Graphs and Linear Functions

A 2-dimensional graph is a visual representation of a relationship between two variables given by an equation or an inequality. Graphs help us solve algebraic problems by analysing the geometric aspects of a problem. While equations are more suitable for precise calculations, graphs are more suitable for showing patterns and trends in a relationship. To fully utilize what graphs can offer, we must first understand the concepts and skills involved in graphing that are discussed in this chapter.

G.1 System of Coordinates, Graphs of Linear Equations and the Midpoint Formula

In this section, we will review the rectangular coordinate system, graph various linear equations and inequalities, and introduce a formula for finding coordinates of the midpoint of a given segment.

The Cartesian Coordinate System

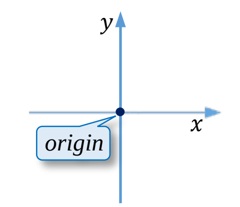


Figure 1a

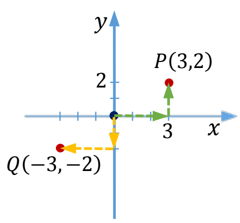


Figure 1b

A rectangular coordinate system, also called a **Cartesian coordinate system** (in honor of French mathematician, ***René Descartes)***, consists of two perpendicular number lines that cross each other at point zero, called the **origin**. Traditionally, one of these number lines, usually called the **x-axis**, is positioned horizontally and directed to the right (see *Figure 1a*). The other number line, usually called **y-axis**, is positioned vertically and directed up. Using this setting, we identify each point P of the plane with an **ordered pair** of numbers (x, y), which indicates the location of this point with respect to the origin. The first number in the ordered pair, the **x-coordinate**, represents the horizontal distance of the point P from the origin. The second number, the **y-coordinate**, represents the vertical distance of the point P from the origin. For example, to locate point $P(3,2)$, we start from the origin, go 3 steps to the right, and then two steps up. To locate point $Q(-3,-2)$, we start from the origin, go 3 steps to the left, and then two steps down (see *Figure 1b*).

Observe that the coordinates of the origin are $(0,0)$. Also, the second coordinate of any point on the x-axis as well as the first coordinate of any point on the y-axis is equal to zero. So, points on the x-axis have the form $(x, 0)$, while points on the y-axis have the form of $(0, y)$.

To **plot** (or **graph**) an ordered pair (x, y) means to place a dot at the location given by the ordered pair.

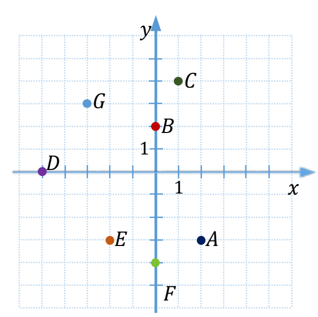
Example 1 ► **Plotting Points in a Cartesian Coordinate System**

Plot the following points:

$A(2,-3)$, $B(0,2)$, $C(1,4)$, $D(-5,0)$,
$E(-2,-3)$, $F(0,-4)$, $G(-3,3)$

Solution ►

Remember! *The order of numbers in an ordered pair is important! The* ***first*** *number represents the* ***horizontal*** *displacement and the* ***second*** *number represents the* ***vertical*** *displacement from the origin.*



Graphs of Linear Equations

A **graph of an equation** in two variables, x and y, is the set of points corresponding to **all ordered pairs** $(\boldsymbol{x}, \boldsymbol{y})$ that **satisfy** the equation (make the equation true). This means that a graph of an equation is the visual representation of the **solution set** of this equation.

To determine if a point (a, b) belongs to the graph of a given equation, we check if the equation is satisfied by $x = a$ and $y = b$.

Example 1 ► **Determining if a Point is a Solution of a Given Equation**

Determine if the points $(5,3)$ and $(-3,-2)$ are solutions of $2x - 3y = 0$.

Solution ► After substituting $x = 5$ and $y = 3$ into the equation $2x - 3y = 0$, we obtain

$$2 \cdot 5 - 3 \cdot 3 = 0$$
$$10 - 9 = 0$$
$$1 = 0,$$

which is not true. Since the coordinates of the point $(5,3)$ do not satisfy the given equation, the point $(\mathbf{5}, \mathbf{3})$ **is not a solution** of this equation.

Note: The fact that the point $(5,3)$ does not satisfy the given equation indicates that it does not belong to the graph of this equation.

However, after substituting $x = -3$ and $y = -2$ into the equation $2x - 3y = 0$, we obtain

$$2 \cdot (-3) - 3 \cdot (-2) = 0$$
$$-6 + 6 = 0$$
$$0 = 0,$$

which is true. Since the coordinates of the point $(-3,-2)$ satisfy the given equation, the point $(-\mathbf{3}, -\mathbf{2})$ **is a solution** of this equation.

Note: The fact that the point $(-3,-2)$ satisfies the given equation indicates that it belongs to the graph of this equation.

To find a solution to a given equation in two variables, we choose a particular value for one of the variables, substitute it into the equation, and then solve the resulting equation for the other variable.

For example, to find a solution to $3x + 2y = 6$, we can choose for example $x = 0$, which leads us to

$$3 \cdot 0 + 2y = 6$$
$$2y = 6$$
$$y = 3.$$

This means that the point $(0, 3)$ satisfies the equation and therefore belongs to the graph of this equation. If we choose a different x-value, for example $x = 1$, the corresponding y-value becomes

$$3 \cdot 1 + 2y = 6$$
$$2y = 3$$
$$y = \frac{3}{2}.$$

So, the point $\left(1, \frac{3}{2}\right)$ also belongs to the graph.

Since any real number could be selected for the x-value, there are infinitely many solutions to this equation. Obviously, we will not be finding all of these infinitely many ordered pairs of numbers in order to graph the solution set to an equation. Rather, based on the location of several solutions that are easy to find, we will look for a pattern and predict the location of the rest of the solutions to complete the graph.

To find more points that belong to the graph of the equation in our example, we might want to solve $3x + 2y = 6$ for y. The equation is equivalent to

$$2y = -3x + 6$$

$$y = -\frac{3}{2}x + 3$$

Observe that if we choose x-values to be multiples of 2, the calculations of y-values will be easier in this case. Here is a table of a few more (x, y) points that belong to the graph:

x	$y = -\frac{3}{2}x + 3$	(x, y)
-2	$-\frac{3}{2}(-2) + 3 = 6$	$(-2, 6)$
2	$-\frac{3}{2}(2) + 3 = 0$	$(2, 0)$
4	$-\frac{3}{2}(4) + 3 = -3$	$(4, -3)$

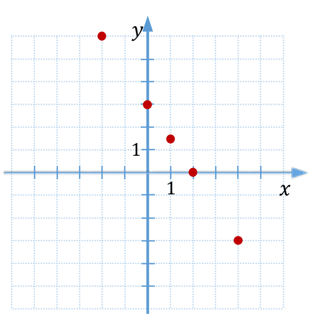


Figure 2a

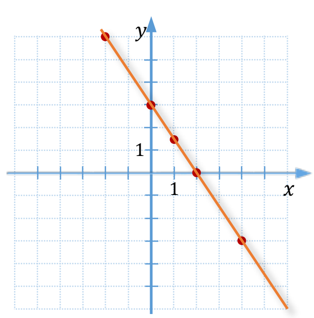


Figure 2b

After plotting the obtained solutions, $(-2, 6)$, $(0, 3)$, $\left(1, \frac{3}{2}\right)$, $(2, 0)$, $(4, -3)$, we observe that the points appear to lie on the same line (see *Figure 2a*). If all the ordered pairs that satisfy the equation $3x + 2y = 6$ were graphed, they would form the line shown in *Figure 2b*. Therefore, if we knew that the graph would turn out to be a line, it would be enough to find just two points (solutions) and draw a line passing through them.

How do we know whether or not the graph of a given equation is a line? It turns out that:

For any equation in two variables, the graph of the equation is a **line** if and only if (iff) the equation is **linear**.

So, the question is how to recognize a linear equation?

Definition 1.1 ► Any equation that can be written in the form

$$\boldsymbol{Ax + By = C}, \text{ where } A, B, C \in \mathbb{R}, \text{ and } A \text{ and } B \text{ are not both } 0,$$

is called a **linear equation** in two variables.

The form $\boldsymbol{Ax + By = C}$ is called **standard form** of a linear equation.

Example 2 ► **Graphing Linear Equations Using a Table of Values**

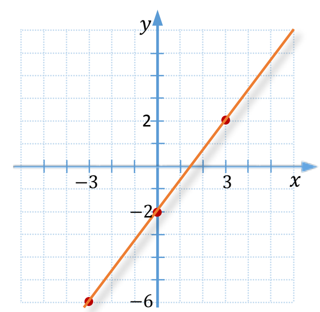
Graph $4x - 3y = 6$ using a table of values.

Solution ► Since this is a linear equation, we expect the graph to be a line. While finding two points satisfying the equation is sufficient to graph a line, it is a good idea to use a third point to guard against errors. To find several solutions, first, let us solve $4x - 3y = 6$ for y:

$$-3y = -4x + 6$$
$$y = \tfrac{4}{3}x - 2$$

We like to choose x-values that will make the calculations of the corresponding y-values relatively easy. For example, if x is a multiple of 3, such as -3, 0 or 3, the denominator of $\frac{4}{3}$ will be reduced. Here is the table of points satisfying the given equation and the graph of the line.

$\boldsymbol{x}$	$\boldsymbol{y = \frac{4}{3}x - 2}$	$(\boldsymbol{x}, \boldsymbol{y})$
$\mathbf{-3}$	$\frac{4}{3}(-3) - 2 = -6$	$(-3, -6)$
$\mathbf{0}$	$\frac{4}{3}(0) - 2 = -2$	$(0, -2)$
$\mathbf{3}$	$\frac{4}{3}(3) - 2 = 2$	$(3, 2)$



To graph a linear equation in standard form, we can develop a table of values as in *Example 2*, or we can use the x- and y-intercepts.

Definition 1.2 ► The $\boldsymbol{x}$**-intercept** is the point (if any) where the line intersects the x-axis. So, the x-intercept has the form $(\boldsymbol{x}, \mathbf{0})$.

The $\boldsymbol{y}$**-intercept** is the point (if any) where the line intersects the y-axis. So, the y-intercept has the form $(\mathbf{0}, \boldsymbol{y})$.

Example 3 ► **Graphing Linear Equations Using $\boldsymbol{x}$- and $\boldsymbol{y}$-intercepts**

Graph $5x - 3y = 15$ by finding and plotting the x- and y-intercepts.

Solution ► To find the x-intercept, we substitute $y = 0$ into $5x - 3y = 15$, and then solve the resulting equation for y. So, we have

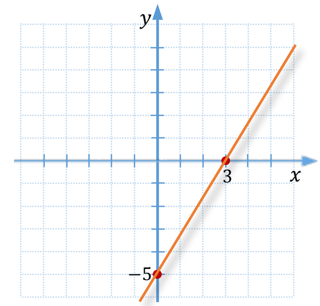
$$5x = 15$$
$$x = 3.$$

To find y-intercept, we substitute $x = 0$ into $5x - 3y = 15$, and then solve the resulting equation for x. So,

$$-3y = 15$$
$$y = -5.$$

Hence, we have

	x	y
x-intercept	**3**	**0**
y-intercept	**0**	**−5**



To find several points that belong to the graph of a linear equation in two variables, it was easier to solve the standard form $Ax + By = C$ for y, as follows

$$By = -Ax + C$$

$$y = -\frac{A}{B}x + \frac{C}{B}.$$

This form of a linear equation is also very friendly for graphing, as the graph can be obtained without any calculations. See *Example 4*.

Any equation $Ax + By = C$, where $B \neq 0$ can be written in the form

$$\boldsymbol{y = mx + b},$$

which is referred to as the **slope-intercept form** of a linear equation.
The value $\boldsymbol{m} = -\frac{\boldsymbol{A}}{\boldsymbol{B}}$ represents the **slope** of the line. Recall that ***slope*** $= \frac{\boldsymbol{rise}}{\boldsymbol{run}}$.
The value $\boldsymbol{b}$ represents the y-intercept, so the point $(\boldsymbol{0}, \boldsymbol{b})$ belongs to the graph of this line.

Example 4 ► **Graphing Linear Equations Using Slope and y-intercept**

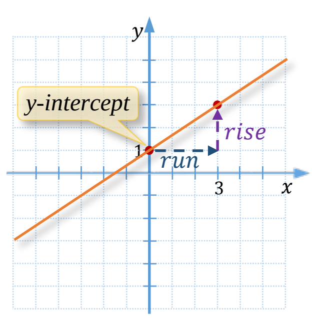
Determine the slope and y-intercept of each line and then graph it.

a. $y = \frac{2}{3}x + 1$ **b.** $5x + 2y = 8$

Solution ► **a.** The slope is the coefficient by x, so it is $\frac{2}{3}$.

The y-intercept equals 1.

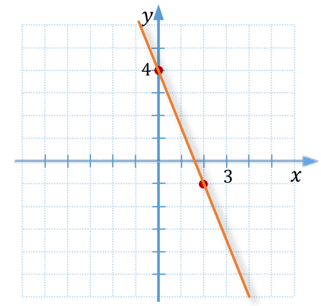
So we plot point $(0,1)$ and then, since $\frac{2}{3} = \frac{rise}{run}$, we rise 2 units and run 3 units to find the next point that belongs to the graph.



b. To see the slope and y-intercept, we solve

$$5x + 2y = 8 \text{ for } y.$$
$$2y = -5x + 8$$
$$y = \frac{-5}{2}x + 4$$

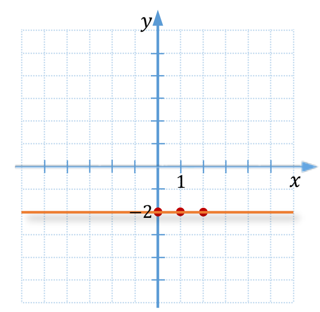
So, the slope is $\frac{-5}{2}$ and the y-intercept is 4. We start from $(0,4)$ and then run 2 units and fall 5 units (because of -5 in the numerator).



Note: Although we can *run* to the right or to the left, depending on the sign in the denominator, we usually **keep the denominator positive and always *run* forward** (to the right). If the slope is negative, we **keep the negative sign in the numerator** and either *rise* or *fall*, depending on this sign. However, when finding additional points of the line, sometimes we can repeat the *run/rise* movement in either way, to the right, or to the left from one of the already known points. For example, in *Example 4a*, we could find the additional point at $(-3,-2)$ by *running* 3 units to the left and 2 units down from $(0,1)$, as the slope $\frac{2}{3}$ can also be seen as $\frac{-2}{-3}$, if needed.

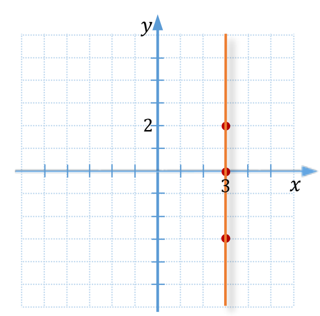
Some linear equations contain just one variable. For example, $x = 3$ or $y = -2$. How would we graph such equations in the xy-plane?

Observe that $\boldsymbol{y = -2}$ can be seen as $y = 0x - 2$, so we can graph it as before, using the **slope** of **zero** and the y-intercept of -2. The graph consists of all points that have y-coordinates equal to -2. Those are the points of type $(x, -2)$, where x is any real number. The graph is a **horizontal line** passing through the point $(0, 2)$.



Note: The horizontal line $\boldsymbol{y = 0}$ is the x-axis.

The equation $\boldsymbol{x = 3}$ doesn't have a slope-intercept representation, but it is satisfied by any point with x-coordinate equal to 3. So, by plotting several points of the type $(3, y)$, where y is any real number, we obtain a **vertical line** passing through the point $(3, 0)$. This particular line doesn't have a y-intercept, and its ***slope*** $= \frac{\textit{rise}}{\textit{run}}$ is considered to be **undefined**. This is because the "*run*" part calculated between any two points on the line is equal to zero and we can't perform division by zero.



Note: The vertical line $\boldsymbol{x = 0}$ is the y-axis.

In general, the graph of any equation of the type

$$\boldsymbol{y} = \boldsymbol{b}, \text{ where } \boldsymbol{b} \in \mathbb{R}$$

is a **horizontal line** with the y-intercept at $\boldsymbol{b}$. The **slope** of such line is **zero**.

The graph of any equation of the type

$$\boldsymbol{x} = \boldsymbol{a}, \text{ where } \boldsymbol{a} \in \mathbb{R}$$

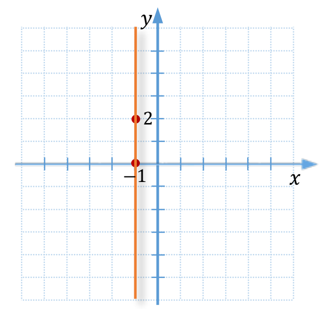
is a **vertical line** with the x-intercept at $\boldsymbol{a}$. The **slope** of such line is **undefined**.

Example 5 ► **Graphing Special Types of Linear Equations**

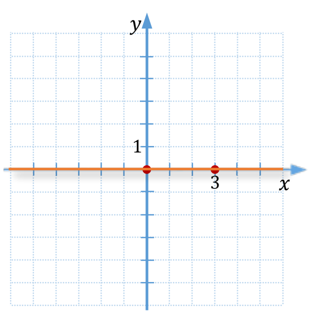
Graph each equation and state its slope.

a. $x = -1$ **b.** $y = 0$ **c.** $y = x$

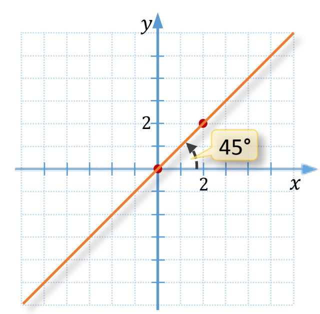
Solution ► **a.** The solutions to the equation $x = -1$ are all pairs of the type $(-1, y)$, so after plotting points like $(-1,0)$, $(-1,2)$, etc., we observe that the graph is a **vertical line** intercepting x-axis at $x = -1$. So the **slope** of this line is **undefined**.



b. The solutions to the equation $y = 0$ are all pairs of the type $(x, 0)$, so after plotting points like $(0,0)$, $(0,3)$, etc., we observe that the graph is a **horizontal line** following the x-axis. The **slope** of this line is **zero**.



c. The solutions to the equation $y = x$ are all pairs of the type (x, x), so after plotting points like $(0,0)$, $(2,2)$, etc., we observe that the graph is a **diagonal line**, passing through the origin and making $45°$ with the x-axis. The **slope** of this line is **1**.

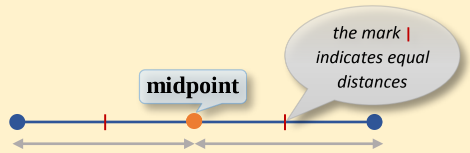


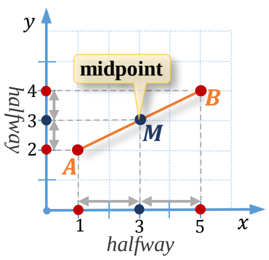
Observation: *A graph of any equation of the type $\boldsymbol{y} = \boldsymbol{mx}$ is a line passing through the origin, as the point* $(0,0)$ *is one of the solutions.*

Midpoint Formula

To find a representative value of a list of numbers, we often calculate the average of these numbers. Particularly, to find an average of, for example, two test scores, 72 and 84, we take half of the sum of these scores. So, the average of 72 and 84 is equal to $\frac{72+84}{2} = \frac{156}{2} = 78$. Observe that 78 lies on a number line exactly halfway between 72 and 84. The idea of taking an average is employed in calculating coordinates of the midpoint of any line segment.

Definition 1.3 ► The **midpoint** of a line segment is the point of the segment that is equidistant from both ends of this segment.





Suppose $A=(x_1,y_1)$, $B=(x_2,y_2)$, and $\boldsymbol{M}$ is the **midpoint** of the line segment $\overline{AB}$. Then the x-coordinate of M lies halfway between the two end x-values, x_1 and x_2, and the y-coordinate of M lies halfway between the two end y-values, y_1 and y_2. So, the coordinates of the midpoint are **averages** of corresponding x-, and y-coordinates:

$$\boldsymbol{M}=\left(\frac{\boldsymbol{x_1+x_2}}{\boldsymbol{2}},\frac{\boldsymbol{y_1+y_2}}{\boldsymbol{2}}\right) \tag{1}$$

Example 6 ► **Finding Coordinates of the Midpoint**

Find the midpoint M of the line segment connecting $P=(-3,7)$ and $Q=(5,-12)$.

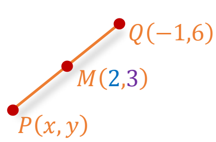
Solution ► The coordinates of the midpoint M are averages of the x- and y-coordinates of the endpoints. So,

$$M=\left(\frac{-3+5}{2},\frac{7+(-12)}{2}\right)=\left(\mathbf{1},-\tfrac{\mathbf{5}}{\mathbf{2}}\right).$$

Example 7 ► **Finding Coordinates of an Endpoint Given the Midpoint and the Other Endpoint**

Suppose segment PQ has its midpoint M at $(2,3)$. Find the coordinates of point P, knowing that $Q=(-1,6)$.

Solution ► Let $P=(x,y)$ and $Q=(-1,6)$. Since $M=(2,3)$ is the midpoint of $\overline{PQ}$, by formula (1), the following equations must hold:



$$\frac{x+(-1)}{2}=2 \quad \text{and} \quad \frac{y+6}{2}=3$$

Multiplying these equations by 2, we obtain

$$x+(-1)=4 \quad \text{and} \quad y+6=6,$$

which results in

$$x=5 \quad \text{and} \quad y=0.$$

Hence, the coordinates of point P are $(\mathbf{5},\mathbf{0})$.

G.1 Exercises

Vocabulary Check *Fill in each blank with the most appropriate term or phrase from the given list:* ***averages, graph, horizontal, linear, line, ordered, origin, root, slope, slope-intercept, solution, undefined, vertical, x, x-axis, x-intercept, y, y-axis, y-intercept, zero.***

1. The point with coordinates $(0,0)$ is called the _____________ of a rectangular coordinate system.

2. Each point P of a plane in a rectangular coordinate system is identified with an ____________ pair of numbers (x,y), where x measures the ______________ displacement of the point P from the origin and y measures the ______________ displacement of the point P from the origin.

3. Any point on the ______________ has the x-coordinate equal to 0.

4. Any point on the ______________ has the y-coordinate equal to 0.

5. A _____________ of an equation consists of all points (x,y) satisfying the equation.

6. To find the x-intercept of a line, we let _____ equal 0 and solve for _____. To find the y-intercept, we let _____ equal 0 and solve for _____.

7. Any equation of the form $Ax + By = C$, where $A, B, C \in \mathbb{R}$, and A and B are not both 0, is called a ___________ equation in two variables. The graph of such equation is a _________ .

8. In the ________________ form of a line, $y = mx + b$, the coefficient m represents the ________ and the free term b represents the _______________ of this line.

9. The slope of a vertical line is _______________ and the slope of a horizontal line is __________ .

10. A point where the graph of an equation crosses the x-axis is called the ______________ of this graph. This point is also refered to as the _________ or _____________ of the equation.

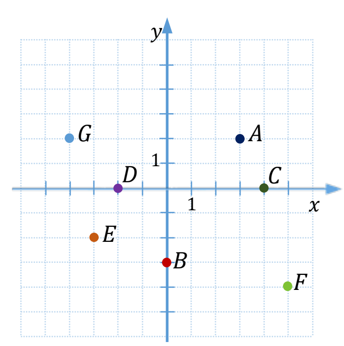
11. The coordinates of the midpoint of a line segment are the ______________ of the x- and y-coordinates of the endpoints of this segment.

Concept Check

12. Plot each point in a rectangular coordinate system.

a. $(1,2)$ **b.** $(-2,0)$ **c.** $(0,-3)$ **d.** $(4,-1)$ **e.** $(-1,-3)$

13. State the coordinates of each plotted point.



Concept Check *Determine if the given ordered pair is a solution of the given equation.*

14. $(-2,2)$; $y = \frac{1}{2}x + 3$ **15.** $(4,-5)$; $3x - 2y = 2$ **16.** $(5,4)$; $4x - 5y = 1$

Concept Check *Graph each equation using the suggested table of values.*

17. $y = 2x - 3$

x	y
0	
1	
2	
3	

18. $y = -\frac{1}{3}x + 2$

x	y
−3	
0	
3	
6	

19. $x + y = 3$

x	y
0	
	0
−1	
	1

20. $4x - 5y = 20$

x	y
0	
	0
2	
	−3

Discussion Point

21. What choices of x-values would be helpful to find points on the graph of $y = \frac{5}{3}x + 4$?

Concept Check *Graph each equation using a table of values.*

22. $y = \frac{1}{3}x$ **23.** $y = \frac{1}{2}x + 2$ **24.** $6x - 3y = -9$ **25.** $6x + 2y = 8$

26. $y = \frac{2}{3}x - 1$ **27.** $y = -\frac{3}{2}x$ **28.** $3x + y = -1$ **29.** $2x = -5y$

30. $-3x = -3$ **31.** $6y - 18 = 0$ **32.** $y = -x$ **33.** $2y - 3x = 12$

Concept Check *Determine the **x-** and **y-intercepts** of each line and then graph it. Find additional points, if needed.*

34. $5x + 2y = 10$ **35.** $x - 3y = 6$ **36.** $8y + 2x = -4$ **37.** $3y - 5x = 15$

38. $y = -\frac{2}{5}x - 2$ **39.** $y = \frac{1}{2}x - \frac{3}{2}$ **40.** $2x - 3y = -9$ **41.** $2x = -y$

Concept Check *Determine the **slope** and **y-intercept** of each line and then graph it.*

42. $y = 2x - 3$ **43.** $y = -3x + 2$ **44.** $y = -\frac{4}{3}x + 1$ **45.** $y = \frac{2}{5}x + 3$

46. $2x + y = 6$ **47.** $3x + 2y = 4$ **48.** $-\frac{2}{3}x - y = 2$ **49.** $2x - 3y = 12$

50. $2x = 3y$ **51.** $y = \frac{3}{2}$ **52.** $y = x$ **53.** $x = 3$

Concept Check *Find the midpoint of each segment with the given endpoints.*

54. $(-8,\ 4)$ and $(-2,-6)$ **55.** $(4,-3)$ and $(-1,3)$ **56.** $(-5,-3)$ and $(7,5)$

57. $(-7,\ 5)$ and $(-2,11)$ **58.** $\left(\frac{1}{2},\ \frac{1}{3}\right)$ and $\left(\frac{3}{2},-\frac{5}{3}\right)$ **59.** $\left(\frac{3}{5},-\frac{1}{3}\right)$ and $\left(\frac{1}{2},-\frac{5}{2}\right)$

Analytic Skills *Segment PQ has the given coordinates for one endpoint P and for its midpoint M. Find the coordinates of the other endpoint Q.*

60. $P(-3,2)$, $M(3,-2)$

61. $P(7,\ 10)$, $M(5,3)$

62. $P(5,-4)$, $M(0,6)$

63. $P(-5,-2)$, $M(-1,4)$

G.2 Slope of a Line and Its Interpretation

Slope (steepness) is a very important concept that appears in many branches of mathematics as well as statistics, physics, business, and other areas. In algebra, slope is used when graphing lines or analysing linear equations or functions. In calculus, the concept of slope is used to describe the behaviour of many functions. In statistics, slope of a regression line explains the general trend in the analysed set of data. In business, slope plays an important role in linear programming. In addition, slope is often used in many practical ways, such as the slope of a road (*grade*), slope of a roof (*pitch*), slope of a ramp, etc.
In this section, we will define, calculate, and provide some interpretations of slope.

Slope

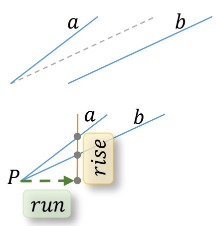


Figure 1a

Given two lines, a and b, how can we tell which one is steeper? One way to compare the steepness of these lines is to move them closer to each other, so that a point of intersection, P, can be seen, as in *Figure 1a*. Then, after running horizontally a few steps from the point P, draw a vertical line to observe how high the two lines have risen. The line that crosses this vertical line at a higher point is steeper. So, for example in *Figure 1a*, line a is steeper than line b. Observe that because we run the same horizontal distance for both lines, we could compare the steepness of the two lines just by looking at the vertical rise. However, since the *run* distance can be chosen arbitrarily, to represent the steepness of any line, we must look at the *rise* (vertical change) in respect to the *run* (horizontal change). This is where the concept of slope as a ratio of *rise* to *run* arises.

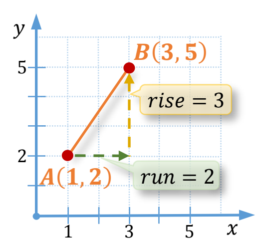


Figure 1b

To measure the slope of a line or a line segment, we choose any two distinct points of such a figure and calculate the ratio of the **vertical change** (***rise***) to the **horizontal change** (***run***) between the two points. For example, the slope between points $A(1,2)$ and $B(3,5)$ equals

$$\frac{rise}{run} = \frac{3}{2},$$

as in *Figure 1a*. If we rewrite this ratio so that the denominator is kept as one,

$$\frac{3}{2} = \frac{1.5}{1} = 1.5,$$

we can think of slope as of the **rate of change in y-values with respect to x-values**. So, a slope of 1.5 tells us that the y-value increases by 1.5 units per every increase of one unit in x-value.

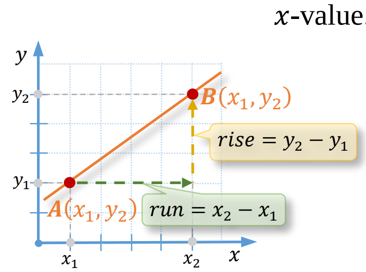


Figure 1c

Generally, the slope of a line passing through two distinct points, $(\boldsymbol{x_1}, \boldsymbol{y_1})$ and $(\boldsymbol{x_2}, \boldsymbol{y_2})$, is the **ratio** of the change in y-values, $y_2 - y_1$, to the change in x-values, $x_2 - x_1$, as presented in *Figure 1c*. Therefore, the formula for calculating slope can be presented as

$$\frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x},$$

where the Greek letter Δ (delta) is used to denote the change in a variable.

Definition 2.1 ► Suppose a line passes through two distinct points $(\boldsymbol{x_1}, \boldsymbol{y_1})$ and $(\boldsymbol{x_2}, \boldsymbol{y_2})$.

If $x_1 \neq x_2$, then the **slope** of this line, often denoted by $\boldsymbol{m}$, is equal to

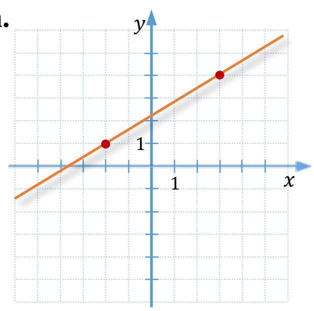
$$\boldsymbol{m = \frac{rise}{run} = \frac{change\ in\ y}{change\ in\ x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}}.$$

If $x_1 = x_2$, then the **slope** of the line is said to be **undefined**.

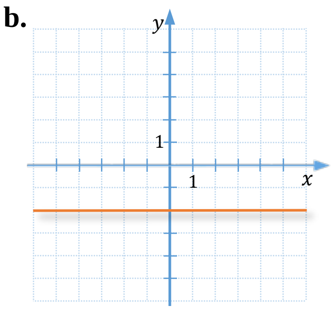
Example 1 ► Determining Slope of a Line, Given Its Graph

Determine the slope of each line.

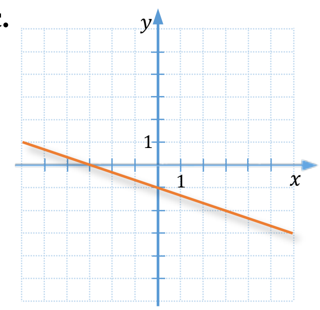
a.



b.

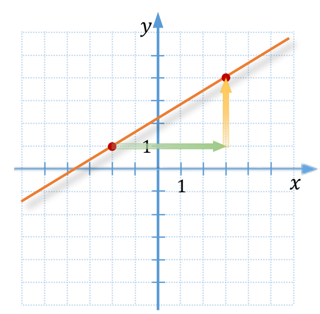


c.



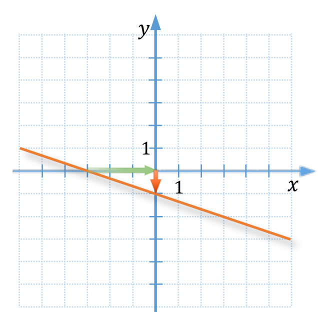
Solution ►

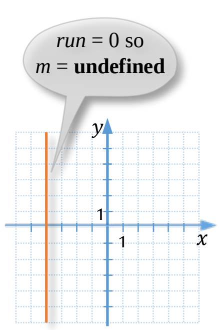
a. To read the slope we choose two distinct points with integral coefficients (often called ***lattice points***), such as the points suggested in the graph. Then, starting from the first point $(-2,1)$ we *run* 5 units and *rise* 3 units to reach the second point $(3,4)$. So, the slope of this line is $m = \frac{\mathbf{5}}{\mathbf{3}}$.



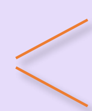
b. This is a horizontal line, so the *rise* between any two points of this line is zero. Therefore the slope of such a line is also **zero**.

c. If we refer to the lattice points $(-3,0)$ and $(0,-1)$, then the *run* is 3 and the *rise* (or rather ***fall***) is -1. Therefore the slope of this line is $m = -\frac{\mathbf{1}}{\mathbf{3}}$.





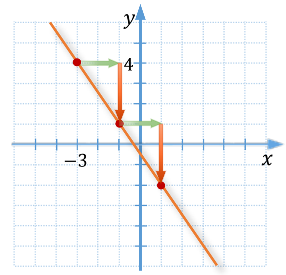
Observation:
A line that **increases** from left to right has a **positive slope**.
A line that **decreases** from left to right has a **negative slope**.
The slope of a **horizontal** line is **zero**.
The slope of a **vertical** line is **undefined**.



Example 2 ► **Graphing Lines, Given Slope and a Point**

Graph the line with slope $-\frac{3}{2}$ that passes through the point $(-3,4)$.

Solution ► irst, plot the point $(-3,4)$. To find another point that belongs to this line, start at the plotted point and run 2 units, then fall 3 units. This leads us to point $(-1,1)$. For better precision, repeat the movement (two across and 3 down) to plot one more point, $(1,-2)$. Finally, draw a line connecting the plotted points.



Example 3 ► **Calculating Slope of a Line, Given Two Points**

Determine the slope of a line passing through the points $(-3,5)$ and $(7,-11)$.

Solution ► The slope of the line passing through $(-3,5)$ and $(7,-11)$ is the quotient

$$\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}=\frac{5-(-11)}{-3-7}=\frac{5+11}{-10}=-\frac{16}{10}=\mathbf{-1.6}$$

Example 4 ► **Determining Slope of a Line, Given Its Equation**

Determine the slope of a line given by the equation $2x-5y=7$.

Solution ► To see the slope of a line in its equation, we change the equation to its slope-intercept form, $y=mx+b$. The slope is the coefficient m. When solving $2x-5y=7$ for y, we obtain

$$-5y=-2x+7$$

$$y=\frac{2}{5}x-\frac{7}{5}.$$

So, the slope of this line is equal to $\frac{2}{5}$.

Example 5 ► **Interpreting Slope as an Average Rate of Change**

On February 11, 2016, the Dow Jones Industrial Average index value was \$15,660.18. On November 11, 2016, this value was \$18,847.66. Using this information, what was the average rate of change in value of the Dow index per month during this period of time?

Solution ► The value of the Dow index has increased by $18{,}847.66 - 15{,}660.18 = 3187.48$ dollars over the 9 months (from February 11 to November 11). So, the slope of the line segment connecting the Dow index values on these two days (as marked on the above chart) equals

$$\frac{3187.48}{9} \cong \mathbf{354.16}\ \$/month$$

This means that the value of the Dow index was increasing on average by 354.16 dollars per month between February 11, 2016 and November 11, 2016.

Observe that the change in value was actually different in each month. Sometimes the change was larger than the calculated slope, but sometimes the change was smaller or even negative. However, the **slope** of the above segment gave us the information about the **average rate of change** in Dow's value during the stated period.

Parallel and Perpendicular Lines

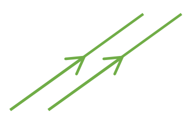


Figure 2

Since slope measures the steepness of lines, and **parallel lines** have the same steepness, then the **slopes** of **parallel lines** are **equal**.

To indicate on a diagram that lines are parallel, we draw on each line arrows pointing in the same direction (see *Figure 2*). To state in mathematical notation that two lines are parallel, we use the $\parallel$ sign.

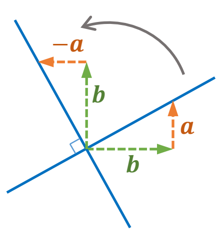


Figure 3

To see how the slopes of perpendicular lines are related, rotate a line with a given slope $\frac{a}{b}$ (where $b \neq 0$) by 90°, as in *Figure 3*. Observe that under this rotation the vertical change $\boldsymbol{a}$ becomes the horizontal change but in opposite direction ($-\boldsymbol{a}$), and the horizontal change $\boldsymbol{b}$ becomes the vertical change. So, the **slope** of the **perpendicular line** is $-\frac{b}{a}$. In other words, **slopes** of **perpendicular lines** are **opposite reciprocals**. Notice that the **product of perpendicular slopes**, $\frac{a}{b} \cdot \left(-\frac{b}{a}\right)$, is equal to $\mathbf{-1}$.

In the case of $b = 0$, the slope is undefined, so the line is vertical. After rotation by 90°, we obtain a horizontal line, with a slope of zero. So a line with a zero slope and a line with an "undefined" slope can also be considered perpendicular.

To indicate on a diagram that two lines are perpendicular, we draw a square at the intersection of the two lines, as in *Figure 3*. To state in mathematical notation that two lines are perpendicular, we use the $\perp$ sign.

In summary, if $\boldsymbol{m_1}$ and $\boldsymbol{m_2}$ are **slopes** of two lines, then the lines are:

- **parallel** iff $\boldsymbol{m_1 = m_2}$, and
- **perpendicular** iff $\boldsymbol{m_1 = -\frac{1}{m_2}}$ (or equivalently, if $\boldsymbol{m_1 \cdot m_2 = -1}$)

  In addition, a **horizontal** line (with a slope of **zero**) is **perpendicular** to a **vertical** line (with **undefined** slope).

Example 6 ► **Determining Whether the Given Lines are Parallel, Perpendicular, or Neither**

For each pair of linear equations, determine whether the lines are parallel, perpendicular, or neither.

a. $3x + 5y = 7$
$5x - 3y = 4$

b. $y = x$
$2x - 2y = 5$

c. $y = 5$
$y = 5x$

Solution ►

a. As seen in *section G1*, the slope of a line given by an equation in standard form, $Ax + By = C$, is equal to $-\frac{A}{B}$. One could confirm this by solving the equation for y and taking the coefficient by x for the slope.
Using this fact, the slope of the line $3x + 5y = 7$ is $-\frac{\mathbf{3}}{\mathbf{5}}$, and the slope of $5x - 3y = 4$ is $\frac{\mathbf{5}}{\mathbf{3}}$. Since these two slopes are opposite reciprocals of each other, the two lines are **perpendicular**.

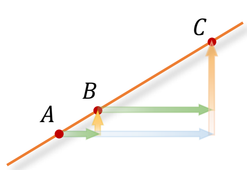
b. The slope of the line $y = x$ is $\mathbf{1}$ and the slope of $2x - 2y = 5$ is also $\frac{2}{2} = \mathbf{1}$. So, the two lines are parallel.

c. The line $y = 5$ can be seen as $y = 0x + 5$, so its slope is $\mathbf{0}$. The slope of the second line, $y = 5x$, is $\mathbf{5}$. So, the two lines are neither parallel nor perpendicular.

Collinear Points

Definition 2.2 ► Points that lie on the same line are called **collinear**.

Two points are always collinear because there is only one line passing through these points. The question is how could we check if a third point is collinear with the given two points? If we have an equation of the line passing through the first two points, we could plug in the coordinates of the third point and see if the equation is satisfied. If it is, the third point is collinear with the other two. But, can we check if points are collinear without referring to an equation of a line?



Notice that if several points lie on the same line, the slope between any pair of these points will be equal to the slope of this line. So, these slopes will be the same. One can also show that if the slopes between any two points in the group are the same, then such points lie on the same line. So, they are collinear.

Points are **collinear** iff the **slope** between each pair of points **is the same**.

Example 7 ► **Determine Whether the Given Points are Collinear**

Determine whether the points $A(-3,7)$, $B(-1,2)$, and $C = (3,-8)$ are collinear.

Solution ► Let m_{AB} represent the slope of $\overline{AB}$ and m_{BC} represent the slope of $\overline{BC}$. Since

$$m_{AB} = \frac{2-7}{-1-(-3)} = -\frac{5}{2} \text{ and } m_{BC} = \frac{-8-2}{3-(-1)} = -\frac{10}{4} = -\frac{5}{2},$$

Then all points A, B, and C lie on the same line. Thus, they are collinear.

Example 8 ► **Finding the Missing Coordinate of a Collinear Point**

For what value of y are the points $P(2,2)$, $Q(-1,y)$, and $R(1,6)$ collinear?

Solution ► For the points P, Q, and R to be collinear, we need the slopes between any two pairs of these points to be equal. For example, the slope m_{PQ} should be equal to the slope m_{PR}. So, we solve the equation

$$m_{PQ} = m_{PR}$$

for y:

$$\frac{y-2}{-1-2} = \frac{6-2}{1-2}$$

$$\frac{y-2}{-3} = -4 \qquad /\cdot(-3)$$

$$y - 2 = 12 \qquad /+2$$

$$y = 14$$

Thus, point Q is collinear with points P and R, if $y = \mathbf{14}$.

G.2 Exercises

Vocabulary Check *Fill in each blank with the most appropriate term or phrase from the given list:* ***slope, undefined, increases, negative, collinear, opposite reciprocals, parallel, zero.***

1. The average rate of change between two points on a graph is measured by the _____________ of the line segment connecting the two points.

2. A vertical line has ________________ slope. The slope of a horizontal line is _________ .

3. A line with a positive slope ________________ from left to right.

4. A decreasing line has a _______________ slope.

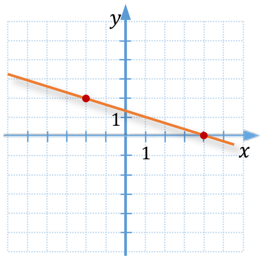
5. If the slope between each pair of points is constantly the same, then the points are __________________.

6. ______________ lines have the same slopes.

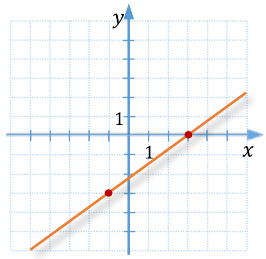
7. The slopes of perpendicular lines are _____________ _______________ .

Concept Check *Given the graph, find the slope of each line.*

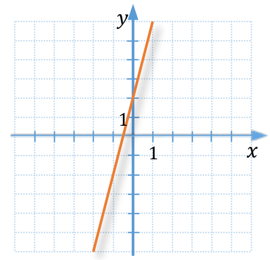
8.



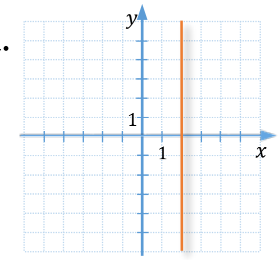
9.



10.



11.



Concept Check *Given the equation, find the slope of each line.*

12. $y = \frac{1}{2}x - 7$

13. $y = -\frac{1}{3}x + 5$

14. $4x - 5y = 2$

15. $3x + 4y = 2$

16. $x = 7$

17. $y = -\frac{3}{4}$

18. $y + x = 1$

19. $-8x - 7y = 24$

20. $-9y - 36 + 4x = 0$

Concept Check *Graph each line satisfying the given information.*

21. passing through $(-2, -4)$ with slope $m = 4$

22. passing through $(-1, -2)$ with slope $m = -3$

23. passing through $(-3, 2)$ with slope $m = \frac{1}{2}$

24. passing through $(-3,4)$ with slope $m = -\frac{2}{5}$

25. passing through $(2, -1)$ with undefined slope

26. passing through $(2, -1)$ with slope $m = 0$

Concept Check

27. Which of the following forms of the slope formula are correct?

a. $m = \frac{y_1 - y_2}{x_2 - x_1}$

b. $m = \frac{y_1 - y_2}{x_1 - x_2}$

c. $m = \frac{x_2 - x_1}{y_2 - y_1}$

d. $m = \frac{y_2 - y_1}{x_2 - x_1}$

Concept Check *Find the slope of the line through each pair of points.*

28. $(-2,2), (4,5)$

29. $(8,7), (2,-1)$

30. $(9,-4), (3,-8)$

31. $(-5,2), (-9,5)$

32. $(-2,3), (7,-12)$

33. $(3,-1), \left(-\frac{1}{2}, \frac{1}{5}\right)$

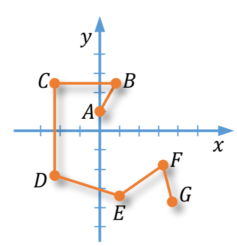
34. $(-5,2), (8,2)$

35. $(-3,4), (-3,10)$

36. $\left(\frac{1}{2}, 6\right), \left(-\frac{2}{3}, \frac{5}{2}\right)$

Concept Check

37. List the line segments in the accompanying figure with respect to their slopes, from the smallest to the largest slope. List the segment with an undefined slope as last.



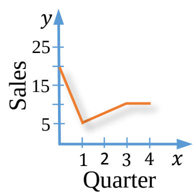
38. **Concept Check** *Match each situation in a–d with the most appropriate graph in A–D.*

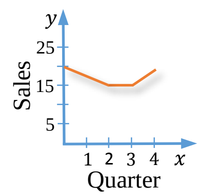
a. Sales rose sharply during the first quarter, leveled off during the second quarter, and then rose slowly for the rest of the year.

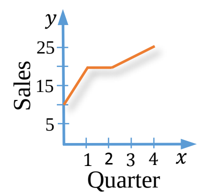
b. Sales fell sharply during the first quarter and then rose slowly during the second and third quarters before leveling off for the rest of the year.

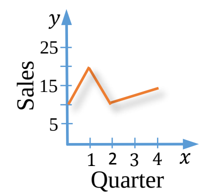
c. Sales rose sharply during the first quarter and then fell to the original level during the second quarter before rising steadily for the rest of the year.

d. Sales fell during the first two quarters of the year, leveled off during the third quarter, and rose during the fourth quarter.

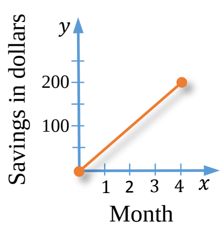
A.


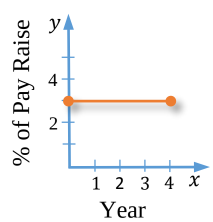
B.


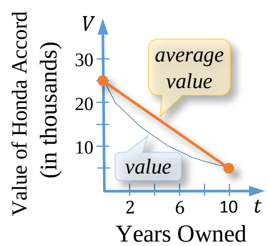
C.


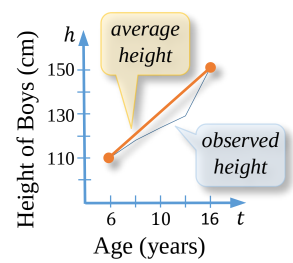
D.


Find and interpret the average rate of change illustrated in each graph.

39.


40.


41.


42.


Analytic Skills *Sketch a graph that models the given situation.*

43. The distance that a cyclist is from home if he is initially 20 miles away from home and arrives home after riding at a constant speed for 2 hours.

44. The distance that an athlete is from home if the athlete runs away from home at 8 miles per hour for 30 minutes and then walks back home at 4 miles per hour.

45. The distance that a person is from home if this individual drives (at a constant speed) to a mall, stays 2 hours, and then drives home, assuming that the distance to the mall is 20 miles and that the trip takes 30 minutes.

46. The amount of water in a 10,000-gallon swimming pool that is filled at the rate of 1000 gallons per hour, left full for 10 hours, and then drained at the rate of 2000 gallons per hour.

Analytic Skills *Solve each problem.*

47. A 80,000-liters swimming pool is being filled at a constant rate. Over a 5-hour period, the water in the pool increases from $\frac{1}{4}$ full to $\frac{5}{8}$ full. At what rate is water entering the pool?

48. An airplane on a 1,800-kilometer trip is flying at a constant rate. Over a 2-hour period, the location of the plane changes from covering $\frac{1}{3}$ of the distance to covering $\frac{3}{4}$ of the distance. What is the speed of the airplane?

Discussion Point

49. Suppose we see a road sign informing that a road grade is 7% for the next 1.5 miles. In meters, what would be the expected change in elevation 1.5 miles down the road? (*Recall:* 1 mile ≈ 1.61 kilometers)

Concept Check *Decide whether each pair of lines is parallel, perpendicular, or neither.*

50. $y = x$
$y = -x$

51. $y = 3x - 6$
$y = -\frac{1}{3}x + 5$

52. $2x + y = 7$
$-6x - 3y = 1$

53. $x = 3$
$x = -2$

54. $3x + 4y = 3$
$3x - 4y = 5$

55. $5x - 2y = 3$
$2x - 5y = 1$

56. $y - 4x = 1$
$x + 4y = 3$

57. $y = \frac{2}{3}x - 2$
$-2x + 3y = 6$

Concept Check *Solve each problem.*

58. Check whether or not the points $(-2, 7)$, $(1, 5)$, and $(3, 4)$ are collinear.

59. The following points, $(2, 2)$, $(-1, k)$, and $(1, 6)$ are collinear. Find the value of k.

G.3 Forms of Linear Equations in Two Variables

Linear equations in two variables can take different forms. Some forms are easier to use for graphing, while others are more suitable for finding an equation of a line given two pieces of information. In this section, we will take a closer look at various forms of linear equations and their utilities.

Forms of Linear Equations

The form of a linear equation that is most useful for graphing lines is the slope-intercept form, as introduced in *section G1*.

Definition 3.1 ► The **slope-intercept form** of the equation of a line with **slope m** and **y-intercept** $(0, b)$ is

$$y = mx + b.$$

Example 1 ► **Writing and Graphing Equation of a Line in Slope-Intercept Form**

Write the equation in slope-intercept form of the line satisfying the given conditions, and then graph this line.

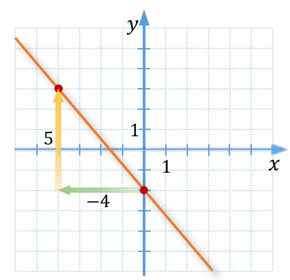
a. slope $-\frac{4}{5}$ and y-intercept $(0, -2)$

b. slope $\frac{1}{2}$ and passing through $(2, -5)$

Solution ►

a. To write this equation, we substitute $m = -\frac{4}{5}$ and $b = -2$ into the slope-intercept form. So, we obtain

$$y = -\frac{4}{5}x - 2.$$



To graph this line, we start with plotting the y-intercept $(0, -2)$. To find the second point, we follow the slope, as in *Example 2, section G2*. According to the slope $-\frac{4}{5} = \frac{-4}{5}$, starting from $(0, -2)$, we could run 5 units to the right and 4 units down, but then we would go out of the grid. So, this time, let the negative sign in the slope be kept in the denominator, $\frac{4}{-5}$. Thus, we run 5 units to the left and 4 units up to reach the point $(0, -2)$. Then we draw the line by connecting the two points.

b. Since $m = \frac{1}{2}$, our equation has a form $y = \frac{1}{2}x + b$. To find b, we substitute point $(2, -5)$ into this equation and solve for b. So

$$-5 = \frac{1}{2}(2) + b$$

gives us

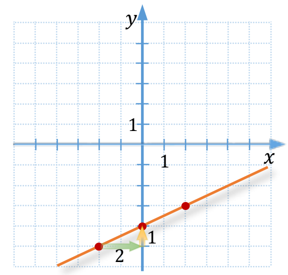
$$-5 = 1 + b$$

and finally

$$b = -6.$$

Therefore, our equation of the line is $\boldsymbol{y = \frac{1}{2}x - 6}$.

We graph it, starting by plotting the given point $(2, -5)$ and finding the second point by following the slope of $\frac{1}{2}$, as described in *Example 2, section G2*.



The form of a linear equation that is most useful when writing equations of lines with unknown y-intercept is the slope-point form.

Definition 3.2 ► The **slope-point form** of the equation of a line with slope $\boldsymbol{m}$ and passing through the point $(\boldsymbol{x_1}, \boldsymbol{y_1})$ is

$$\boldsymbol{y - y_1 = m(x - x_1)}.$$

This form is based on the defining property of a line. A line can be defined as a set of points with a constant slope m between any two of these points. So, if (x_1, y_1) is a given (fixed) point of the line and (x, y) is any (variable) point of the line, then, since the slope is equal to m for all such points, we can write the equation

$$m = \frac{y - y_1}{x - x_1}.$$

After multiplying by the denominator, we obtain the slope-point formula, as in *Definition 3.2*.

Example 2 ► **Writing Equation of a Line Using Slope-Point Form**

Use the slope-point form to write an equation of the line satisfying the given conditions. Leave the answer in the slope-intercept form and then graph the line.

a. slope $-\frac{2}{3}$ and passing through $(1, -3)$

b. passing through points $(2, 5)$ and $(-1, -2)$

Solution ► **a.** To write this equation, we plug the slope $m = -\frac{2}{3}$ and the coordinates of the point $(1, -3)$ into the slope-point form of a line. So, we obtain

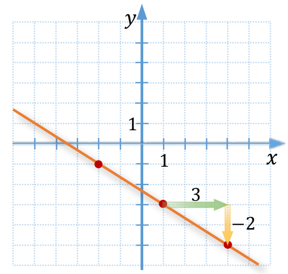
$$y - (-3) = -\frac{2}{3}(x - 1)$$

$$y + 3 = -\frac{2}{3}x + \frac{2}{3} \qquad /-3$$

$$y = -\frac{2}{3}x + \frac{2}{3} - \frac{9}{3}$$

$$\boldsymbol{y = -\frac{2}{3}x - \frac{7}{3}}$$

To graph this line, we start with plotting the point $(1, -3)$ and then apply the slope of $-\frac{2}{3}$ to find additional points that belong to the line.



b. This time the slope is not given, so we will calculate it using the given points, $(2, 5)$ and $(-1, -2)$. Thus,

$$m = \frac{\Delta y}{\Delta x} = \frac{-2 - 5}{-1 - 2} = \frac{-7}{-3} = \frac{7}{3}$$

Then, using the calculated slope and one of the given points, for example $(2, 5)$, we write the slope-point equation of the line

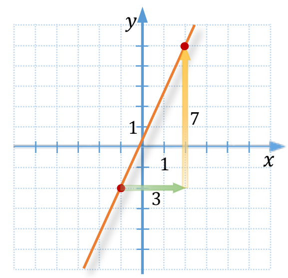
$$y - 5 = \frac{7}{3}(x - 2)$$

and solve it for y:

$$y - 5 = \frac{7}{3}x - \frac{14}{3} \qquad /-5$$

$$y = \frac{7}{3}x - \frac{14}{3} + \frac{15}{3}$$

$$\boldsymbol{y = \frac{7}{3}x + \frac{1}{3}}$$



To graph this line, it is enough to connect the two given points.

One of the most popular forms of a linear equation is the standard form. This form is helpful when graphing lines based on x- and y-intercepts, as illustrated in *Example 3, section G1*.

Definition 3.3 ► The **standard form** of a linear equation is

$$\boldsymbol{Ax + By = C},$$

Where $A, B, C \in \mathbb{R}$, A and B are not both 0, and $A \geq 0$.

When writing linear equations in standard form, the expectation is to use a **nonnegative coefficient** $\boldsymbol{A}$ and **clear any fractions**, if possible. For example, to write $-x + \frac{1}{2}y = 3$ in standard form, we multiply the equation by (-2), to obtain $2x - y = -6$. In addition, we prefer to write equations in simplest form, where the greatest common factor of A, B, and C is 1. For example, we prefer to write $2x - y = -6$ rather than any multiple of this equation, such as $4x - 2y = -12$, or $6x - 3y = -18$.

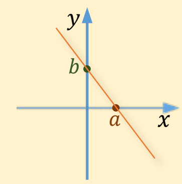
Observe that if $B \neq 0$ then the **slope** of the line given by the equation $\boldsymbol{Ax + By = C}$ is $-\frac{A}{B}$.
This is because after solving this equation for y, we obtain $\boldsymbol{y = -\frac{A}{B}x + \frac{C}{B}}$.
If $B = 0$, then the slope is **undefined**, as we are unable to divide by zero.

The form of a linear equation that is most useful when writing equations of lines based on their x- and y-intercepts is the intercept form.

Definition 3.4 ► The **intercept form** of a linear equation is

$$\boldsymbol{\frac{x}{a} + \frac{y}{b} = 1},$$

where $\boldsymbol{a}$ is the **x-intercept** and $\boldsymbol{b}$ is the **y-intercept** of the line.



We should be able to convert a linear equation from one form to another.

Example 3 ► **Converting a Linear Equation to a Different Form**

a. Write the equation $3x + 7y = 2$ in slope-intercept form.

b. Write the equation $y = \frac{3}{5}x + \frac{7}{2}$ in standard form.

c. Write the equation $\frac{x}{4} - \frac{y}{3} = 1$ in standard form.

Solution ► **a.** To write the equation $3x + 7y = 2$ in slope-intercept form, we solve it for y.

$$3x + 7y = 2 \qquad /-3x$$

$$7y = -3x + 2 \qquad /\div 7$$

$$\boldsymbol{y = -\frac{3}{7}x + \frac{2}{7}}$$

b. To write the equation $y = \frac{3}{5}x + \frac{7}{2}$ in standard form, we bring the x-term to the left side of the equation and multiply the equation by the LCD, with the appropriate sign.

$$y = \frac{3}{5}x + \frac{7}{2} \qquad /-\frac{3}{5}x$$

$$-\frac{3}{5}x + y = \frac{7}{2} \qquad /\cdot(-10)$$

$$\boldsymbol{6x - 10y = -35}$$

c. To write the equation $\frac{x}{4} - \frac{y}{3} = 1$ in standard form, we multiply it by the LCD, with the appropriate sign.

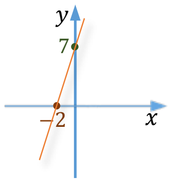
$$\frac{x}{4} - \frac{y}{3} = 1 \qquad /\cdot 12$$

$$\boldsymbol{3x - 4y = 12}$$

Example 4 ► **Writing Equation of a Line Using Intercept Form**

Write an equation of the line passing through points $(0, -2)$ and $(7, 0)$. Leave the answer in standard form.

Solution ► Since point $(0,-2)$ is the y-intercept and point $(7,0)$ is the x-intercept of our line, to write the equation of the line we can use the intercept form with $a=-2$ and $b=7$. So, we have

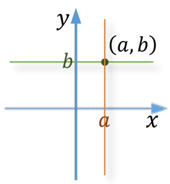


$$\frac{x}{-2}+\frac{y}{7}=1.$$

To change this equation to standard form, we multiply it by the LCD $=-14$. Thus,

$$\mathbf{7x-2y=-14}.$$

Equations representing horizontal or vertical lines are special cases of linear equations in standard form, and as such, they deserve special consideration.



The **horizontal line** passing through the point (a,b) has equation $\boldsymbol{y=b}$,
while the **vertical line** passing through the same point has equation $\boldsymbol{x=a}$.

The equation of a **horizontal line**, $\boldsymbol{y=b}$, can be shown in standard form as $0x+y=b$. Observe, that the slope of such a line is $-\frac{0}{1}=0$.

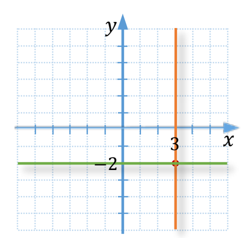
The equation of a **vertical line**, $\boldsymbol{x=a}$, can be shown in standard form as $x+0y=a$. Observe, that the slope of such a line is $-\frac{1}{0}=$ undefined.

Example 5 ► **Writing Equations of Horizontal and Vertical Lines**

Find equations of the vertical and horizontal lines that pass through the point $(3,-2)$. Then, graph these two lines.

Solution ► Since x-coordinates of all points of the vertical line, including $(3,-2)$, are the same, then these x-coordinates must be equal to 3. So, the equation of the vertical line is $x=3$.

Since y-coordinates of all points of a horizontal line, including $(3,-2)$, are the same, then these y-coordinates must be equal to -2. So, the equation of the horizontal line is $y=-2$.



Here is a summary of the various forms of linear equations.

Forms of Linear Equations

Equation	Description	When to Use
$\boldsymbol{y=mx+b}$	**Slope-Intercept Form** slope is $\boldsymbol{m}$ y-intercept is $(0,\boldsymbol{b})$	This form is ideal for graphing by using the y-intercept and the slope.
$\boldsymbol{y-y_1=m(x-x_1)}$	**Slope-Point Form** slope is $\boldsymbol{m}$ the line passes through $(\boldsymbol{x_1},\boldsymbol{y_1})$	This form is ideal for finding the equation of a line if the slope and a point on the line, or two points on the line, are known.

$\boldsymbol{Ax + By = C}$	**Standard Form** slope is $-\frac{A}{B}$, if $B \neq 0$ x-intercept is $\left(\frac{C}{A}, 0\right)$, if $A \neq 0$. y-intercept is $\left(0, \frac{C}{B}\right)$, if $B \neq 0$.	This form is useful for graphing, as the x- and y-intercepts, as well as the slope, can be easily found by dividing appropriate coefficients.
$\boldsymbol{\frac{x}{a} + \frac{y}{b} = 1}$	**Intercept Form** slope is $-\frac{b}{a}$ x-intercept is $(\boldsymbol{a}, 0)$ y-intercept is $(0, \boldsymbol{b})$	This form is ideal for graphing, using the x- and y-intercepts.
$\boldsymbol{y = b}$	**Horizontal Line** slope is 0 y-intercept is $(0, \boldsymbol{b})$	This form is used to write equations of, for example, horizontal asymptotes.
$\boldsymbol{x = a}$	**Vertical Line** slope is undefined x-intercept is $(\boldsymbol{a}, 0)$	This form is used to write equations of, for example, vertical asymptotes.

Note: Except for the equations for a horizontal or vertical line, all of the above forms of linear equations can be converted into each other via algebraic transformations.

Writing Equations of Parallel and Perpendicular Lines

Recall that the slopes of parallel lines are the same, and slopes of perpendicular lines are opposite reciprocals. See *section G2*.

Example 6 ► **Writing Equations of Parallel Lines Passing Through a Given Point**

Find the slope-intercept form of a line parallel to $y = -2x + 5$ that passes through the point $(-4,5)$. Then, graph both lines on the same grid.

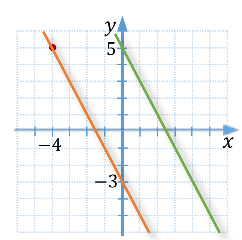
Solution ► Since the line is parallel to $y = -2x + 5$, its slope is -2. So, we plug the slope of -2 and the coordinates of the point $(-4,5)$ into the slope-point form of a linear equation.

$$y - 5 = -2(x + 4)$$

This can be simplified to the slope-intercept form, as follows:

$$y - 5 = -2x - 8$$

$$\boldsymbol{y = -2x - 3}$$



As shown in the accompanying graph, the line $y = -2x - 3$ (in orange) is parallel to the line $y = -2x + 5$ (in green) and passes through the given point $(-4,5)$.

Example 7 ► **Writing Equations of Perpendicular Lines Passing Through a Given Point**

Find the slope-intercept form of a line perpendicular to $2x - 3y = 6$ that passes through the point $(1,4)$. Then, graph both lines on the same grid.

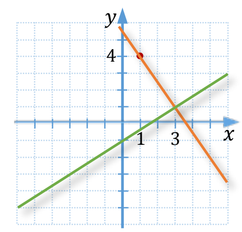
Solution ► The slope of the given line, $2x - 3y = 3$, is $\frac{2}{3}$. To find the slope of a perpendicular line, we take the opposite reciprocal of $\frac{2}{3}$, which is $-\frac{3}{2}$. Since we already know the slope and the point, we can plug these pieces of information into the slope-point formula. So, we have

$$y - 4 = -\frac{3}{2}(x - 1)$$

$$y - 4 = -\frac{3}{2}x + \frac{3}{2} \qquad /+4$$

$$y = -\frac{3}{2}x + \frac{3}{2} + \frac{8}{2}$$

$$\boldsymbol{y = -\frac{3}{2}x + \frac{11}{2}}$$



As shown in the accompanying graph, the line $2x - 3y = 6$ (in orange) is indeed perpendicular to the line $y = -\frac{3}{2}x + \frac{11}{2}$ (in green) and passes through the given point $(1,4)$.

Linear Equations in Applied Problems

Linear equations can be used to model a variety of applications in sciences, business, and other areas. Here are some examples.

Example 8 ► **Given the Rate of Change and the Initial Value, Determine the Linear Model Relating the Variables**

A young couple buys furniture for \$2000, agreeing to pay \$200 down and \$100 at the end of each month until the entire debt is paid off.

a. Write an equation to express the amount paid off, P, in terms of the number of monthly payments, m.

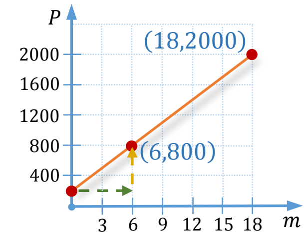
b. Graph the equation found in part **a.**

c. Use the graph to estimate how long it will take to pay off the debt.

Solution ► **a.** Since each month the couple pays \$100, after m months, the amount paid off by the monthly installments is $100m$. If we add the initial payment of \$200, the equation representing the amount paid off can be written as

$$P = 100m + 200$$

b. To graph this equation, we use the slope-intercept method. Starting with the P-intercept of 200, we run 1 and rise 100, repeating this process as many times as needed to hit a lattice point on the chosen scale. So, as shown in the accompanying graph, the line passes through points $(6, 800)$ and $(18, 2000)$.



c. As shown in the graph, \$2000 will be paid off in 18 months.

Example 9 ► **Finding a Linear Equation that Fits the Data Given by Two Ordered Pairs**

Gabriel Daniel Fahrenheit invented the mercury thermometer in 1717. The thermometer shows that water freezes at 32°F and boils at 212°F. In 1742, Anders Celsius invented the Celsius temperature scale. On this scale, water freezes at 0°C and boils at 100°C. Determine a linear equation that can be used to predict the Celsius temperature, C, when the Fahrenheit temperature, F, is known.

Solution ►

To predict the Celsius temperature, C, knowing the Fahrenheit temperature, F, we treat the variable C as dependent on the variable F. So, we consider C as the second coordinate when setting up the ordered pairs, (F, C), of given data. The corresponding freezing temperatures give us the pair $(32,0)$ and the boiling temperatures give us the pair $(212,100)$. To find the equation of a line passing through these two points, first, we calculate the slope, and then, we use the slope-point formula. So, the slope is

$$m = \frac{100 - 0}{212 - 32} = \frac{100}{180} = \frac{\mathbf{5}}{\mathbf{9}},$$

and using the point $(32,0)$, the equation of the line is

$$\boldsymbol{C = \frac{5}{9}(F - 32)}$$

Example 10 ► **Determining if the Given Set of Data Follows a Linear Pattern**

Determine whether the data given in each table follow a linear pattern. If they do, find the slope-intercept form of an equation of the line passing through all the given points.

a.

x	1	3	5	7	9
y	12	16	20	24	28

b.

x	10	20	30	40	50
y	15	21	26	30	35

Solution ►

a. The set of points follows a linear pattern if the slopes between consecutive pairs of these points are the same. These slopes are the ratios of increments in y-values to increments in x-values. Notice that the increases between successive x-values of the given points are constantly equal to 2. So, to check if the points follow a linear pattern, it is enough to check if the increases between successive y-values are also constant. Observe that the numbers in the list 12, 16, 20, 24, 28 steadily increase by 4. Thus, the given set of data follow a linear pattern.

To find an equation of the line passing through these points, we use the slope, which is $\frac{4}{2} = 2$, and one of the given points, for example $(1,12)$. By plugging these pieces of information into the slope-point formula, we obtain

$$y - 12 = 2(x - 1),$$

which after simplifying becomes

$$y - 12 = 2x - 2 \qquad /+12$$

$$\mathbf{y = 2x + 10}$$

b. Observe that the increments between consecutive x-values of the given points are constantly equal to 10, while the increments between consecutive y-values in the list 15, 21, 26, 30, 35 are 6, 5, 4, 5. So, they are not constant. Therefore, the given set of data does not follow a linear pattern.

Example 11 ► **Finding a Linear Model Relating the Number of Items Bought at a Fixed Amount**

A manager for a country market buys apples at $0.25 each and pears at $0.50 each. Write a linear equation in standard form relating the number of apples, a, and pears, p, she can buy for $80. Then,

a. graph the equation and

b. using the graph, find at least 3 points (a, p) satisfying the equation, and interpret their meanings in the context of the problem.

Solution ► It costs $0.25a$ dollars to buy a apples. Similarly, it costs $0.50p$ dollars to buy p pears. Since the total charge is $80, we have

$$0.25a + 0.50p = 80$$

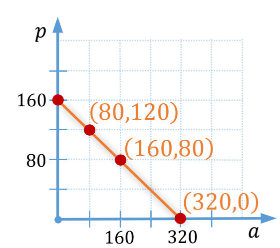
We could convert the coefficients into integers by multiplying the equation by a hundred. So, we obtain

$$25a + 50p = 8000,$$

which, after dividing by 25, turns into

$$\mathbf{a + 2p = 320}.$$

a. To graph this equation, we will represent the number of apples, a, on the horizontal axis and the number of pears, p, on the vertical axis, respecting the alphabetical order of labelling the axes. Using the intercept method, we connect points $(320,0)$ and $(0,160)$.



b. Aside of the intercepts, $(320,0)$ and $(0,160)$, the graph shows us a few more points that satisfy the equation. In particular, $(\mathbf{80}, \mathbf{120})$ and $(\mathbf{160}, \mathbf{80})$ are points of the graph. If a point (a, p) of the graph has integral coefficients, it tells us that for $80, the manager could buy a apples and p pears. For example, the point $(\mathbf{80}, \mathbf{120})$ tells us that the manager can buy **80 apples** and **120 pears** for **$80**.

G.3 Exercises

Vocabulary Check *Fill in each blank with the most appropriate term or phrase from the given list:* ***b, coefficients, intercept, parallel, slope-point, standard,*** ***x-intercept, x = a, y-intercept, y = b***.

1. When graphing a linear equation written in the slope-intercept form, we first plot the _______________.

2. To write a linear equation when two points on the line are given, we usually use the _______________ form.

3. When writing a linear equation in ____________ form, we start with a positive x-term followed by the y-term. Also, if possible, we clear all the fractional _____________.

4. The equation of a vertical line passing through the point (a, b) is _________.

5. The equation of a horizontal line passing through the point (a, b) is _________.

6. The linear equation $\frac{x}{a} + \frac{y}{b} = 1$ is written in the ____________ form. In this form, the value a represents the ___________, while the value ___ represents the y-intercept.

7. Two lines that have no points in common are ______________.

Concept Check *Write each equation in standard form.*

8. $y = -\frac{1}{2}x - 7$

9. $y = \frac{1}{3}x + 5$

10. $\frac{x}{5} + \frac{y}{-4} = 1$

11. $y - 7 = \frac{3}{2}(x - 3)$

12. $y - \frac{5}{2} = -\frac{2}{3}(x + 6)$

13. $2y = -0.21x + 0.35$

Concept Check *Write each equation in slope-intercept form.*

14. $3y = \frac{1}{2}x - 5$

15. $\frac{x}{3} + \frac{y}{5} = 1$

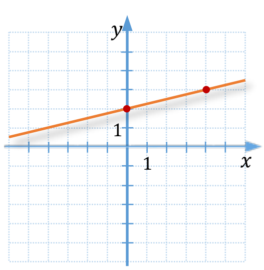
16. $4x - 5y = 10$

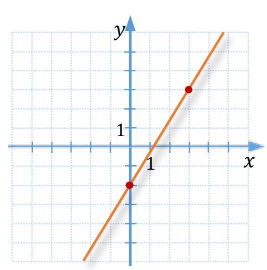
17. $3x + 4y = 7$

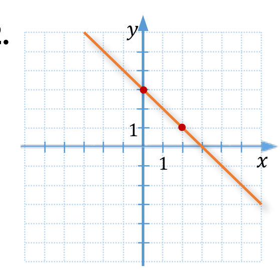
18. $y + \frac{3}{2} = \frac{2}{5}(x + 2)$

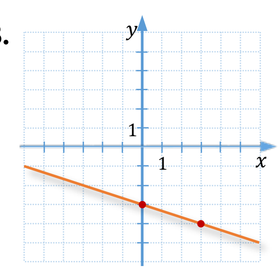
19. $y - \frac{1}{2} = -\frac{2}{3}\left(x - \frac{1}{2}\right)$

Concept Check *Write an equation in slope-intercept form of the line shown in each graph.*

20.


21.


22.


23.


Find an equation of the line that satisfies the given conditions. Write the equation in ***slope-intercept*** *and* ***standard*** *form.*

24. through $(-3,2)$, with slope $m = \frac{1}{2}$

25. through $(-2,3)$, with slope $m = -4$

26. with slope $m = \frac{3}{2}$ and y-intercept at -1

27. with slope $m = -\frac{1}{5}$ and y-intercept at 2

28. through $(-1,-2)$, with y-intercept at -3

29. through $(-4,5)$, with y-intercept at $\frac{3}{2}$

30. through $(2,-1)$ and $(-4,6)$

31. through $(3,7)$ and $(-5,1)$

32. through $\left(-\frac{4}{3},-2\right)$ and $\left(\frac{4}{5},\frac{2}{3}\right)$

33. through $\left(\frac{4}{3},\frac{3}{2}\right)$ and $\left(-\frac{1}{2},\frac{4}{3}\right)$

Find an equation of the line that satisfies the given conditions.

34. through $(-5,7)$, with slope 0

35. through $(-2,-4)$, with slope 0

36. through $(-1,-2)$, with undefined slope

37. through $(-3,4)$, with undefined slope

38. through $(-3,6)$ and horizontal

39. through $\left(-\frac{5}{3},-\frac{7}{2}\right)$ and horizontal

40. through $\left(-\frac{3}{4},-\frac{3}{2}\right)$ and vertical

41. through $(5,-11)$ and vertical

Concept Check *Write an equation in standard form for each of the lines described. In each case make a sketch of the given line and the line satisfying the conditions.*

42. through $(7,2)$ and parallel to $3x - y = 4$

43. through $(4,1)$ and parallel to $2x + 5y = 10$

44. through $(-2,3)$ and parallel to $-x + 2y = 6$

45. through $(-1,-3)$ and parallel to $-x + 3y = 12$

46. through $(-1,2)$ and parallel to $y = 3$

47. through $(-1,2)$ and parallel to $x = -3$

48. through $(6,2)$ and perpendicular to $2x - y = 5$

49. through $(0,2)$ and perpendicular to $5x + y = 15$

50. through $(-2,4)$ and perpendicular to $3x + y = 6$

51. through $(-4,-1)$ and perpendicular to $x - 3y = 9$

52. through $(3,-4)$ and perpendicular to $x = 2$

53. through $(3,-4)$ and perpendicular to $y = -3$

Analytic Skills *For each situation, write an equation in the form y = mx + b, and then answer the question of the problem.*

54. Membership in the Midwest Athletic Club costs \$99, plus \$41 per month. Let x represent the number of months and y represent the cost. How much does one-year membership cost?

55. A cell phone plan includes 900 anytime minutes for \$60 per month, plus a one-time activation fee of \$36. A cell phone is included at no additional charge. Let x represent the number of months of service and y represent the cost. If you sign a 1-yr contract, how much will this cell phone plan cost?

56. There is a \$30 fee to rent a chainsaw, plus \$6 per day. Let x represent the number of days the saw is rented and y represent the total charge to the renter, in dollars. If the total charge is \$138, for how many days is the saw rented?

57. A rental car costs \$50 plus \$0.12 per kilometer. Let x represent the number of kilometers driven and y represent the total charge to the renter, in dollars. How many kilometers was the car driven if the renter paid \$84.20?

Analytic Skills *Solve each problem.*

58. At its inception, a professional organization had 26 members. Three years later, the organization had grown to 83 members. If membership continues to grow at the same rate, find an equation that represents the number n of members in the organization after t years.

59. Thirty minutes after a truck driver passes the 142-km marker on a freeway, he passes the 170-km marker. Find an equation that shows the distance d he drives in t hr.

60. The average annual cost of a private college or university is shown in the table. This cost includes tuition, fees, room, and board.

Year *y*	**2007**	**2016**
Cost *C*	\$37000	\$72000

a. Find the slope-intercept form of a line that passes through these two data points.
b. Interpret the slope in the context of the problem.
c. To the nearest thousand, estimate the cost of private college or university in 2020.

61. The life expectancy for a person born in 1900 was 48 years, and in 2000 it was 77 years. To the nearest year, estimate the life expectancy for someone born in 1970.

62. After 2 years, the amount in a savings account earning simple interest was \$1070. After 5 years, the amount in the account was \$1175. Find an equation that represents the amount A in the account after t years.

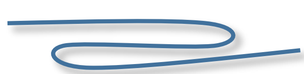
63. A real-estate agent receives a flat monthly salary plus a 0.5% commission on her monthly home sales. In a particular month, her home sales were \$500,000, and her total monthly income was \$4300.

a. Write an equation in slope-intercept form that shows the real-estate agent's total monthly income I in terms of her monthly home sales s.
b. Graph the equation on the coordinate plane.
c. What does the I-intercept represent in the context of the problem?
d. What does the slope represent in the context of the problem?

64. A taxi company charges a flat meter fare of \$1.25 plus an additional fee for each kilometer (or part thereof) driven. A passenger pays \$10.25 for a 6-kilometer taxi ride.

a. Find an equation in slope-intercept form that models the total meter fare f in terms of the number k of kilometers driven.
b. Graph the equation on the coordinate plane.
c. What does the slope of the graph of the equation in part **a.** represent in this situation?
d. How many kilometers were driven if a passenger pays \$20.75?

65. Fold a string like this:



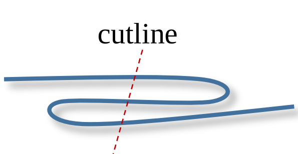


Figure 3.1

Count how many pieces of string you would have after cutting the string as shown in *Figure 3.1*. Predict how many pieces of string you would have if you made 2, 3, or more such cuts. Complete the table below and determine whether or not the data in the table follow a linear pattern. Can you find an equation that predicts the number of pieces if you know the number of cuts?

# of cuts	**0**	**1**	**2**	**3**	**4**	**5**
# of pieces						

G.4 Linear Inequalities in Two Variables Including Systems of Inequalities

In many real-life situations, we are interested in a range of values satisfying certain conditions rather than in one specific value. For example, when exercising, we like to keep the heart rate between 120 and 140 beats per minute. The systolic blood pressure of a healthy person is usually between 100 and 120 mmHg (millimeters of mercury). Such conditions can be described using inequalities. Solving systems of inequalities has its applications in many practical business problems, such as how to allocate resources to achieve a maximum profit or a minimum cost. In this section, we study graphical solutions of linear inequalities and systems of linear inequalities.

Linear Inequalities in Two Variables

Definition 4.1 ► Any inequality that can be written as

$$\boldsymbol{Ax+By<C},\ \ \boldsymbol{Ax+By\le C},\ \ \boldsymbol{Ax+By>C},\ \ \boldsymbol{Ax+By\ge C},\ \text{or}\ \boldsymbol{Ax+By\ne C},$$

where $A, B, C \in \mathbb{R}$ and A and B are not both 0, is a **linear inequality in two variables**.

To **solve** an inequality in two variables, x and y, means to **find <u>all</u> ordered pairs** $(\boldsymbol{x}, \boldsymbol{y})$ satisfying the inequality.

Inequalities in two variables arise from many situations. For example, suppose that the number of full-time students, f, and part-time students, p, enrolled in upgrading courses at the University of the Fraser Valley is at most 1200. This situation can be represented by the inequality

$$f+p\le 1200.$$

Some of the solutions (f,p) of this inequality are: $(1000, 200)$, $(1000, 199)$, $(1000, 198)$, $(600, 600)$. $(550, 600)$, $(1100, 0)$, and many others.

The solution sets of inequalities in two variables contain infinitely many ordered pairs of numbers which, when graphed in a system of coordinates, fulfill specific regions of the coordinate plane. That is why it is more beneficial to present such solutions in the form of a graph rather than using set notation. To graph the region of points satisfying the inequality $f+p\le 1200$, we may want to solve it first for p,

$$p\le -f+1200,$$

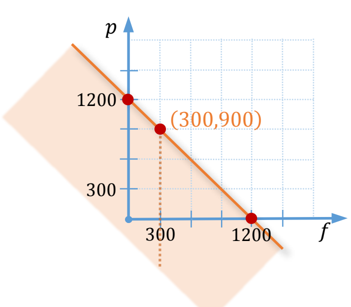


Figure 1a

and then graph the related equation, $p=-f+1200$, called the **boundary line**. Notice, that setting f to, for instance, 300 results in the inequality

$$p\le -300+1200=900.$$

So, any point with the first coordinate of 300 and the second coordinate of 900 or less satisfies the inequality (see the dotted half-line in *Figure 1a*). Generally, observe that any point with the first coordinate f and the second coordinate $-f+1200$ or less satisfies the inequality. Since the union of all half-lines that start from the boundary line and go down is the whole half-plane below the boundary line, we shade it as the solution set to the discussed inequality (see *Figure 1a*). The solution set also includes the points of the boundary line, as the inequality includes equation.

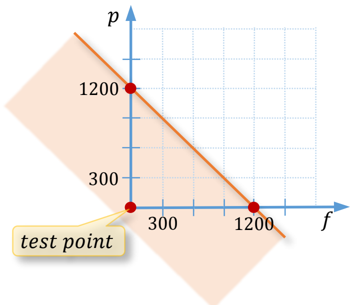


Figure 1b

The above strategy can be applied to any linear inequality in two variables. Hence, one can conclude that the solution set to a given linear inequality in two variables consists of **all points of one of the half-planes** obtained by cutting the coordinate plane by the corresponding boundary line. This fact allows us to find the solution region even faster. After graphing the boundary line, to know which half-plane to shade as the solution set, it is enough to check just one point, called a **test point**, chosen outside of the boundary line. In our example, it was enough to test for example point $(0,0)$. Since $0 \le -0 + 1200$ is a true statement, then the point $(0,0)$ belongs to the solution set. This means that the half-plane containing this test point must be the solution set to the given inequality, so we shade it.

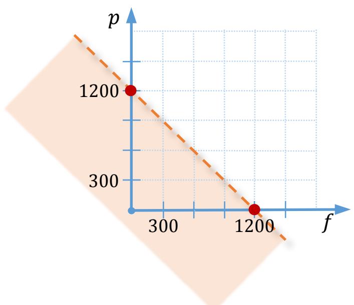


Figure 1c

The solution set of the strong inequality $p < -f + 1200$ consists of the same region as in *Figure 1b*, except for the points on the boundary line. This is because the points of the boundary line satisfy the equation $p = -f + 1200$, but not the inequality $p < -f + 1200$. To indicate this on the graph, we draw the boundary line using a dashed line (see *Figure 1c*).

In summary, to graph the solution set of a linear inequality in two variables, follow the steps:

1. Draw the graph of the corresponding **boundary line**.
   Make the line **solid** if the inequality involves ≤ or ≥.
   Make the line **dashed** if the inequality involves < or >.
2. Choose a **test point** outside of the line and substitute the coordinates of that point into the inequality.
3. If the test point satisfies the original inequality, **shade the half-plane containing the point.**
   If the test point does not satisfy the original inequality, **shade the other half-plane** (the one that does not contain the point).

Example 1 ► **Determining if a Given Ordered Pair of Numbers is a Solution to a Given Inequality**

Determine if the points $(3,1)$ and $(2,1)$ are solutions to the inequality $5x - 2y > 8$.

Solution ► An ordered pair is a solution to the inequality $5x - 2y > 8$ if its coordinates satisfy this inequality. So, to determine whether the pair $(3,1)$ is a solution, we substitute 3 for x and 1 for y. The inequality becomes

$$5 \cdot 3 - 2 \cdot 1 > 8,$$

which simplifies to the true inequality $13 > 8$.
Thus, $(3,1)$ is a solution to $5x - 2y > 8$.

However, replacing x by 2 and y by 1 results in $5 \cdot 2 - 2 \cdot 1 > 8$, or equivalently $8 > 8$. Since 8 is not larger than 8, the point $(2,1)$ does not satisfy the inequality. Thus, $(2,1)$ is not a solution to $5x - 2y > 8$.

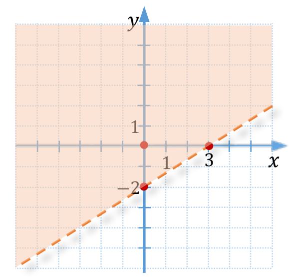
Example 2 ► **Graphing Linear Inequalities in Two Variables**

Graph the solution set of each inequality in two variables.

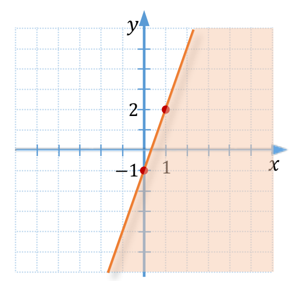
a. $2x - 3y < 6$ **b.** $y \leq 3x - 1$

c. $x \geq -3$ **d.** $y \neq x$

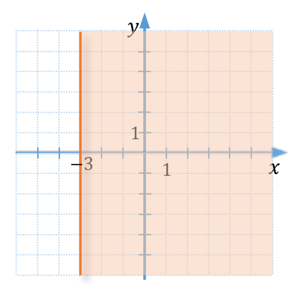
Solution ►



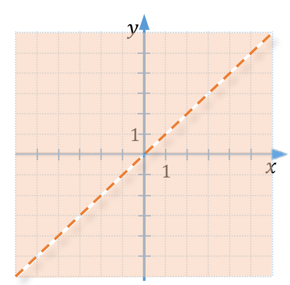
a. First, we graph the boundary line $2x - 3y = 6$, using the x- and y-intercepts: $(3,0)$ and $(0,-2)$. Since the inequality $<$ does not involve an equation, the line is marked as **dashed**, which indicates that the points on the line are not the solutions of the inequality. Then, we choose the point $(0,0)$ for the test point. Since $2 \cdot 0 - 3 \cdot 0 < 6$ is a true statement, then we shade the half-plane containing $(0,0)$ as the solution set.



b. First, we graph the boundary line $y \leq 3x - 1$, using the slope and y-intercept. Since the inequality $\leq$ contains an equation, the line is marked as **solid**. This indicates that the points on the line belong to solutions of the inequality. To decide which half-plane to shade as the solution region, we observe that y is lower than or equal to the $3x - 1$, which tells us that the solution points lie below or on the boundary line. So, we shade the half-plane below the line.



c. As before, to graph $x \geq -3$, first, we graph the solid vertical line $x = -3$, and then we shade the half-plane consisting of points with x-coordinates larger or equal to -3. So the solution set is the half-plane to the right of the boundary line, including this line.



d. The solution set of the inequality $y \neq x$ consists of all points that do not satisfy the equation $y = x$. This means that we mark the boundary line as dashed and shade the rest of the points of the coordinate plane.

Systems of Linear Inequalities

Let us refer back to our original problem about the full-time and part-time students that was modelled by the inequality $f + p \leq 1200$. Since f and p represent the number of students, it is reasonable to assume that $f \geq 0$ and $p \geq 0$. This means that we are really interested in solutions to the system of inequalities

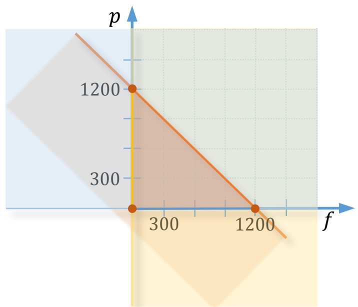


Figure 2

$$\begin{cases} p \leq -f + 1200 \\ f \geq 0 \\ p \geq 0 \end{cases}$$

To find this solution set, we graph each inequality in the same coordinate system. The solutions to the first inequality are marked in orange, the second inequality, in yellow, and the third inequality, in blue (see *Figure 2*). The intersection of the three shadings, orange, yellow, and blue, results in the brown triangular region, including the border lines and the vertices. This is the overall solution set to our system of inequalities. It tells us that the coordinates of any point from the triangular region, including its boundary, could represent the actual number of full-time and part-time students enrolled in upgrading courses during the given semester.

To graph the solution set to a system of inequalities, follow the steps:

1. Using different shadings, graph the solution set to each inequality in the system, drawing the solid or dashed boundary lines, whichever applies.

2. Shade the **intersection** of the solution sets more strongly if the inequalities were connected by the word "**and**". Mark each intersection point of boundary lines with a **filled in** circle if **both** lines are **solid**, or with a **hollow** circle if at least one of the lines is dashed.

or

Shade the **union** of the solution sets more strongly if the inequalities were connected by the word "**or**". Mark each intersection of boundary lines with a **hollow** circle if **both** lines are **dashed**, or with a **filled in** circle if at least one of the lines is **solid**.

Remember that a brace indicates the "*and*" connection!

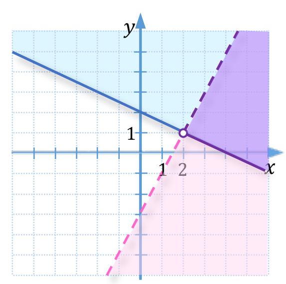
Example 3 ► **Graphing Systems of Linear Inequalities in Two Variables**

Graph the solution set to each system of inequalities in two variables.

a. $\begin{cases} y < 2x - 3 \\ y \geq -\frac{1}{2}x + 1 \end{cases}$ **b.** $y > x + 2$ or $y \leq 1$

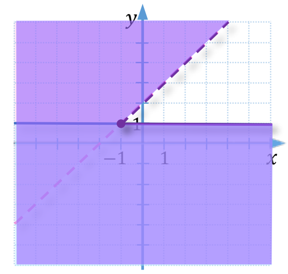
Solution ►

a. First, we graph the solution set to $y < 2x - 3$ in pink, and the solution set to $y \geq -\frac{1}{2}x + 2$ in blue. Since both inequalities must be satisfied, the solution set of the system is the **intersection** of the solution sets of individual inequalities. So, we shade the overlapping region, in purple, indicating the solid or dashed border lines. Since the



intersection of the boundary lines lies on a dashed line, it does not satisfy one of the inequalities, so it is not a solution to the system. Therefore, we mark it with a hollow circle.

b. As before, we graph the solution set to $y > x + 2$ in pink, and the solution set to $y \le 1$ in blue. Since the two inequalities are connected with the word "or", we look for the **union** of the two solutions. So, we shade the overall region, in purple, indicating the solid or dashed border lines. Since the intersection of these lines belongs to a solid line, it satisfies one of the inequalities, so it is also a solution of this system. Therefore, we mark it by a filled in circle.



Absolute Value Inequalities in Two Variables

As shown in *section L6*, absolute value linear inequalities can be written as systems of linear inequalities. So we can graph their solution sets, using techniques described above.

Example 4 ► **Graphing Absolute Value Linear Inequalities in Two Variables**

Rewrite the following absolute value inequalities as systems of linear inequalities and then graph them.

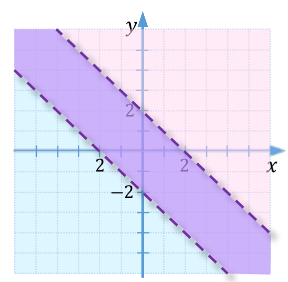
a. $|x + y| < 2$ **b.** $|x + 2| \ge y$ **c.** $|x - 1| \ge 2$

Solution ► **a.** First, we rewrite the inequality $|x + y| < 2$ in the equivalent form of the system of inequalities,

$$-2 < x + y < 2.$$

The solution set to this system is the intersection of the solutions to $-2 < x + y$ and $x + y < 2$. For easier graphing, let us rewrite the last two inequalities in the explicit form

$$\begin{cases} y > -x - 2 \\ y < -x + 2 \end{cases}$$



So, we graph $y > -x - 2$ in pink, $y < -x + 2$ in blue, and the final solution, in purple. Since both inequalities are strong (do not contain equation), the boundary lines are dashed.

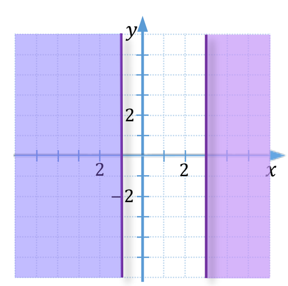
b. We rewrite the inequality $|x - 1| \ge 2$ in the form of the system of inequalities,

$$x - 1 \ge 2 \text{ or } x - 1 \le -2,$$

or equivalently as

$$x \ge 3 \text{ or } x \le -1.$$

Thus, the solution set to this system is the union of the solutions to $x \ge 3$, marked in pink, and $x \le -1$, marked in



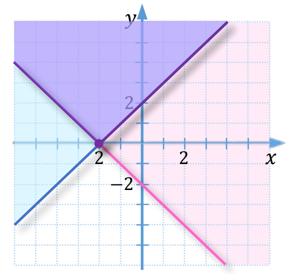
blue. The overall solution to the system is marked in purple and includes the boundary lines.

c. We rewrite the inequality $|x+2| \leq y$ in the form of the system of inequalities,

$$-y \leq x+2 \leq y,$$

or equivalently as

$$y \geq -x-2 \text{ and } y \geq x+2.$$



Thus, the solution set to this system is the intersection of the solutions to $y \geq -x-2$, marked in pink, and $y \geq x+2$, marked in blue. The overall solution to the system, marked in purple, includes the border lines and the vertex.

G.4 Exercises

Vocabulary Check *Fill in each blank with the most appropriate term from the given list:* ***above, below, boundary, dashed, intersection, satisfies, solid, test, union***.

1. To graph the solution set to the inequality $y > x+3$, first, we graph the _______________ line $y = x+3$. Since equation is not a part of the inequality $>$, the boundary line is marked as a ____________ line.

2. The solution set to the inequality $y > x+3$ lies _______________ the boundary line.

3. The solution set to the inequality $y \leq x+3$ lies _______________ the boundary line.

4. The boundary line of the solution region to the inequality $y \leq x+3$ is graphed as a ____________ line because the equality is a part of the inequality $\leq$.

5. To decide which half-plane to shade when graphing solutions to the inequality $5x-3y \geq 15$, we use a _______ point that does not lie on the boundary line. We shade the half-plane that includes the test point if it ______________ the inequality. In case the chosen test point doesn't satisfy the inequality, we shade the opposite half-plane.

6. To graph the solution set to a system of inequalities with the connecting word "and" we shade the ______________ of solutions to individual inequalities.

7. To graph the solution set to a system of inequalities with the connecting word "or" we shade the ______________ of solutions to individual inequalities.

Concept Check *For each inequality, determine if the given points belong to the solution set of the inequality.*

8. $y \geq -4x+3$; $(1,-1)$, $(1,0)$

9. $2x-3y < 6$; $(3,0)$, $(2,-1)$

10. $y > -2$; $(0,0)$, $(-1,-1)$

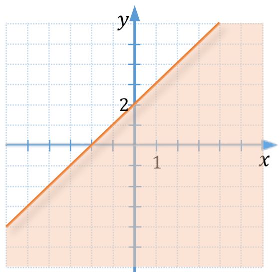
11. $x \geq -2$; $(-2,1)$, $(-3,1)$

Concept Check

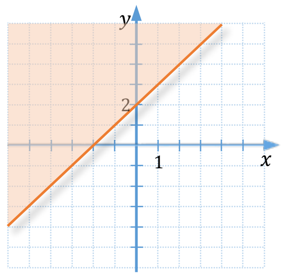
12. Match the given inequalities with the graphs of their solution sets.

a. $y \geq x + 2$ **b.** $y < -x + 2$ **c.** $y \leq x + 2$ **d.** $y > -x + 2$

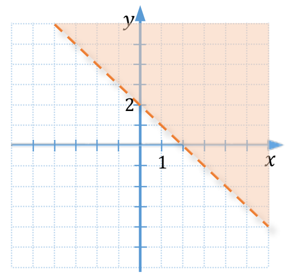
I



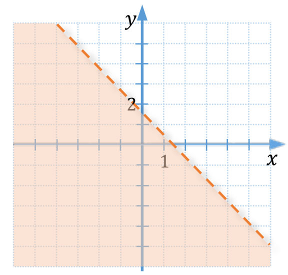
II



III



IV



Concept Check *Graph each linear inequality in two variables.*

13. $y \geq -\frac{1}{2}x + 3$ **14.** $y \leq \frac{1}{3}x - 2$ **15.** $y < 2x - 4$

16. $y > -x + 3$ **17.** $y \geq -3$ **18.** $y < 4.5$

19. $x > 1$ **20.** $x \leq -2.5$ **21.** $x + 3y > -3$

22. $5x - 3y \leq 15$ **23.** $y - 3x \geq 0$ **24.** $3x - 2y < -6$

25. $3x \leq 2y$ **26.** $3y \neq 4x$ **27.** $y \neq 2$

Graph each compound inequality.

28. $\begin{cases} x + y \geq 3 \\ x - y < 4 \end{cases}$ **29.** $\begin{cases} x \geq -2 \\ y \leq -2x + 3 \end{cases}$ **30.** $\begin{cases} x - y < 2 \\ x + 2y \geq 8 \end{cases}$

31. $\begin{cases} 2x - y < 2 \\ x + 2y > 6 \end{cases}$ **32.** $\begin{cases} 3x + y \leq 6 \\ 3x + y \geq -3 \end{cases}$ **33.** $\begin{cases} y < 3 \\ x + y < 5 \end{cases}$

34. $3x + 2y > 2$ or $x \geq 2$ **35.** $x + y > 1$ or $x + y < 3$

36. $y \geq -1$ or $2x + y > 3$ **37.** $y > x + 3$ or $x > 3$

Analytic Skills *For each problem, write a system of inequalities describing the situation and then graph the solution set in the xy-plane.*

38. At a movie theater, tickets cost \$8 and a bag of popcorn costs \$4. Let x be the number of tickets bought and y be the number of bags of popcorn bought. Graph the region in the xy-plane that represents all possible combinations of tickets and bags of popcorn that cost \$32 or less.

39. Suppose that candy costs \$3 per pound and cashews cost \$5 per pound. Let x be the number of pounds of candy bought and y be the number of pounds of cashews bought. Graph the region in the xy-plane that represents all possible weight combinations that cost less than \$15.

G.5 Concept of Function, Domain, and Range

In mathematics, we often investigate relationships between two quantities. For example, we might be interested in the average daily temperature in Abbotsford, BC, over the last few years, the amount of water wasted by a leaking tap over a certain period of time, or particular connections among a group of bloggers. The relations can be described in many different ways: in words, by a formula, through graphs or arrow diagrams, or simply by listing the ordered pairs of elements that are in the relation. A group of relations, called *functions*, will be of special importance in further studies. In this section, we will define functions, examine various ways of determining whether a relation is a function, and study related concepts such as *domain* and *range*.

Relations, Domains, and Ranges

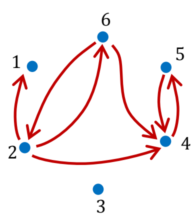
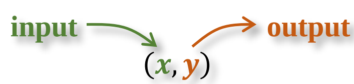


Figure 1

Consider a relation of knowing each other in a group of 6 people, represented by the arrow diagram shown in *Figure 1*. In this diagram, the points 1 through 6 represent the six people and an arrow from point x to point y tells us that the person x knows the person y. This correspondence could also be represented by listing the ordered pairs (x, y) whenever person x knows person y. So, our relation can be shown as the set of points

$$\{(2,1), (2,4), (2,6), (4,5), (5,4), (6,2), (6,4)\}$$

The x-coordinate of each pair (x, y) is called the **input**, and the y-coordinate is called the **output**.



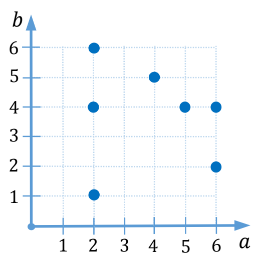


Figure 2a

The ordered pairs of numbers can be plotted in a system of coordinates, as in *Figure 2a*. The obtained graph shows that some inputs are in a relation with many outputs. For example, input 2 is in a relation with output 1, and 4, and 6. Also, the same output, 4, is assigned to many inputs. For example, the output 4 is assigned to the input 2, and 5, and 6.

The set of all the inputs of a relation is its **domain**. Thus, the domain of the above relation consists of all first coordinates

$$\{2, 4, 5, 6\}$$

The set of all the outputs of a relation is its **range**. Thus, the range of our relation consists of all second coordinates

$$\{1, 2, 4, 5, 6\}$$

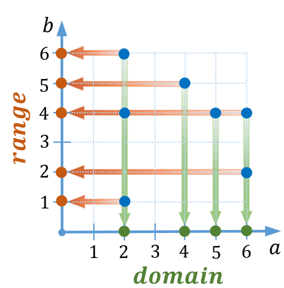
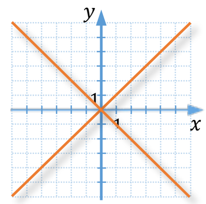


Figure 2b

The domain and range of a relation can be seen on its graph through the **perpendicular projection** of the graph **onto the horizontal axis**, for the **domain**, and **onto the vertical axis**, for the **range**. See *Figure 2b*.

In summary, we have the following definition of a relation and its domain and range:

Definition 5.1 ► A **relation** is any **set of ordered pairs**. Such a set establishes a **correspondence** between the **input** and **output** values. In particular, any subset of a coordinate plane represents a relation.
The **domain** of a relation consists of all **inputs** (**first coordinates**).
The **range** of a relation consists of all **outputs** (**second coordinates**).



Relations can also be given by an equation or an inequality. For example, the equation

$$|y| = |x|$$

describes the set of points in the xy-plane that lie on two diagonals, $y = x$ and $y = -x$. In this case, the domain and range for this relation are both the set of real numbers because the projection of the graph onto each axis covers the entire axis.

Functions, Domains, and Ranges

Relations that have exactly one output for every input are of special importance in mathematics. This is because as long as we know the rule of a correspondence defining the relation, the output can be uniquely determined for every input. Such relations are called **functions**. For example, the linear equation $y = 2x + 1$ defines a function, as for every input x, one can calculate the corresponding y-value in a unique way. Since both the input and the output can be any real number, the domain and range of this function are both the set of real numbers.

Definition 5.2 ► A **function** is a relation that assigns **exactly one** **output** value in the **range** to each **input** value of the **domain**.

If (x, y) is an ordered pair that belongs to a function, then x can be any arbitrarily chosen input value of the domain of this function, while y must be the uniquely determined value that is assigned to x by this function. That is why x is referred to as an **independent** variable while y is referred to as the **dependent** variable (because the y-value depends on the chosen x-value).

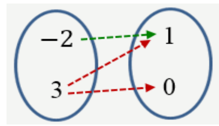
independent variable → (x, y) → **dependent variable**

How can we recognize if a relation is a function?

If the relation is given as a set of ordered pairs, it is enough to check if there are no two pairs with the same inputs. For example:

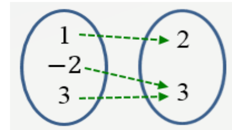
$\{(2,1), (2,4), (1,3)\}$ **relation**	$\{(2,1), (1,3), (4,1)\}$ **function**
The pairs $(2,1)$ and $(2,4)$ have the same inputs. So, there are **two y-values** assigned to the x-value 2, which makes it not a function.	There are no pairs with the same inputs, so each x-value is associated with exactly one pair and consequently with exactly one y-value. This makes it a function.

If the relation is given by a diagram, we want to check if no point from the domain is assigned to two points in the range. For example:



relation

There are **two arrows** starting from 3. So, there are two y-values assigned to 3, which makes it <u>not</u> a function.



function

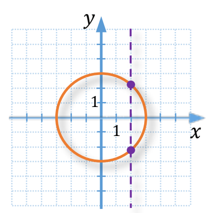
Only one arrow starts from each point of the domain, so each x-value is associated with exactly one y-value. Thus this is a function.

If the relation is given by a <u>graph</u>, we use **The Vertical Line Test**:

VERTICAL LINE TEST

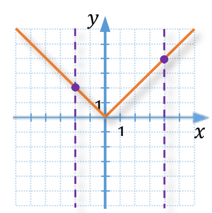
A relation is a **function** if **<u>no</u> vertical line** intersects the graph more than once.

For example:



relation

There is a vertical line that intersects the graph **twice**. So, there are two y-values assigned to an x-value, which makes it <u>not</u> a function.



function

Any vertical line intersects the graph only **once**. So, by The Vertical Line Test, this is a function.

If the relation is given by an <u>equation</u>, we check whether the y-value can be determined uniquely. For example:

$x^2 + y^2 = 1$

relation

Both points $(0,1)$ and $(0,-1)$ belong to the relation. So, there are **two y-values** assigned to 0, which makes it <u>not</u> a function.

$y = \sqrt{x}$

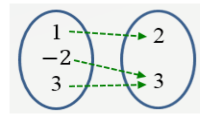
function

The y-value is uniquely defined as the square root of the x-value, for $x \geq 0$. So, this is a function.

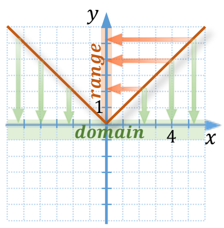
In general, to determine if a given relation is a function we analyse the relation to see whether or not it assigns two different y-values to the same x-value. If it does, it is just a relation, <u>not</u> a function. If it doesn't, it is a function.

Since functions are a special type of relation, the **domain and range of a function** can be determined the same way as in the case of a relation.
Let us look at domains and ranges of the above examples of functions.

The domain of the function $\{(2,1),(1,3),(4,1)\}$ is the set of the first coordinates of the ordered pairs, which is $\{1,2,4\}$. The range of this function is the set of second coordinates of the ordered pairs, which is $\{1,3\}$.

The domain of the function defined by the diagram 

is the first set of points, particularly $\{1,-2,3\}$.
The range of this function is the second set of points, which is $\{2,3\}$.



The domain of the function given by the accompanying graph is the projection of the graph onto the x-axis, which is the set of all real numbers $\mathbb{R}$.
The range of this function is the projection of the graph onto the y-axis, which is the interval of points larger or equal to zero, $[0,\infty)$.

The domain of the function given by the equation $y=\sqrt{x}$ is the set of nonnegative real numbers, $[0,\infty)$, since the square root of a negative number is not real.
The range of this function is also the set of nonnegative real numbers, $[0,\infty)$, as the value of a square root is never negative.

Example 1 ► **Determining Whether a Relation is a Function and Finding Its Domain and Range**

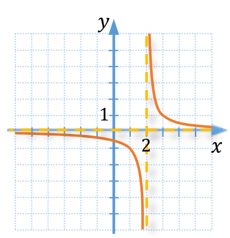
Decide whether each relation defines a function, and give the domain and range.

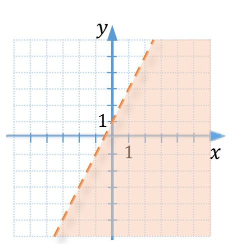
a. $y=\frac{1}{x-2}$ **b.** $y<2x+1$

c. $x=y^2$ **d.** $y=\sqrt{2x-1}$

Solution ►

a. Since $\frac{1}{x-2}$ can be calculated uniquely for every x from its domain, the relation $y=\frac{1}{x-2}$ is a function.
The domain consists of all real numbers that make the denominator, $x-2$, different than zero. Since $x-2=0$ for $x=2$, then the domain, D, is the set of all real numbers except for 2. We write $D=\mathbb{R}\setminus\{2\}$.
Since a fraction with nonzero numerator cannot be equal to zero, the range of $y=\frac{1}{x-2}$ is the set of all real numbers except for 0. We write $range=\mathbb{R}\setminus\{0\}$.

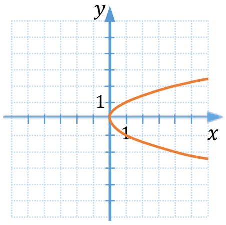




b. The inequality $y<2x+1$ is not a function as for every x-value there are many y-values that are lower than $2x+1$. Particularly, points $(0,0)$ and $(0,-1)$ satisfy the inequality and show that the y-value is not unique for $x=0$.

In general, because of the many possible y-values, <u>no</u> inequality defines a function.

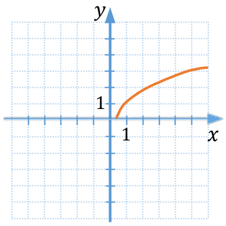
Since there are no restrictions on x-values, the domain of this relation is the set of all real numbers, $\mathbb{R}$. The range is also the set of all real numbers, $\mathbb{R}$, as observed in the accompanying graph.



c. Here, we can show two points, $(1,1)$ and $(1,-1)$, that satisfy the equation, which contradicts the requirement of a single y-value assigned to each x-value. So, this relation is <u>not</u> a function.

Since x is a square of a real number, it cannot be a negative number. So the domain consists of all nonnegative real numbers. We write, $D = [0, \infty)$. However, y can be any real number, so $range = \mathbb{R}$.

d. The equation $y = \sqrt{2x-1}$ represents a function, as for every x-value from the domain, the y-value can be calculated in a unique way.



The domain of this function consists of all real numbers that would make the radicand $2x - 1$ nonnegative. So, to find the domain, we solve the inequality:

$$2x - 1 \geq 0$$
$$2x \geq 1$$
$$x \geq \frac{1}{2}$$

Thus, $D = \left[\frac{1}{2}, \infty\right)$. As for the range, since the values of a square root are nonnegative, we have $range = [0, \infty)$

G.5 Exercises

Vocabulary Check *Fill in each blank with the most appropriate term from the given list:* ***domain, function, inputs, outputs, range, set, Vertical, x-axis, y-axis****.*

1. A relation is a ______ of ordered pairs, or equivalently, a correspondence between the elements of the set of inputs called the _____________ and the set of outputs, called the __________.

2. A relation with exactly one output for every input is called a _______________.

3. A graph represents a function iff (*if and only if*) it satisfies the ____________ Line Test.

4. The domain of a relation or function is the set of all ___________. To find the domain of a relation or function given by a graph in an xy-plane we project the graph perpendicularly onto the ____________.

5. The range of a relation or function is the set of all ___________. To find the range of a relation or function given by a graph in an xy-plane we project the graph perpendicularly onto the _____________.

Concept Check *Decide whether each relation defines a function, and give its domain and range.*

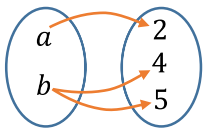
6. $\{(2,4), (0,2), (2,3)\}$

7. $\{(3,4), (1,2), (2,3)\}$

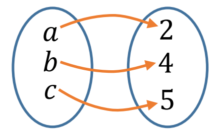
8. $\{(2,3), (3,4), (4,5), (5,2)\}$

9. $\{(1,1), (1,-1), (2,5), (2,-5)\}$

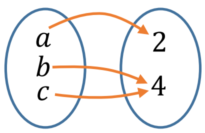
10.



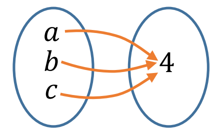
11.



12.



13.



14.

x	y
0	1
0	−1
1	2
1	−2

15.

x	y
−1	4
0	2
1	0
2	−2

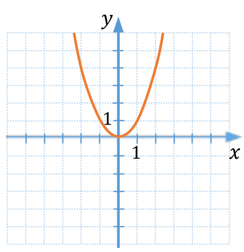
16.

x	y
3	1
6	2
9	1
12	2

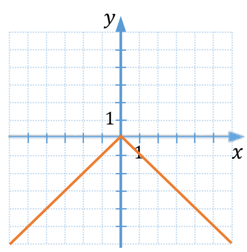
17.

x	y
−2	3
−2	0
−2	−3
−2	−6

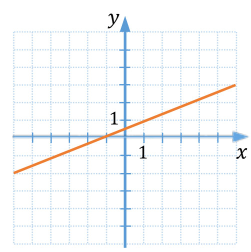
18.



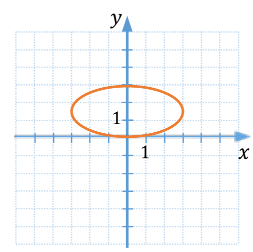
19.



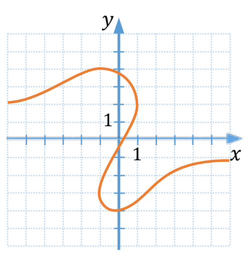
20.



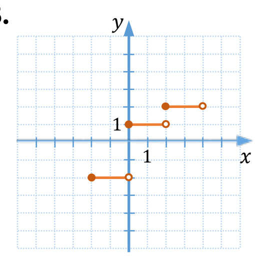
21.



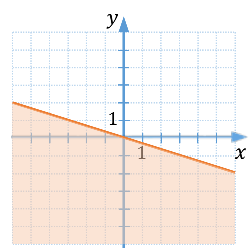
22.



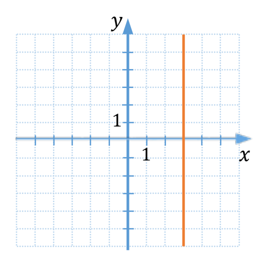
23.



24.



25.



Find the domain of each relation and decide whether the relation defines y as a function of x.

26. $y = 3x + 2$

27. $y = 5 - 2x$

28. $y = |x| - 3$

29. $x = |y| + 1$

30. $y^2 = x^2$

31. $y^2 = x^4$

32. $x = y^4$

33. $y = x^3$

34. $y = -\sqrt{x}$

35. $y = \sqrt{2x - 5}$

36. $y = \frac{1}{x+5}$

37. $y = \frac{1}{2x-3}$

38. $y = \frac{x-3}{x+2}$

39. $y = \frac{1}{|2x-3|}$

40. $y \le 2x$

41. $y - 3x \ge 0$

42. $y \ne 2$

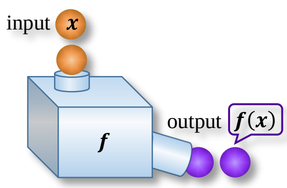
43. $x = -1$

44. $y = x^2 + 2x + 1$

45. $xy = -1$

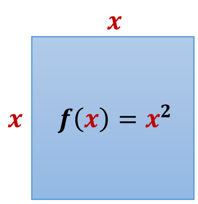
46. $x^2 + y^2 = 4$

G.6 Function Notation and Evaluating Functions



A function is a correspondence that assigns a single value of the range to each value of the domain. Thus, a function can be seen as an input-output machine, where the input is taken independently from the domain, and the output is the corresponding value of the range. The rule that defines a function is often written as an equation, with the use of x and y for the independent and dependent variables, for instance, $y = 2x$ or $y = x^2$. To emphasize that y depends on x, we write $\boldsymbol{y = f(x)}$, where f is the name of the function. The expression $\boldsymbol{f(x)}$, read as "$\boldsymbol{f}$ ***of*** $\boldsymbol{x}$", represents the dependent variable assigned to the particular x. Such notation shows the dependence of the variables as well as allows for using different names for various functions. It is also handy when evaluating functions. In this section, we introduce and use *function notation*, and show how to evaluate functions at specific input-values.

Function Notation



Consider the equation $y = x^2$, which relates the length of a side of a square, x, and its area, y. In this equation, the y-value depends on the value x, and it is uniquely defined. So, we say that y is a function of x. Using function notation, we write

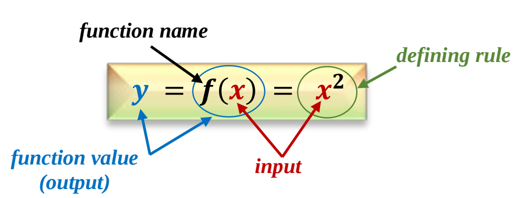
$$f(x) = x^2$$

The expression $f(x)$ is just another name for the dependent variable y, and it shouldn't be confused with a product of f and x. Even though $f(x)$ is really the same as y, we often write $f(x)$ rather than just y, because the notation $f(x)$ carries more information. Particularly, it tells us the name of the function so that it is easier to refer to the particular one when working with many functions. It also indicates the independent value for which the dependent value is calculated. For example, using function notation, we find the area of a square with a side length of 2 by evaluating $f(2) = 2^2 = 4$. So, 4 is the area of a square with a side length of 2.

The statement $f(2) = 4$ tells us that the pair $(2,4)$ belongs to function f, or equivalently, that 4 is assigned to the input of 2 by the function f. We could also say that function f attains the value 4 at 2.

If we calculate the value of function f for $x = 3$, we obtain $f(3) = 3^2 = 9$. So the pair $(3,9)$ also belongs to function f. This way, we may produce many ordered pairs that belong to f and consequently, make a graph of f.

Here is what each part of **function notation** represents:



Note: Functions are customarily denoted by a single letter, such as f, g, h, but also by abbreviations, such as sin, cos, or tan.

Function Values

Function notation is handy when evaluating functions for several input values. To evaluate a function given by an equation at a specific x-value from the domain, we substitute the x-value into the defining equation. For example, to evaluate $f(x) = \frac{1}{x-1}$ at $x = 3$, we calculate

$$f(3) = \frac{1}{3-1} = \frac{1}{2}$$

So $f(3) = \frac{1}{2}$, which tells us that when $x = 3$, the y-value is $\frac{1}{2}$, or equivalently, that the point $\left(3, \frac{1}{2}\right)$ belongs to the graph of the function f.

Notice that function f cannot be evaluated at $x = 1$, as it would make the denominator $(x - 1)$ equal to zero, which is not allowed. We say that $f(1) = DNE$ (read: *Does Not Exist*). Because of this, the domain of function f, denoted D_f, is $\mathbb{R}\setminus\{1\}$.

Graphing a function usually requires evaluating it for several x-values and then plotting the obtained points. For example, evaluating $f(x) = \frac{1}{x-1}$ for $x = \frac{3}{2}, 2, 5, \frac{1}{2}, 0, -1$, gives us

$$f\left(\frac{3}{2}\right) = \frac{1}{\frac{3}{2}-1} = \frac{1}{\frac{1}{2}} = 2$$

$$f(2) = \frac{1}{2-1} = \frac{1}{1} = 1$$

$$f(5) = \frac{1}{5-1} = \frac{1}{4}$$

$$f\left(\frac{1}{2}\right) = \frac{1}{\frac{1}{2}-1} = \frac{1}{-\frac{1}{2}} = -2$$

$$f(0) = \frac{1}{0-1} = -1$$

$$f(-1) = \frac{1}{-1-1} = -\frac{1}{2}$$

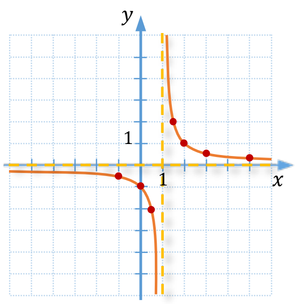


Figure 1

Thus, the points $\left(\frac{3}{2}, 2\right)$, $(2, 1)$, $\left(3, \frac{1}{2}\right)$, $\left(5, \frac{1}{4}\right)$, $\left(\frac{1}{2}, -2\right)$, $(0, -1)$, $\left(-1, -\frac{1}{2}\right)$ belong to the graph of f. After plotting them in a system of coordinates and predicting the pattern for other x-values, we produce the graph of function f, as in *Figure 1*.

Observe that the graph seems to be approaching the vertical line $x = 1$ as well as the horizontal line $y = 0$. These two lines are called ***asymptotes*** and are not a part of the graph of function f; however, they shape the graph. Asymptotes are customarily graphed by dashed lines.

Sometimes a function is given not by an equation but by a graph, a set of ordered pairs, a word description, etc. To evaluate such a function at a given input, we simply apply the function rule to the input.

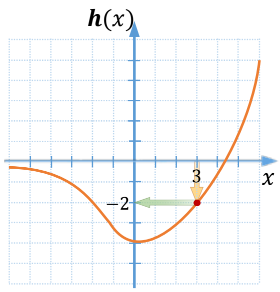


Figure 2a

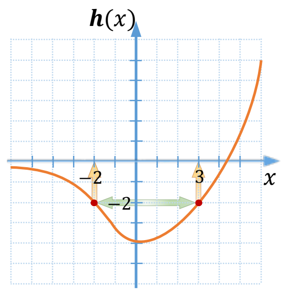


Figure 2b

For example, to find the value of function h, given by the graph in *Figure 2a*, for $x = 3$, we read the second coordinate of the intersection point of the vertical line $x = 3$ with the graph of h. Following the arrows in *Figure 2*, we conclude that $h(3) = -2$.

Notice that to find the x-value(s) for which $h(x) = -2$, we reverse the above process. This means: we read the first coordinate of the intersection point(s) of the horizontal line $y = -2$ with the graph of h. By following the reversed arrows in *Figure 2b*, we conclude that $h(x) = -2$ for $x = 3$ and for $x = -2$.

Example 1 ► **Evaluating Functions**

Evaluate each function at $x = 2$ and write the answer using function notation.

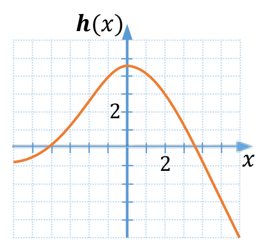
a. $f(x) = 3 - 2x$

b. function f squares the input

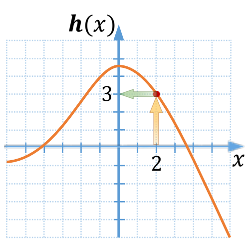
c.

x	$g(x)$
-1	2
2	5
3	-1

d.



Solution ►



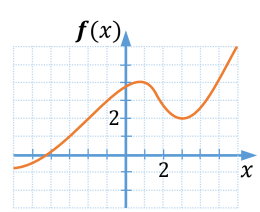
a. Following the formula, we have $f(2) = 3 - 2(2) = 3 - 4 = \mathbf{-1}$

b. Following the word description, we have $f(2) = 2^2 = \mathbf{4}$

c. $g(2)$ is the value in the second column of the table that corresponds to 2 from the first column. Thus, $g(2) = \mathbf{5}$.

d. As shown in the graph, $h(2) = \mathbf{3}$.

Example 2 ► **Finding from a Graph the x-value for a Given $f(x)$-value**

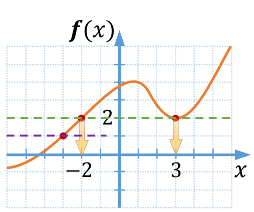


Given the graph, find all x-values for which

a. $f(x) = 1$

b. $f(x) = 2$

Solution ►



a. The purple line $y = 1$ cuts the graph at $x = -3$, so $f(x) = 1$ for $x = -3$.

b. The green line $y = 2$ cuts the graph at $x = -2$ and $x = 3$, so $f(x) = 2$ for $x \in \{-2, 3\}$.

Example 3 **Evaluating Functions and Expressions Involving Function Values**

Suppose $f(x) = \frac{1}{2}x - 1$ and $g(x) = x^2 - 5$. Evaluate each expression.

a. $f(4)$ **b.** $g(-2)$ **c.** $g(a)$ **d.** $f(2a)$

e. $g(a-1)$ **f.** $3f(-2)$ **g.** $g(2+h)$ **h.** $f(2+h) - f(2)$

Solution

a. Replace x in the equation $f(x) = \frac{1}{2}x - 1$ by the value 4. So,
$f(4) = \frac{1}{2}(4) - 1 = 2 - 1 = \mathbf{1}$.

b. Replace x in the equation $g(x) = x^2 - 5$ by the value -2, using parentheses around the -2. So, $g(-2) = (-2)^2 - 5 = 4 - 5 = \mathbf{-1}$.

c. Replace x in the equation $g(x) = x^2 - 5$ by the input a. So, $g(a) = \boldsymbol{a^2 - 5}$.

d. Replace x in the equation $f(x) = \frac{1}{2}x - 1$ by the input $2a$. So,
$f(2a) = \frac{1}{\cancel{2}}(\cancel{2}a) - 1 = \boldsymbol{a - 1}$.

$(a-1)^2$
$= (a-1)(a-1)$
$= a^2 - a - a + 1$
$= a^2 - 2a + 1$

e. Replace x in the equation $g(x) = x^2 - 5$ by the input $(a - 1)$, using parentheses around the input. So, $g(a-1) = (a-1)^2 - 5 = a^2 - 2a + 1 - 5 = \boldsymbol{a^2 - 2a - 4}$.

f. The expression $3f(-2)$ means three times the value of $f(-2)$, so we calculate
$3f(-2) = 3 \cdot \left(\frac{1}{\cancel{2}}(-\cancel{2}) - 1\right) = 3(-1-1) = 3(-2) = \mathbf{-6}$.

$(2+h)^2$
$= (2+h)(2+h)$
$= 4 + 2h + 2h + h^2$
$= 4 + 4h + h^2$

g. Replace x in the equation $g(x) = x^2 - 5$ by the input $(2 + h)$, using parentheses around the input. So, $g(2+h) = (2+h)^2 - 5 = 4 + 4h + h^2 - 5 = \boldsymbol{h^2 + 4h - 1}$.

h. When evaluating $f(2+h) - f(2)$, focus on evaluating $f(2+h)$ first and then, to subtract the expression $f(2)$, use a bracket just after the subtraction sign. So,

$$f(2+h) - f(2) = \underbrace{\frac{1}{2}(2+h) - 1}_{f(2+h)} - \underbrace{\left[\frac{1}{2}(2) - 1\right]}_{f(2)} = 1 + \frac{1}{2}h - 1 - [1-1] = \boldsymbol{\frac{1}{2}h}$$

Note: To perform the perfect squares in the solution to *Example 3e* and *3g*, we follow the **perfect square formula** $\boldsymbol{(a+b)^2 = a^2 + 2ab + b^2}$ or $\boldsymbol{(a-b)^2 = a^2 - 2ab + b^2}$. One can check that this formula can be obtained as a result of applying the distributive law, often referred to as the *FOIL* method, when multiplying two binomials (see the examples in callouts in the left margin). However, we prefer to use the perfect square formula rather than the *FOIL* method, as it makes the calculation process more efficient.

Function Notation in Graphing and Application Problems

By *Definition 1.1* in *section G1*, a linear equation is an equation of the form $Ax + By = C$. The graph of any linear equation is a line, and any nonvertical line satisfies the Vertical Line Test. Thus, any linear equation $Ax + By = C$ with $B \neq 0$ defines a linear function.

How can we write this function using function notation?

Since $y = f(x)$, we can replace the variable y in the equation $Ax + By = C$ with $f(x)$ and then solve for $f(x)$. So, we obtain

$$Ax + B \cdot f(x) = C \quad /-Ax$$

$$B \cdot f(x) = -Ax + C \quad /\div B$$

$$f(x) = -\frac{A}{B}x + \frac{C}{B}$$

Alternatively, we can solve the original equation for y and then replace y with $f(x)$.

must assume that $B \neq 0$

Definition 6.1 ► Any function that can be written in the form

$$\boldsymbol{f(x) = mx + b},$$

where m and b are real numbers, is called a **linear function**. The value $\boldsymbol{m}$ represents the **slope** of the graph, and the value $\boldsymbol{b}$ represents the **$\boldsymbol{y}$-intercept** of this function. The **domain** of any linear function is the set of all real numbers, $\mathbb{R}$.

In particular:

Definition 6.2 ► A linear function with slope $\boldsymbol{m = 0}$ takes the form

$$\boldsymbol{f(x) = b},$$

where b is a real number, and is called a **constant function**.

Note: Similarly as the domain of any linear function, the **domain** of a constant function is the set $\mathbb{R}$. However, the **range** of a constant function is the one element set $\{\boldsymbol{b}\}$, while the range of any nonconstant linear function is the set $\mathbb{R}$.

Generally, any equation in two variables, x and y, that defines a function can be written using function notation by solving the equation for y and then letting $y = f(x)$. For example, to rewrite the equation $-4x^2 + 2y = 5$ **explicitly** as a function f of x, we solve for y,

$$2y = 4x^2 + 5 \quad \text{(implicit form)}$$

$$y = 2x^2 + \frac{5}{2}, \quad \text{(explicit form)}$$

and then replace y by $f(x)$. So, $f(x) = 2x^2 + \frac{5}{2}$.

Using function notation, the graph of a function is defined as follows:

Definition 6.3 ► The graph of a function $\boldsymbol{f}$ of $\boldsymbol{x}$ is the set of ordered pairs $(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x}))$ for every input x form the domain $\boldsymbol{D_f}$ of the function. This can be stated as

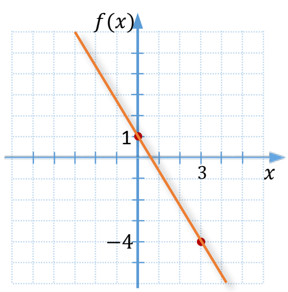
$$\textbf{\textit{graph}}\ \textit{of}\ \boldsymbol{f} = \{(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x})) | \boldsymbol{x} \in \boldsymbol{D_f}\}$$

Example 4 ► **Function Notation in Writing and Graphing Functions**

Each of the given equations represents a function of x. Rewrite the formula in explicit form, using function notation. Then graph this function and state its domain and range.

a. $5x + 3 \cdot f(x) = 3$ **b.** $|x| - y = -3$

Solution ► **a.** After solving the equation for $f(x)$,



$$5x + 3 \cdot f(x) = 3$$

$$3 \cdot f(x) = -5x + 3$$

$$f(x) = -\tfrac{5}{3}x + 1,$$

we observe that the function f is linear. So, we can graph it using the slope-intercept method. The graph confirms that the domain and range of this function are both the set of all real numbers, $\mathbb{R}$.

b. After solving the equation for y,

$$|x| - y = 2$$

$$|x| - 2 = y,$$

we obtain the function $f(x) = |x| - 2$.

If we are not sure how the graph of this function looks like, we may evaluate $f(x)$ for several x-values, plot the obtained points, and observe the pattern. For example, let $x = -2, -1, 0, 1, 2$. We fill in the table of values,

$\boldsymbol{x}$	$\boldsymbol{\lvert x\rvert - 2 = f(x)}$	$\boldsymbol{(x, f(x))}$
-2	$\lvert -2\rvert - 2 = 0$	$(-2,0)$
-1	$\lvert -1\rvert - 2 = -1$	$(-1,-1)$
0	$\lvert 0\rvert - 2 = -2$	$(0,-2)$
1	$\lvert 1\rvert - 2 = -1$	$(1,-1)$
2	$\lvert 2\rvert - 2 = 0$	$(2,0)$

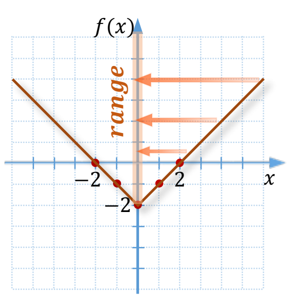


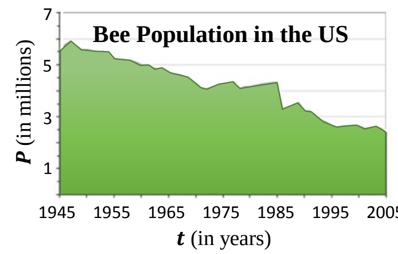
Figure 3

and plot the points listed in the third column. One may evaluate $f(x)$ for several more x-values, plot the additional points, and observe that these points form a V-shape with a vertex at $(0, -2)$. By connecting the points as in *Figure 3*, we obtain the graph of function $f(x) = |x| - 2$.

Since one can evaluate the function $f(x) = |x| - 2$ for any real x, the domain of f is the set $\mathbb{R}$. The range can be observed by projecting the graph perpendicularly onto the vertical axis. So, the range is the interval $[-2, \infty)$, as shown in *Figure 3*.

Example 5 ► A Function in Applied Situations

The accompanying graph defines the function of bee population P, in millions, with respect to time t, in years.



a. Use the graph to approximate the value of $P(1945)$ and $P(2005)$. Interpret each value in the context of the problem.

b. Estimate the average rate of change in the bee population over the years $1945 - 2005$, and interpret the result in the context of the problem.

c. Approximately in what year is $P(t) = 5$? Approximately in what year is $P(t) = 3$? Interpret each situation in the context of the problem.

d. What is the general tendency of the function $P(t)$ over the years $1945 - 2005$?

e. Assuming the same declining tendency of the function P will continue, using the graph, estimate the year in which we could expect the extinction of bees in the US.

Solution ►

a. One may read from the graph that $P(1945) \approx 5.5$ and $P(2005) \approx 2.5$ (see the orange line in *Figure 4a*). The first equation tells us that in 1945 there were approximately 5.5 million bees in the US. The second equation indicates that in 2005 there were approximately 2.5 million bees in the US.

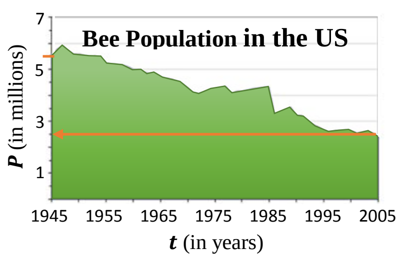


Figure 4a

b. The average rate of change is represented by the slope. Since the change in bee population over the years $1945 - 2005$ is $2.5 - 5.5 = -3$ million, and the change in time $1945 - 2005 = 50$ years, then the slope is $-\frac{3}{50} = -0.06$ million per year. This means that on average, 60,000 bees died each year between 1945 and 2005, in the US.

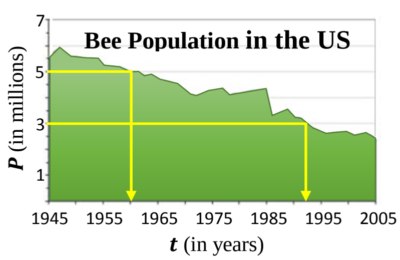


Figure 4b

c. As indicated by yellow arrows in *Figure 4b*, $P(t) = 5$ for $t = 1960$ and $P(t) = 3$ for $t \approx 1992$. The first result tells us that the bee population in the US was 5 million in 1960. The second result tells us that the bee population in the US was 3 million in approximately 1992.

d. The general tendency of function $P(t)$ over the years $1945 - 2005$ is declining.

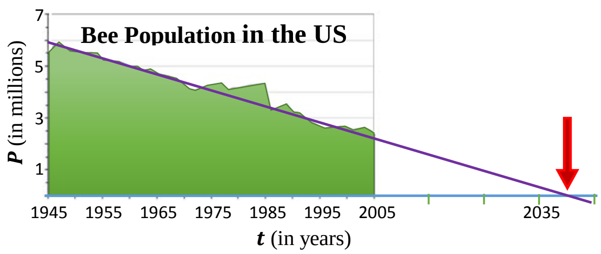
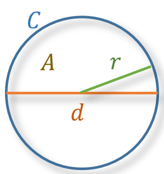


Figure 4c

e. Assuming the same declining tendency, to estimate the year in which the bee population in the US will disappear, we extend the t-axis and the approximate line of tendency (see the purple line in *Figure 4c*) to see where they intersect. After extending of the scale on the t-axis, we predict that the bee population will disappear around the year 2040.

Example 6 ► Constructing Functions



Consider a circle with area A, circumference C, radius r, and diameter d.

a. Write A as a function of r.
b. Write r as a function of A.
c. Write d as a function of C.
d. Write d as a function of A.

Solution ►

a. Using the formula for the area of a circle, $A = \pi r^2$, the function A of r is $\boldsymbol{A(r) = \pi r^2}$.

b. To express r as a function of A, we solve the area formula for r.

$$A = \pi r^2$$
$$\frac{A}{\pi} = r^2$$
$$\sqrt{\frac{A}{\pi}} = r$$

So, the function r of A is $\boldsymbol{r(A) = \sqrt{\frac{A}{\pi}}}$.

c. To write d as a function of C, we start by connecting the formula for the circumference C in terms of r and the formula that expresses d in terms of r. Since

$$C = 2\pi r \quad \text{and} \quad d = 2r,$$

then after letting $r = \frac{d}{2}$ in the first equation, we obtain

$$C = 2\pi r = \cancel{2}\pi \cdot \frac{d}{\cancel{2}} = \pi d,$$

which after solving for d, gives us $d = \frac{C}{\pi}$. Hence, our function d of C is $\boldsymbol{d(C) = \frac{C}{\pi}}$.

d. Since $d = 2r$ and $r = \sqrt{\frac{A}{\pi}}$ (as developed in the solution to *Example 6b*),

then $d = 2\sqrt{\frac{A}{\pi}}$. Thus, the function d of A is $\boldsymbol{d(A) = 2\sqrt{\frac{A}{\pi}}}$.

G.6 Exercises

Vocabulary Check *Fill in each blank with the most appropriate term from the given list:* ***constant, d, function, graph, input, linear, output,*** $\mathbb{R}$.

1. The notation $y = f(x)$ is called ___________ notation. The notation $f(3)$ represents the ___________ value of the function f for the ___________ 3 and it shouldn't be confused with "f times 3".

2. Any function of the form $f(x) = mx + b$ is called a ___________ function.

3. The domain of any linear function is ____.

4. The range of any linear function that is not ____________ is $\mathbb{R}$.

5. If $f(a) = b$, the point (a, b) is on the _________ of f.

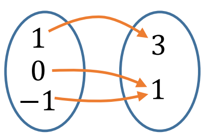
6. If (c, d) is on the graph of g, then $g(c) =$ ____.

Concept Check *For each function, find* ***a)*** $f(-1)$ *and* ***b)*** *all* x*-values such that* $f(x) = 1$.

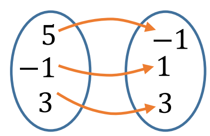
7. $\{(2,4), (-1,2), (3,1)\}$

8. $\{(-1,1), (1,2), (2,1)\}$

9.



10.



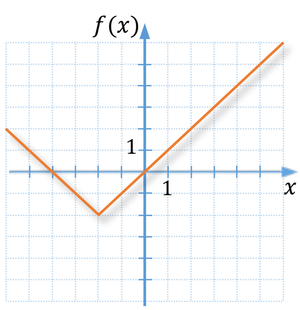
11.

x	$f(x)$
−1	4
0	2
2	1
4	−1

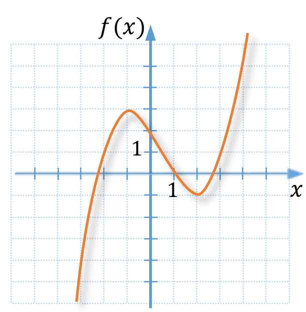
12.

x	$f(x)$
−3	1
−1	2
1	2
3	1

13.



14.



Concept Check *Let* $f(x) = -3x + 5$ *and* $g(x) = -x^2 + 2x - 1$. *Find the following.*

15. $f(1)$

16. $g(0)$

17. $g(-1)$

18. $f(-2)$

19. $f(p)$

20. $g(a)$

21. $g(-x)$

22. $f(-x)$

23. $f(a+1)$ **24.** $g(a+2)$ **25.** $g(x-1)$ **26.** $f(x-2)$

27. $f(2+h)$ **28.** $g(1+h)$ **29.** $g(a+h)$ **30.** $f(a+h)$

31. $f(3)-g(3)$ **32.** $g(a)-f(a)$ **33.** $3g(x)+f(x)$ **34.** $f(x+h)-f(x)$

Concept Check *Fill in each blank with the correct response.*

35. The graph of the equation $2x + y = 4$ is a ___________. One point that lies on this graph is (3, ___). If we solve the equation for y and use function notation, we obtain $f(x) =$ ___________ . For this function, $f(3) =$ ______ , meaning that the point (___ , ___) lies on the graph of the function.

Graph each function. Give the domain and range.

36. $f(x) = -2x + 5$ **37.** $g(x) = \frac{1}{3}x + 2$ **38.** $h(x) = -3x$

39. $F(x) = x$ **40.** $G(x) = 0$ **41.** $H(x) = 2$

42. $x - h(x) = 4$ **43.** $-3x + f(x) = -5$ **44.** $2 \cdot g(x) - 2 = x$

45. $k(x) = |x-3|$ **46.** $m(x) = 3 - |x|$ **47.** $q(x) = x^2$

48. $Q(x) = x^2 - 2x$ **49.** $p(x) = x^3 + 1$ **50.** $s(x) = \sqrt{x}$

Solve each problem.

51. A taxicab driver charges \$1.75 per kilometer.

a. Fill in the table with the correct charge $f(x)$ for a trip of x kilometers.
b. Find the linear function that gives the amount charged $f(x) =$ _________.
c. Graph $f(x)$ for the domain $\{0, 1, 2, 3\}$.

x	$f(x)$
0	
1	
2	
3	

52. The table represents a linear function.

a. What is $f(3)$?
b. If $f(x) = -2.5$, what is the value of x?
c. What is the slope of this line?
d. What is the y-intercept of this line?
e. Using your answers to parts **c.** and **d.**, write an equation for $f(x)$.

x	$f(x)$
0	3.5
1	2.3
2	1.1
3	−0.1
4	−1.3
5	−2.5

53. A car rental is \$150 plus \$0.20 per kilometer. Let x represent the number of kilometers driven and $f(x)$ represent the total cost to rent the car.

a. Write a linear function that models this situation.
b. Find $f(250)$ and interpret your answer in the context of this problem.
c. Find the value of x satisfying the equation $f(x) = 230$ and interpret it in the context of this problem.

54. A window cleaner charges \$50 per visit plus \$35 per hour.

a. Express the total charge, C, as a function of the number of hours worked, n.

b. Find $C(4)$ and interpret your answer in the context of this problem.

c. If the window cleaner charged $295 for his job, what was the number of hours for which he has charged?

55. Refer to the accompanying graph to answer the questions below.

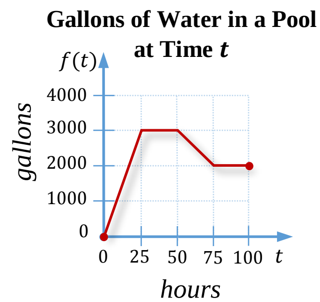
a. What numbers are possible values of the independent variable? The dependent variable?

b. For how long is the water level increasing? Decreasing?

c. How many gallons of water are in the pool after 90 hr?

d. Call this function f. What is $f(0)$? What does it mean?

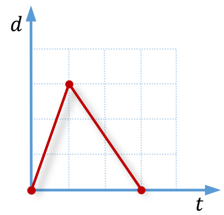
e. What is $f(25)$? What does it mean?



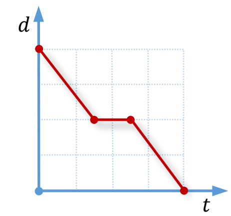
Analytic Skills

The graph represents the distance that a person is from home while walking on a straight path. The t-axis represents time, and the d-axis represents distance. Interpret the graph by describing the person's movement.

56.



57.



58. Consider a square with area A, side s, perimeter P, and diagonal d. (*Hint for question 58 c and d*: apply the Pythagorean equation $a^2 + b^2 = c^2$, where c is the hypotenuse of a right angle triangle with arms a and b.)

a. Write A as a function of s.

b. Write s as a function of P.

c. Write d as a function of s.

d. Write d as a function of P.