F.4 Solving Polynomial Equations and Applications of Factoring

Many application problems involve solving polynomial equations. In Chapter L, we studied methods for solving linear, or first-degree, equations. Solving higher degree polynomial equations requires other methods, which often involve factoring. In this chapter, we study solving polynomial equations using the zero-product property, graphical connections between roots of an equation and zeros of the corresponding function, and some application problems involving polynomial equations or formulas that can be solved by factoring.

Zero-Product Property

Recall that to solve a linear equation, for example $2x + 1 = 0$, it is enough to isolate the variable on one side of the equation by applying reverse operations. Unfortunately, this method usually does not work when solving higher degree polynomial equations. For example, we would not be able to solve the equation $x^2 - x = 0$ through the reverse operation process, because the variable x appears in different powers.

So … how else can we solve it?

In this particular example, it is possible to guess the solutions. They are $x = 0$ and $x = 1$.

But how can we solve it algebraically?

It turns out that factoring the left-hand side of the equation $x^2 - x = 0$ helps. Indeed, $x(x - 1) = 0$ tells us that the product of x and $x - 1$ is 0. Since the product of two quantities is 0, at least one of them must be 0. So, either $x = 0$ or $x - 1 = 0$, which solves to $x = 1$.

The equation discussed above is an example of a second degree polynomial equation, more commonly known as a quadratic equation.

Definition 4.1 ► A **quadratic equation** is a second degree polynomial equation in one variable that can be written in the form,

$$\boldsymbol{ax^2 + bx + c = 0},$$

where a, b, and c are real numbers and $a \neq 0$. This form is called **standard form**.

One of the methods of solving such equations involves factoring and the zero-product property that is stated below.

Zero-Product Theorem ► For any real numbers $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$\boldsymbol{ab = 0} \text{ if and only if } \boldsymbol{a = 0} \text{ or } \boldsymbol{b = 0}$$

This means that any product containing a factor of 0 is equal to 0, and conversely, if a product is equal to 0, then at least one of its factors is equal to 0.

Proof ► The implication "if $\boldsymbol{a = 0}$ or $\boldsymbol{b = 0}$, then $\boldsymbol{ab = 0}$" is true by the multiplicative property of zero.

To prove the implication "if $\boldsymbol{ab = 0}$, then $\boldsymbol{a = 0}$ or $\boldsymbol{b = 0}$", let us assume first that $a \neq 0$. (As, if $a = 0$, then the implication is already proven.)

Since $a \neq 0$, then $\frac{1}{a}$ exists. Therefore, both sides of $ab = 0$ can be multiplied by $\frac{1}{a}$ and we obtain

$$\frac{1}{a} \cdot ab = \frac{1}{a} \cdot 0$$

$$b = 0,$$

which concludes the proof.

Attention: The zero-product property works only for a product equal to **0**. For example, the fact that $\boldsymbol{ab = 1}$ does not mean that either a or b equals to 1.

Example 1 ► **Using the Zero-Product Property to Solve Polynomial Equations**

Solve each equation.

a. $(x-3)(2x+5) = 0$ **b.** $2x(x-5)^2 = 0$

Solution ► **a.** Since the product of $x-3$ and $2x+5$ is equal to zero, then by the zero-product property at least one of these expressions must equal to zero. So,

$$x - 3 = 0 \quad \text{or} \quad 2x + 5 = 0$$

which results in $x = 3$ or $2x = -5$

$$x = -\frac{5}{2}$$

Thus, $\left\{-\frac{5}{2}, 3\right\}$ is the solution set of the given equation.

b. Since the product $2x(x-5)^2$ is zero, then either $x = 0$ or $x - 5 = 0$, which solves to $x = 5$. Thus, the solution set is equal to $\{\mathbf{0}, \mathbf{5}\}$.

Note 1: The factor of 2 does not produce any solution, as 2 is never equal to 0.

Note 2: The perfect square $(x-5)^2$ equals to 0 if and only if the base $x-5$ equals to 0.

Solving Polynomial Equations by Factoring

To solve polynomial equations of second or higher degree by factoring, we

- **arrange** the polynomial **in decreasing order** of powers **on one side** of the equation,
- keep the **other side** of the equation **equal to 0**,
- **factor** the polynomial **completely**,
- use the zero-product property to **form linear equations for each factor**,
- **solve** the linear equations to find the roots (solutions) to the original equation.

Example 2 ► **Solving Quadratic Equations by Factoring**

Solve each equation by factoring.

a. $x^2 + 9 = 6x$

b. $15x^2 - 12x = 0$

c. $(x + 2)(x - 1) = 4(3 - x) - 8$

d. $(x - 3)^2 = 36x^2$

Solution ►

a. To solve $x^2 + 9 = 6x$ by factoring we need one side of this equation equal to 0. So, first, we move the $6x$ term to the left side of the equation,

$$x^2 + 9 - 6x = 0,$$

and arrange the terms in decreasing order of powers of x,

$$x^2 - 6x + 9 = 0.$$

Then, by observing that the resulting trinomial forms a perfect square of $x - 3$, we factor

$$(x - 3)^2 = 0,$$

which is equivalent to

$$x - 3 = 0,$$

and finally

$$x = 3.$$

So, the solution is $x = \mathbf{3}$**.**

b. After factoring the left side of the equation $15x^2 - 12x = 0$,

$$3x(5x - 4) = 0,$$

we use the zero-product property. Since 3 is never zero, the solutions come from the equations

$$x = \mathbf{0} \quad \text{or} \quad 5x - 4 = 0.$$

Solving the second equation for x, we obtain

$$5x = 4,$$

and finally

$$x = \mathbf{\frac{4}{5}}.$$

So, the solution set consists of **0** and $\mathbf{\frac{4}{5}}$.

c. To solve $(x + 2)(x - 1) = 4(3 - x) - 8$ by factoring, first, we work out the brackets and arrange the polynomial in decreasing order of exponents on the left side of the equation. So, we obtain

$$x^2 + x - 2 = 12 - 4x - 8$$

$$x^2 + 5x - 6 = 0$$

$$(x + 6)(x - 1) = 0$$

Now, we can read the solutions from each bracket, that is, $x = \mathbf{-6}$ and $x = \mathbf{1}$.

Observation: In the process of solving a linear equation of the form $ax + b = 0$, first we subtract b and then we divide by a. So the solution, sometimes referred to as the root, is $x = -\frac{b}{a}$. This allows us to read the solution directly from the equation. For example, the solution to $x - 1 = 0$ is $x = 1$ and the solution to $2x - 1 = 0$ is $x = \frac{1}{2}$.

d. To solve $(x-3)^2 = 36x^2$, we bring all the terms to one side and factor the obtained difference of squares, following the formula $a^2 - b^2 = (a+b)(a-b)$. So, we have

$$(x-3)^2 - 36x^2 = 0$$

$$(x-3+6x)(x-3-6x) = 0$$

$$(7x-3)(-5x-3) = 0$$

Then, by the zero-product property,

$$7x - 3 = 0 \text{ or } -5x - 3,$$

which results in

$$x = \mathbf{\frac{3}{7}} \text{ or } x = -\mathbf{\frac{3}{5}}.$$

Example 3 ► **Solving Polynomial Equations by Factoring**

Solve each equation by factoring.

a. $2x^3 - 2x^2 = 12x$ **b.** $x^4 + 36 = 13x^2$

Solution ► **a.** First, we bring all the terms to one side of the equation and then factor the resulting polynomial.

$$2x^3 - 2x^2 = 12x$$

$$2x^3 - 2x^2 - 12x = 0$$

$$2x(x^2 - x - 6) = 0$$

$$2x(x-3)(x+2) = 0$$

By the zero-product property, the factors $x, (x-3)$ and $(x+2)$, give us the corresponding solutions, $0, 3,$ and -2. So, the solution set of the given equation is $\{\mathbf{0}, \mathbf{3}, \mathbf{-2}\}$.

b. Similarly as in the previous examples, we solve $x^4 + 36 = 13x^2$ by factoring and using the zero-product property. Since

$$x^4 - 13x^2 + 36 = 0$$

$$(x^2-4)(x^2-9)=0$$

$$(x+2)(x-2)(x+3)(x-3)=0,$$

then, the solution set of the original equation is $\{\mathbf{-2}, \mathbf{2}, \mathbf{-3}, \mathbf{3}\}$

Observation: n-th degree polynomial equations may have up to n roots (solutions).

Factoring in Applied Problems

Factoring is a useful strategy when solving applied problems. For example, factoring is often used in **solving formulas** for a variable, in **finding roots** of a polynomial function, and generally, in any problem involving **polynomial equations** that can be solved by factoring.

Example 4 ► **Solving Formulas with the Use of Factoring**

Solve each formula for the specified variable.

a. $A=2\boldsymbol{h}w+2wl+2l\boldsymbol{h}$, for $\boldsymbol{h}$ **b.** $s=\frac{2\boldsymbol{t}+3}{\boldsymbol{t}}$, for $\boldsymbol{t}$

Solution ► **a.** To solve $A=2\boldsymbol{h}w+2wl+2l\boldsymbol{h}$ for $\boldsymbol{h}$, we want to keep both terms containing $\boldsymbol{h}$ on the same side of the equation and bring the remaining terms to the other side. Here is an equivalent equation,

$$A-2wl=2\boldsymbol{h}w+2l\boldsymbol{h},$$

which, for convenience, could be written starting with h-terms:

$$2\boldsymbol{h}w+2l\boldsymbol{h}=A-2wl$$

Now, factoring $\boldsymbol{h}$ out causes that $\boldsymbol{h}$ appears in only one place, which is what we need to isolate it. So,

$$(2w+2l)\boldsymbol{h}=A-2wl \qquad / \div(2w+2l)$$

$$\boldsymbol{h}=\frac{\boldsymbol{A-2wl}}{\boldsymbol{2w+2l}}$$

Notice: In the above formula, there is nothing that can be simplified. Trying to reduce 2 or $2w$ or l would be an error, as there is no essential common factor that can be carried out of the numerator.

b. When solving $s=\frac{2\boldsymbol{t}+3}{\boldsymbol{t}}$ for $\boldsymbol{t}$, our goal is to, firstly, keep the variable $\boldsymbol{t}$ in the numerator and secondly, to keep it in a single place. So, we have

$$s=\frac{2\boldsymbol{t}+3}{\boldsymbol{t}} \qquad / \cdot t$$

$$s\boldsymbol{t}=2\boldsymbol{t}+3 \qquad / -2t$$

factor t

$$st - 2t = 3$$

$$t(s-2) = 3 \qquad / \div (s-2)$$

$$t = \frac{3}{s-2}.$$

Example 5 ► **Finding Roots of a Polynomial Function**

A small rocket is launched from the ground vertically upwards with an initial velocity of 128 feet per second. Its height in feet after t seconds is a function defined by

$$h(t) = -16t^2 + 128t.$$

After how many seconds will the rocket hit the ground?

Solution ► The rocket hits the ground when its height is 0. So, we need to find the time t for which $h(t) = 0$. Therefore, we solve the equation

$$-16t^2 + 128t = 0$$

for t. From the factored form

$$-16t(t-8) = 0$$

we conclude that the rocket is on the ground at times 0 and 8 seconds. So the rocket hits the ground **after 8 seconds** from its launch.

Example 6 ► **Solving an Application Problem with the Use of Factoring**

The height of a triangle is 1 meter less than twice the length of the base. The area is 14 m^2. What are the measures of the base and the height?

Solution ► Let b and h represent the base and the height of the triangle, correspondingly. The first sentence states that h is 1 less than 2 times b. So, we record

$$h = 2b - 1.$$

Using the formula for area of a triangle, $A = \frac{1}{2}bh$, and the fact that $A = 14$, we obtain

$$14 = \frac{1}{2}b(2b-1).$$

Since this is a one-variable quadratic equation, we will attempt to solve it by factoring, after bringing all the terms to one side of the equation. So, we have

to clear the fraction, multiply each term by 2 before working out the bracket

$$0 = \frac{1}{2}b(2b-1) - 14 \qquad / \cdot 2$$

$$0 = b(2b-1) - 28$$

$$0 = 2b^2 - b - 28$$

$$0 = (2b + 7)(b - 4),$$

which by the zero-product property gives us $b = -\frac{7}{2}$ or $b = 4$. Since b represents the length of the base, it must be positive. So, the base is **4** meters long and the height is $h = 2b - 1 = 2 \cdot 4 - 1 = \mathbf{7}$ meters long.

F.4 Exercises

Vocabulary Check *Complete each blank with the most appropriate term from the given list:* ***factored, linear, n, zero, zero-product.***

1. The _________________________ property states that if $ab = 0$ then either $a = 0$ or $b = 0$.

2. When a quadratic equation is solved by factoring, the zero-product property is used to form two _______________ equations.

3. The zero-product property can be applied only when one side of the equation is equal to ___________ and the other side is in a _________________ form.

4. An n-th degree polynomial equation may have up to _____ solutions.

Concept Check *True or false.*

5. If $xy = 0$ then $x = 0$ or $y = 0$.

6. If $ab = 1$ then $a = 1$ or $b = 1$.

7. If $x + y = 0$ then $x = 0$ or $y = 0$.

8. If $a^2 = 0$ then $a = 0$.

9. If $x^2 = 1$ then $x = 1$.

Concept Check

10. Which of the following equations is **not** in proper form for using the zero-product property.

a. $x(x - 1) + 3(x - 1) = 0$

b. $(x + 3)(x - 1) = 0$

c. $x(x - 1) = 3(x - 1)$

d. $(x + 3)(x - 1) = -3$

Solve each equation.

11. $3(x - 1)(x + 4) = 0$

12. $2(x + 5)(x - 7) = 0$

13. $(3x + 1)(5x + 4) = 0$

14. $(2x - 3)(4x - 1) = 0$

15. $x^2 + 9x + 18 = 0$

16. $x^2 - 18x + 80 = 0$

17. $2x^2 = 7 - 5x$

18. $3k^2 = 14k - 8$

19. $x^2 + 6x = 0$

20. $6y^2 - 3y = 0$

21. $(4 - a)^2 = 0$

22. $(2b + 5)^2 = 0$

23. $0 = 4n^2 - 20n + 25$

24. $0 = 16x^2 + 8x + 1$

25. $p^2 - 32 = -4p$

26. $19a + 36 = 6a^2$

27. $x^2 + 3 = 10x - 2x^2$

28. $3x^2 + 9x + 30 = 2x^2 - 2x$

29. $(3x + 4)(3x - 4) = -10x$

30. $(5x + 1)(x + 3) = -2(5x + 1)$

31. $4(y - 3)^2 - 36 = 0$

32. $3(a + 5)^2 - 27 = 0$

33. $(x - 3)(x + 5) = -7$

34. $(x + 8)(x - 2) = -21$

35. $(2x - 1)(x - 3) = x^2 - x - 2$

36. $4x^2 + x - 10 = (x - 2)(x + 1)$

37. $4(2x + 3)^2 - (2x + 3) - 3 = 0$

38. $5(3x - 1)^2 + 3 = -16(3x - 1)$

39. $x^3 + 2x^2 - 15x = 0$

40. $6x^3 - 13x^2 - 5x = 0$

41. $25x^3 = 64x$

42. $9x^3 = 49x$

43. $y^4 - 26y^2 + 25 = 0$

44. $n^4 - 50n^2 + 49 = 0$

45. $x^3 - 6x^2 = -8x$

46. $x^3 - 2x^2 = 3x$

47. $a^3 + a^2 - 9a - 9 = 0$

48. $2x^3 - x^2 - 2x + 1 = 0$

49. $5x^3 + 2x^2 - 20x - 8 = 0$

50. $2x^3 + 3x^2 - 18x - 27 = 0$

Discussion Point

51. A student tried to solve the equation $x^3 = 9x$ by first dividing each side by x, obtaining $x^2 = 9$. She then solved the resulting equation by the zero-product property and obtained the solution set $\{-3,3\}$. Is this a complete solution? Explain your reasoning.

Analytic Skills

52. Given that $f(x) = x^2 + 14x + 50$, find all values of x such that $f(x) = 5$.

53. Given that $g(x) = 2x^2 - 15x$, find all values of x such that $g(x) = -7$.

54. Given that $f(x) = 2x^2 + 3x$ and $g(x) = -6x + 5$, find all values of x such that $f(x) = g(x)$.

55. Given that $g(x) = 2x^2 + 11x - 16$ and $h(x) = 5 + 2x - x^2$, find all values of x such that $g(x) = h(x)$.

Solve each equation for the specified variable.

56. $\boldsymbol{P}rt = A - \boldsymbol{P}$, for $\boldsymbol{P}$

57. $3\boldsymbol{s} + 2p = 5 - r\boldsymbol{s}$, for $\boldsymbol{s}$

58. $5a + b\boldsymbol{r} = \boldsymbol{r} - 2c$, for $\boldsymbol{r}$

59. $E = \frac{R+\boldsymbol{r}}{\boldsymbol{r}}$, for $\boldsymbol{r}$

60. $z = \frac{x+2y}{y}$, for $\boldsymbol{y}$

61. $c = \frac{-2t+4}{t}$, for $\boldsymbol{t}$

Analytic Skills

Use factoring the GCF out to ***solve*** *each formula* ***for the indicated variable****.*

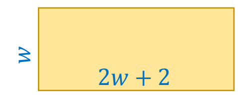
62. An object is thrown downwards, with an initial speed of 16 ft/s, from the top of a building 480 ft high. If the distance travelled by the object, in ft, is given by the function $d(t) = vt + 16t^2$, where v is the initial speed in ft/s, and t is the time in seconds, then how many seconds later will the object hit the ground?

63. A sandbag is dropped from a hot-air balloon 900 ft above the ground. The height, h, of the sandbag above the ground, in feet, after t seconds is given by the function $h(t) = 900 - 16t^2$. When will the sandbag hit the ground?

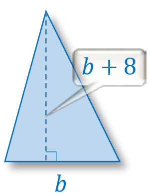
64. The sum of a number and its square is 72. Find the number.

65. The sum of a number and its square is 210. Find the number.

66. The length of a rectangle is 2 meters more than twice the width. The area of the rectangle is 84 m^2. Find the length and width of the rectangle.



67. An envelope is 4 cm longer than it is wide. The area is 96 cm^2. Find its length and width.



68. The height of a triangle is 8 cm more than the length of the base. The area of the triangle is 64 cm^2. Find the base and height of the triangle.

69. A triangular sail is 9 m taller than it is wide. The area is 56 m^2. Find the height and the base of the sail.

70. A gardener decides to build a stone pathway of uniform width around her flower bed. The flower bed measures 10 ft by 12 ft. If she wants the area of the bed and the pathway together to be 224 ft^2, how wide should she make the pathway?

71. Suppose a rectangular flower bed is 3 m longer than it is wide. What are the dimensions of the flower bed if its area is 108 m^2 ?

72. A picture frame measures 12 cm by 20 cm, and 84 cm^2 of picture shows. Find the width of the frame.

73. A picture frame measures 14 cm by 20 cm, and 160 cm^2 of picture shows. Find the width of the frame.

74. If each of the sides of a square is lengthened by 6 cm, the area becomes 144 cm^2. Find the length of a side of the original square.

75. If each of the sides of a square is lengthened by 4 m, the area becomes 49 m^2. Find the length of a side of the original square.