the area of the blue square, b^2, can be rearranged to a rectangle with the length of $(a+b)$ and the width of $(a-b)$.

To factor a difference of squares $\boldsymbol{a^2-b^2}$, first, identify $\boldsymbol{a}$ and $\boldsymbol{b}$, which are the expressions being squared, and then, form two factors, the sum $(\boldsymbol{a+b})$, and the difference $(\boldsymbol{a-b})$, as illustrated in the example below.

Example 1 ► **Factoring Differences of Squares**

Factor each polynomial completely.

a. $25x^2-1$ **b.** $3.6x^4-0.9y^6$

c. x^4-81 **d.** $16-(a-2)^2$

Solution ► **a.** First, we rewrite each term of $25x^2-1$ as a perfect square of an expression.

$$25x^2-1=(\underset{a}{5x})^2-\underset{b}{1}^2$$

Then, treating $5x$ as the a and 1 as the b in the difference of squares formula $a^2-b^2=(a+b)(a-b)$, we factor:

$$\begin{array}{ccccc} a^2 - b^2 &=& (a+b)(a-b) \\ \downarrow \quad \downarrow & & \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ 25x^2 - 1 = (5x)^2 - 1^2 &=& (\mathbf{5x+1})(\mathbf{5x-1}) \end{array}$$

b. First, we factor out 0.9 to leave the coefficients in a perfect square form. So,

$$3.6x^4 - 0.9y^6 = 0.9(4x^4 - y^6)$$

Then, after writing the terms of $4x^4 - y^6$ as perfect squares of expressions that correspond to a and b in the difference of squares formula $a^2 - b^2 = (a+b)(a-b)$, we factor

$$0.9(4x^4 - y^6) = 0.9[(\underset{a}{2x^2})^2 - (\underset{b}{y^3})^2] = \mathbf{0.9(2x^2 + y^3)(2x^2 - y^3)}$$

c. Similarly as in the previous two examples, $x^4 - 81$ can be factored by following the difference of squares pattern. So,

$$x^4 - 81 = (x^2)^2 - (9)^2 = (x^2 + 9)(x^2 - 9)$$

However, this factorization is not complete yet. Notice that $x^2 - 9$ is also a difference of squares, so the original polynomial can be factored further. Thus,

$$x^4 - 81 = (x^2 + 9)(x^2 - 9) = \mathbf{(x^2 + 9)(x + 3)(x - 3)}$$

Attention: The sum of squares, $x^2 + 9$, cannot be factored using real coefficients.

Generally, except for a common factor, a quadratic binomial of the form $\mathbf{a^2 + b^2}$ is **not factorable** over the real numbers.

Recall that
$\mathbf{a^2 + b^2 \neq (a + b)^2}$

d. Following the difference of squares formula, we have

$$\begin{aligned} 16 - (a-2)^2 &= 4^2 - (a-2)^2 \\ &= [4 + (a-2)][4 - (a-2)] \\ &= (4 + a - 2)(4 - a + 2) \quad \text{work out the inner brackets} \\ &= \mathbf{(2 + a)(6 - a)} \quad \text{combine like terms} \end{aligned}$$

Remember to use brackets after the negative sign!

Perfect Squares

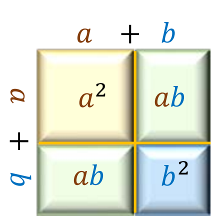


Figure 3.2

Another frequently used special factoring formula is the **perfect square** of a sum or a difference.

$$\mathbf{a^2 + 2ab + b^2 = (a + b)^2}$$

or

$$\mathbf{a^2 - 2ab + b^2 = (a - b)^2}$$

Figure 3.2 shows the geometric interpretation of the perfect square of a sum. We encourage the reader to come up with a similar interpretation of the perfect square of a difference.

To factor a perfect square trinomial $\boldsymbol{a^2 \pm 2ab + b^2}$, we find $\boldsymbol{a}$ and $\boldsymbol{b}$, which are the expressions being squared. Then, depending on the middle sign, we use $\boldsymbol{a}$ and $\boldsymbol{b}$ to form the perfect square of the sum $(\boldsymbol{a}+\boldsymbol{b})^2$, or the perfect square of the difference $(\boldsymbol{a}-\boldsymbol{b})^2$.

Example 2 ► **Identifying Perfect Square Trinomials**

Decide whether the given polynomial is a perfect square.

a. $9x^2 + 6x + 4$ **b.** $9x^2 + 4y^2 - 12xy$

c. $25p^4 + 40p^2 - 16$ **d.** $49y^6 + 84xy^3 + 36x^2$

Solution ► **a.** Observe that the outside terms of the trinomial $9x^2 + 6x + 4$ are perfect squares, as $9x^2 = (3x)^2$ and $4 = 2^2$. So, the trinomial would be a perfect square if the middle terms would equal $2 \cdot 3x \cdot 2 = 12x$. Since this is not the case, our trinomial is **not a perfect square**.

Attention: Except for a common factor, trinomials of the type $\boldsymbol{a^2 \pm ab + b^2}$ are **not factorable** over the real numbers!

b. First, we arrange the trinomial in decreasing order of the powers of x. So, we obtain $9x^2 - 12xy + 4y^2$. Then, since $9x^2 = (3x)^2$, $4y^2 = (2y)^2$, and the middle term (except for the sign) equals $2 \cdot 3x \cdot 2 = 12x$, we claim that the trinomial **is a perfect square**. Since the middle term is negative, this is the perfect square of a difference. So, the trinomial $9x^2 - 12xy + 4y^2$ can be seen as

$$\begin{array}{c} a^2 - 2\,a\,b + b^2 = (a - b)^2 \\ \downarrow \\ (3x)^2 - 2 \cdot 3x \cdot 2y + (2y)^2 = (3x - 2y)^2 \end{array}$$

c. Even though the coefficients of the trinomial $25p^4 + 40p^2 - 16$ and the distribution of powers seem to follow the pattern of a perfect square, the last term is negative, which makes it **not a perfect square**.

d. Since $49y^6 = (7y^3)^2$, $36x^2 = (6x)^2$, and the middle term equals $2 \cdot 7y^3 \cdot 6x = 84xy^3$, we claim that the trinomial **is a perfect square**. Since the middle term is positive, this is the perfect square of a sum. So, the trinomial $9x^2 - 12xy + 4y^2$ can be seen as

$$\begin{array}{c} a^2 + 2\,a\,b + b^2 = (a - b)^2 \\ \downarrow \\ (7y^3)^2 + 2 \cdot 7y^3 \cdot 6x + (6x)^2 = (7y^3 - 6x)^2 \end{array}$$

Example 3 ► **Factoring Perfect Square Trinomials**

Factor each polynomial completely.

a. $25x^2 + 10x + 1$ **b.** $a^2 - 12ab + 36b^2$

c. $m^2 - 8m + 16 - 49n^2$ **d.** $-4y^2 - 144y^8 + 48y^5$

Solution ►

a. The outside terms of the trinomial $25x^2 + 10x + 1$ are perfect squares of $5a$ and 1, and the middle term equals $2 \cdot 5x \cdot 1 = 10x$, so we can follow the perfect square formula. Therefore,

$$25x^2 + 10x + 1 = (\mathbf{5x} + \mathbf{1})^{\mathbf{2}}$$

b. The outside terms of the trinomial $a^2 - 12ab + 36b^2$ are perfect squares of a and $6b$, and the middle term (disregarding the sign) equals $2 \cdot a \cdot 6b = 12ab$, so we can follow the perfect square formula. Therefore,

$$a^2 - 12ab + 36b^2 = (\boldsymbol{a} - \mathbf{6}\boldsymbol{b})^{\mathbf{2}}$$

c. Observe that the first three terms of the polynomial $m^2 - 8m + 16 - 49n^2$ form a perfect square of $m - 6$ and the last term is a perfect square of $7n$. So, we can write

$$m^2 - 8m + 16 - 49n^2 = (m - 6)^2 - (7n)^2$$

This is not in factored form yet!

Notice that this way we have formed a difference of squares. So we can factor it by following the difference of squares formula

$$(m - 6)^2 - (7n)^2 = (\boldsymbol{m} - \mathbf{6} - \mathbf{7}\boldsymbol{n})(\boldsymbol{m} - \mathbf{6} + \mathbf{7}\boldsymbol{n})$$

d. As in any factoring problem, first we check the polynomial $-4y^2 - 144y^8 + 48y^5$ for a common factor, which is $4y^2$. To leave the leading term of this polynomial positive, we factor out $-4y^2$. So, we obtain

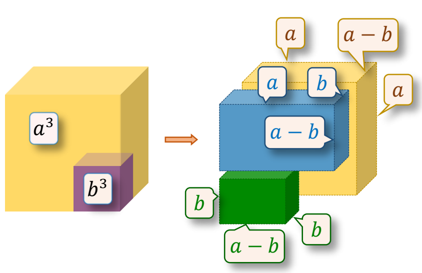
$$-4y^2 - 144y^8 + 48y^5$$
$$= -4y^2\,(1 + 36y^6 - 12y^3)$$
$$= -4y^2\,(36y^6 - 12y^3 + 1)$$

arrange the polynomial in decreasing powers

$$= \mathbf{-4y^2\,(6y^3 - 1)^2}$$

fold to the perfect square form

Sum or Difference of Cubes



$a^3 - b^3 = a^2(a - b) + ab(a - b) + b^2(a - b)$

$= (a - b)(a^2 + ab + b^2)$

Figure 3.3

The last special factoring formula to discuss in this section is the **sum or difference of cubes**.

$$\boldsymbol{a^3 + b^3 = (a + b)(a^2 - ab + b^2)}$$

or

$$\boldsymbol{a^3 - b^3 = (a - b)(a^2 + ab + b^2)}$$

The reader is encouraged to confirm these formulas by multiplying the factors in the right-hand side of each equation. In addition, we offer a geometric visualization of one of these formulas, the difference of cubes, as shown in *Figure 3.3*.

Hints for memorization of the sum or difference of cubes formulas:

- The binomial factor is a copy of the sum or difference of the terms that were originally cubed.
- The trinomial factor follows the pattern of a perfect square, except that the **middle term is single**, not doubled.
- The signs in the factored form follow the pattern ***S****ame*-***O****pposite*-***P****ositive* (**SOP**).

Example 4 ► **Factoring Sums or Differences of Cubes**

Factor each polynomial completely.

a. $8x^3 + 1$ **b.** $27x^7y - 125xy^4$

c. $2n^6 - 128$ **d.** $(p-2)^3 + q^3$

Solution ► **a.** First, we rewrite each term of $8x^3 + 1$ as a perfect cube of an expression.

$$\overset{a}{\downarrow}\quad \overset{b}{\downarrow}$$
$$8x^3 + 1 = (2x)^2 + 1^2$$

Then, treating $2x$ as the a and 1 as the b in the sum of cubes formula $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$, we factor:

$$\mathbf{a^3 + b^3 = (a + b)(a^2 - a\ b + b^2)}$$
$$8x^3 + 1 = (2x)^2 + 1^2 = (2x + 1)((2x)^2 - 2x \cdot 1 + 1^2)$$
$$= \mathbf{(2x + 1)(4x^2 - 2x + 1)}$$

Quadratic trinomials of the form $\boldsymbol{a^2 \pm ab + b^2}$ are **not factorable**!

Notice that the trinomial $4x^2 - 2x + 1$ in not factorable anymore.

b. Since the two terms of the polynomial $27x^7y - 125xy^4$ contain the common factor xy, we factor it out and obtain

$$27x^7y - 125xy^4 = xy(27x^6 - 125y^3)$$

Observe that the remaining polynomial is a difference of cubes, $(3x^2)^3 - (5y)^3$. So, we factor,

$$27x^7y - 125xy^4 = xy[(3x^2)^3 - (5y)^3]$$
$$\mathbf{(a + b)(a^2 - a\ b + b^2)}$$
$$= xy(3x^2 - 5y)[(3x^2)^2 + 3x^2 \cdot 5y + (5y)^2]$$
$$= \mathbf{xy(3x^2 - 5y)(9x^4 + 15x^2y + 25y^2)}$$

c. After factoring out the common factor 2, we obtain

$$2n^6 - 128 = 2(n^6 - 64)$$

Difference of squares or **difference of cubes?**

Notice that $n^6 - 64$ can be seen either as a difference of squares, $(n^3)^2 - 8^2$, or as a difference of cubes, $(n^2)^3 - 4^3$. It turns out that applying the **difference of squares** formula first **leads us to a complete factorization** while starting with the difference of cubes does not work so well here. See the two approaches below.

$$(n^3)^2 - 8^2$$
$$= (n^3 + 8)(n^3 - 8)$$
$$= (n + 2)(n^2 - 2n + 4)(n - 2)(n^2 + 2n + 4)$$

4 prime factors, so the factorization is complete

$$(n^2)^3 - 4^3$$
$$= (n^2 - 4)(n^4 + 4n^2 + 16)$$
$$= (n + 2)(n - 2)(n^4 + 4n^2 + 16)$$

There is no easy way of factoring this trinomial!

Therefore, the original polynomial should be factored as follows:

$$2n^6 - 128 = 2(n^6 - 64) = 2[(n^3)^2 - 8^2] = 2(n^3 + 8)(n^3 - 8)$$
$$= \mathbf{2(n + 2)(n^2 - 2n + 4)(n - 2)(n^2 + 2n + 4)}$$

d. To factor $(p - 2)^3 + q^3$, we follow the sum of cubes formula $(a + b)(a^2 - ab + b^2)$ by assuming $a = p - 2$ and $b = q$. So, we have

$$(p - 2)^3 + q^3 = (p - 2 + q)\,[(p - 2)^2 - (p - 2)q + q^2]$$
$$= (p - 2 + q)\,[p^2 - 2pq + 4 - pq + 2q + q^2]$$
$$= \mathbf{(p - 2 + q)\,[p^2 - 3pq + 4 + 2q + q^2]}$$

General Strategy of Factoring

Recall that a polynomial with integral coefficients is factored completely if all of its factors are prime over the integers.

How to Factorize Polynomials Completely?

1. Factor out all **common factors**. Leave the remaining polynomial with a positive leading term and integral coefficients, if possible.

2. Check the number of terms. If the polynomial has

- **more than three terms**, try to factor by **grouping**; a four term polynomial may require 2-2, 3-1, or 1-3 types of grouping.
- **three terms**, factor by **guessing**, **decomposition**, or follow the **perfect square** formula, if applicable.
- **two terms**, follow the **difference of squares**, or **sum or difference of cubes** formula, if applicable. Remember that sum of squares, $\boldsymbol{a^2 + b^2}$, is **not factorable** over the real numbers, except for possibly a common factor.

3. Keep in mind the **special factoring formulas**:

Difference of Squares	$\boldsymbol{a^2 - b^2 = (a+b)(a-b)}$
Perfect Square of a Sum	$\boldsymbol{a^2 + 2ab + b^2 = (a+b)^2}$
Perfect Square of a Difference	$\boldsymbol{a^2 - 2ab + b^2 = (a-b)^2}$
Sum of Cubes	$\boldsymbol{a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)}$
Difference of Cubes	$\boldsymbol{a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)}$

4. **Keep factoring** each of the obtained factors until all of them are **prime** over the integers.

Example 5 ► **Multiple-step Factorization**

Factor each polynomial completely.

a. $80x^5 - 5x$ **b.** $4a^2 - 4a + 1 - b^2$

c. $(5r+8)^2 - 6(5r+8) + 9$ **d.** $(p-2q)^3 + (p+2q)^3$

Solution ►

a. First, we factor out the GCF of $80x^5$ and $-5x$, which equals to $5x$. So, we obtain

$$80x^5 - 5x = 5x(16x^4 - 1)$$

repeated difference of squares

Then, we notice that $16x^4 - 1$ can be seen as the difference of squares $(4x^2)^2 - 1^2$. So, we factor further

$$80x^5 - 5x = 5x(4x^2+1)(4x^2-1)$$

The first binomial factor, $4x^2 + 1$, cannot be factored any further using integral coefficients as it is the sum of squares, $(2x)^2 + 1^2$. However, the second binomial factor, $4x^2 - 1$, is still factorable as a difference of squares, $(2x)^2 - 1^2$. Therefore,

$$80x^5 - 5x = \boldsymbol{5x\left(4x^2+1\right)(2x+1)(2x-1)}$$

This is a complete factorization as all the factors are prime over the integers.

3-1 type of grouping

b. The polynomial $4a^2 - 4a + 1 - b^2$ consists of four terms, so we might be able to factor it by grouping. Observe that the 2-2 type of grouping has no chance to succeed, as the first two terms involve only the variable a while the second two terms involve only the variable b. This means that after factoring out the common factor in each group, the remaining binomials would not be the same. So, the 2-2 grouping would not lead us to a factorization. However, the 3-1 type of grouping should help. This is because the first three terms form the perfect square, $(2a-1)^2$, and there is a subtraction before the last term b^2, which is also a perfect square. So, in the end, we can follow the difference of squares formula to complete the factoring process.

$$\underbrace{4a^2 - 4a + 1} - \underbrace{b^2} = (2a-1)^2 - b^2$$
$$= \boldsymbol{(2a-1-b)(2a-1+b)}$$

factoring by substitution

c. To factor $(5r+8)^2 - 6(5r+8) + 9$, it is convenient to substitute a new variable, say $\boldsymbol{a}$, for the expression $5r+8$. Then,

$$(5r+8)^2 - 6(5r+8) + 9 = \boldsymbol{a}^2 - 6\boldsymbol{a} + 9$$

perfect square!

$$= (\boldsymbol{a} - 3)^2$$

$$= (5r + 8 - 3)^2$$

go back to the original variable

$$= (5r+5)^2$$

Remember to represent the new variable by a **different letter** than the original variable!

Notice that $5r+5$ can still be factored by taking the 5 out. So, for a complete factorization, we factor further

$$(5r+5)^2 = \left(5(r+1)\right)^2 = \mathbf{25}(\boldsymbol{r}+\mathbf{1})^2$$

multiple special formulas and simplifying

d. To factor $(p-2q)^3 + (p+2q)^3$, we follow the sum of cubes formula $(a+b)(a^2 - ab + b^2)$ by assuming $a = p - 2q$ and $b = p + 2q$. So, we have

$$(p-2q)^3 + (p+2q)^3$$

$$= (p - \cancel{2q} + p + \cancel{2q})\,[(p-2q)^2 - (p-2q)(p+2q) + (p+2q)^2]$$

$$= 2p\,[p^2 - \cancel{4pq} + 4q^2 - (p^2 - 4q^2) + p^2 + \cancel{4pq} + 4q^2]$$

$$= 2p\,(2p^2 + 8q^2 - p^2 + 4q^2) = \mathbf{2}\boldsymbol{p}\left(\boldsymbol{p}^2 + \mathbf{12}\boldsymbol{q}^2\right)$$

F.3 Exercises

Vocabulary Check *Complete each blank with one of the suggested words, or the most appropriate term or phrase from the given list:* ***difference of cubes, difference of squares, perfect square, sum of cubes, sum of squares.***

1. If a binomial is a ____________________________ its factorization has the form $(a+b)(a-b)$.

2. Trinomials of the form $a^2 \pm 2ab + b^2$ are ________________________ trinomials.

3. The product $(a+b)(a^2 - ab + b^2)$ is the factorization of the ______________________.

4. The product $(a-b)(a^2 + ab + b^2)$ is the factorization of the ______________________.

5. A _____________________ is not factorable.

6. Quadratic trinomials of the form $a^2 \pm ab + b^2$ __________ (*are / are not*) factorable.

Concept Check *Determine whether each polynomial is a perfect square, a difference of squares, a sum or difference of cubes, or neither.*

7. $0.25x^2 - 0.16y^2$

8. $x^2 - 14x + 49$

9. $9x^4 + 4x^2 + 1$

10. $4x^2 - (x + 4)^2$

11. $125x^3 - 64$

12. $y^{12} + 0.008x^3$

13. $-y^4 + 16x^4$

14. $64 + 48x^3 + 9x^6$

15. $25x^6 - 10x^3y^2 + y^4$

16. $-4x^6 - y^6$

17. $-8x^3 + 27y^6$

18. $81x^2 - 16x$

Concept Check

19. The binomial $4x^2 + 64$ is an example of a sum of two squares that can be factored. Under what conditions can the sum of two squares be factored?

20. Insert the correct signs into the blanks.

a. $8 + a^3 = (2 \_\_ a)(4 \_\_ 2a \_\_ a^2)$

b. $b^3 - 1 = (b \_\_ 1)(b^2 \_\_ b \_\_ 1)$

Factor each polynomial completely, if possible.

21. $x^2 - y^2$

22. $x^2 + 2xy + y^2$

23. $x^3 - y^3$

24. $16x^2 - 100$

25. $4z^2 - 4z + 1$

26. $x^3 + 27$

27. $4z^2 + 25$

28. $y^2 + 18y + 81$

29. $125 - y^3$

30. $144x^2 - 64y^2$

31. $n^2 + 20nm + 100m^2$

32. $27a^3b^6 + 1$

33. $9a^4 - 25b^6$

34. $25 - 40x + 16x^2$

35. $p^6 - 64q^3$

36. $16x^2z^2 - 100y^2$

37. $4 + 49p^2 + 28p$

38. $x^{12} + 0.008y^3$

39. $r^4 - 9r^2$

40. $9a^2 - 12ab - 4b^2$

41. $\frac{1}{8} - a^3$

42. $0.04x^2 - 0.09y^2$

43. $x^4 + 8x^2 + 1$

44. $-\frac{1}{27} + t^3$

45. $16x^6 - 121x^2y^4$

46. $9 + 60pq + 100p^2q^2$

47. $-a^3b^3 - 125c^6$

48. $36n^{2t} - 1$

49. $9a^8 - 48a^4b + 64b^2$

50. $9x^3 + 8$

51. $(x + 1)^2 - 49$

52. $\frac{1}{4}u^2 - uv + v^2$

53. $2t^4 - 128t$

54. $81 - (n + 3)^2$

55. $x^{2n} + 6x^n + 9$

56. $8 - (a + 2)^3$

57. $16z^4 - 1$

58. $5c^3 + 20c^2 + 20c$

59. $(x + 5)^3 - x^3$

60. $a^4 - 81b^4$

61. $0.25z^2 - 0.7z + 0.49$

62. $(x - 1)^3 + (x + 1)^3$

63. $(x - 2y)^2 - (x + y)^2$

64. $0.81p^8 + 9p^4 + 25$

65. $(x + 2)^3 - (x - 2)^3$

Factor each polynomial completely.

66. $3y^3 - 12x^2y$

67. $2x^2 + 50a^2 - 20ax$

68. $x^3 - xy^2 + x^2y - y^3$

69. $y^2 - 9a^2 + 12y + 36$

70. $64u^6 - 1$

71. $7m^3 + m^6 - 8$

72. $-7n^2 + 2n^3 + 4n - 14$

73. $a^8 - b^8$

74. $y^9 - y$

75. $(x^2 - 2)^2 - 4(x^2 - 2) - 21$

76. $8(p - 3)^2 - 64(p - 3) + 128$

77. $a^2 - b^2 - 6b - 9$

78. $25(2a - b)^2 - 9$

79. $3x^2y^2z + 25xyz^2 + 28z^3$

80. $x^{8a} - y^2$

81. $x^6 - 2x^5 + x^4 - x^2 + 2x - 1$

82. $4x^2y^4 - 9y^4 - 4x^2z^4 + 9z^4$

83. $c^{2w+1} + 2c^{w+1} + c$