Linear Equations and Inequalities

One of the main concepts in Algebra is solving equations or inequalities. This is because solutions to most application problems involve setting up and solving equations or inequalities that describe the situation presented in the problem. In this unit, we will study techniques of solving linear equations and inequalities in one variable, linear forms of absolute value equations and inequalities, and applications of these techniques in word problems.

L1 Linear Equations in One Variable

When two algebraic expressions are compared by an equal sign $(=)$, an **equation** is formed. An equation can be interpreted as a scale that balances two quantities. It can also be seen as a mathematical sentence with the verb "equals" or the verb phrase "is equal to". For example, the equation $3x - 1 = 5$ corresponds to the sentence:

One less than three times an unknown number equals five.

Unless we know the value of the unknown number (the variable x), we are unable to determine whether or not the above sentence is a true or false statement. For example, if $x = 1$, the equation $3x - 1 = 5$ becomes a false statement, as $3 \cdot 1 - 1 \neq 5$ (the "scale" is not in balance); while, if $x = 2$, the equation $3x - 1 = 5$ becomes a true statement, as $3 \cdot 2 - 1 = 5$ (the "scale" is in balance). For this reason, such sentences (equations) are called **open sentences**. Each variable value that satisfies an equation (i.e., makes it a true statement) is a **solution** (i.e., a ***root***, or a ***zero***) of the equation. An equation is **solved** by finding its **solution set**, the set of all solutions.

Attention:

* ***Equations*** *can be* ***solved*** *by finding the variable value(s) satisfying the equation.*

Example: $\overbrace{2x + x - 1}^{\text{left side}} \underset{\uparrow \text{ equal sign}}{=} \overbrace{5}^{\text{right side}}$ *can be solved for x*

* ***Expressions*** *can only be* ***simplified*** *or* ***evaluated***

Example: $2x + x - 1$ *can be simplified to* $3x - 1$
or evaluated for a particular x-value

Example 1 ► **Distinguishing Between Expressions and Equations**

Decide whether each of the following is an expression or an equation.

a. $4x - 16$ **b.** $4x - 16 = 0$

Solution ► **a.** $4x - 16$ is an **expression** as it does not contain any symbol of equality.

This expression can be ***evaluated*** *(for instance, if* $x = 4$*, the expression assumes the value 0), or it can be written in a different form. For example, we could* ***factor*** *it. So, we could write*

$$4x - 16 = 4(x - 4).$$

Notice that the equal symbol (=) in the above line does not indicate an equation, but rather an equivalency between the two expressions, $4x - 16$ *and* $4(x - 4)$.

b. $4x - 16 = 0$ is an **equation** as it contains an equal symbol $(=)$ that connects two sides of the equation.

To solve this equation we could factor the left-hand side expression,

$$4(x-4) = 0,$$

and then from the zero product property, we have

$$x - 4 = 0,$$

which leads us to the solution

$$x = \mathbf{4}.$$

Attention: *Even though the two algebraic forms, $4x - 16$ and $4x - 16 = 0$ are related to each other, it is important that we neither **voluntarily add** the "$= 0$" part when we want to change the form of the expression, nor **voluntarily drop** the "$= 0$" part when we solve the equation.*

Example 2 ► **Determining if a Given Number is a Solution to an Equation**

Determine whether the number -2 is a solution to the given equation.

a. $4x = 10 + x$ **b.** $x^2 - 4 = 0$

Solution ► **a.** To determine whether -2 is a solution to the equation $4x = 10 + x$, we substitute -2 in place of the variable x and find out whether the resulting equation is a true statement. This gives us

$$4(-2) \stackrel{?}{=} 10 + (-2)$$
$$-8 \stackrel{?}{=} 8$$

Since the resulting equation is not a true statement, the number **-2 is not a solution** to the given equation.

b. After substituting -2 for x in the equation $x^2 - 4 = 0$, we obtain

$$(-2)^2 - 4 \stackrel{?}{=} 0$$
$$4 - 4 \stackrel{?}{=} 0,$$

which becomes $0 = 0$, a true statement. Therefore, the number **-2 is a solution** to the given equation.

Equations can be classified with respect to the number of solutions. There are **identities**, **conditional** equations, and **contradictions** (or ***inconsistent*** equations).

Definition 1.1 ► An **identity** is an equation that is satisfied by every real number for which the expressions on both sides of the equation are defined. Some examples of identities are

$$2x + 5x = 7x, \quad x^2 - 4 = (x+2)(x-2), \quad \text{or} \quad \frac{x}{x} = 1.$$

The solution set of the first two identities is the set of all real numbers, $\mathbb{R}$. *However, since the expression* $\frac{x}{x}$ *is undefined for* $x = 0$, *the solution set of the equation* $\frac{x}{x} = 1$ *is the set of all nonzero real numbers,* $\{\boldsymbol{x} | \boldsymbol{x} \neq \mathbf{0}\}$.

A **conditional** equation is an equation that is satisfied by at least one real number, but is not an identity. This is the most commonly encountered type of equation. Here are some examples of conditional equations:

$$3x - 1 = 5, \quad x^2 - 4 = 0, \quad \text{or} \quad \sqrt{x} = 2.$$

The solution set of the first equation is $\{2\}$; *of the second equation is* $\{-2,2\}$; *and of the last equation is* $\{4\}$.

A **contradiction** (an ***inconsistent*** equation) is an equation that has **no solution**. Here are some examples of contradictions:

$$5 = 1, \quad 3x - 3x = 8, \quad \text{or} \quad 0x = 1.$$

The solution set of any contradiction is the empty set, $\emptyset$.

Example 3 ► **Recognizing Conditional Equations, Identities, and Contradictions**

Determine whether the given equation is *conditional*, an *identity*, or a *contradiction*.

a. $x = x$ **b.** $x^2 = 0$ **c.** $\frac{1}{x} = 0$

Solution ►

a. This equation is satisfied by any real number. Therefore, it is an **identity**.

b. This equation is satisfied by $x = 0$, as $0^2 = 0$. However, any nonzero real number when squared becomes a positive number. So, the left side of the equation $x^2 = 0$ does not equal to zero for a nonzero x. That means that a nonzero number does not satisfy the equation. Therefore, the equation $x^2 = 0$ has exactly one solution, $x = 0$. So, the equation is **conditional**.

c. A fraction equals zero only when its numerator equals to zero. Since the numerator of $\frac{1}{x}$ does not equal to zero, then no matter what the value of x would be, the left side of the equation will never equal zero. This means that there is no x-value that would satisfy the equation $\frac{1}{x} = 0$. Therefore, the equation has no solution, which means it is a **contradiction**.

Attention: *Do not confuse the solution* $x = 0$ *to the equation in Example 3b with an empty set* $\emptyset$. *An empty set means that there is no solution.* $x = 0$ *means that there is one solution equal to zero.*

Solving Linear Equations in One Variable

In this section, we will focus on solving linear (up to the first degree) equations in one variable. Before introducing a formal definition of a linear equation, let us recall the definition of a term, a constant term, and a linear term.

Definition 1.2 ► A **term** is a **product** of numbers, letters, and possibly other algebraic expressions.

Examples of terms: $2,\ -3x,\ \frac{2}{3}(x+1),\ 5x^2y,\ -5\sqrt{x}$

A **constant term** is a number or a product of numbers.

Examples of constant terms: $2,\ -3,\ \frac{2}{3},\ 0,\ -5\pi$

A **linear term** is a product of numbers and the first power of a single variable.

Examples of linear terms: $-3x,\ \frac{2}{3}x,\ x,\ -5\pi x$

Definition 1.3 ► A **linear** equation is an equation with only **constant** or **linear terms**. A linear equation in one variable can be written in the form $\boldsymbol{Ax+B=0}$, for some real numbers A and B, and a variable x.

Here are some examples of *linear* equations: $2x+1=0,\quad 2=5,\quad 3x-7=6+2x$

Here are some examples of *nonlinear* equations: $x^2=16,\quad x+\sqrt{x}=-1,\quad 1+\frac{1}{x}=\frac{1}{x+1}$

So far, we have been finding solutions to equations mostly by guessing a value that would make the equation true. To find a methodical way of solving equations, observe the relations between equations with the same solution set. For example, equations

$$3x-1=5,\ \ 3x=6,\ \ and\ \ x=2$$

all have the same solution set $\{2\}$. While the solution to the last equation, $x=2$, is easily "seen" to be 2, the solution to the first equation, $3x-1=5$, is not readily apparent. Notice that the second equation is obtained by adding 1 to both sides of the first equation. Similarly, the last equation is obtained by dividing the second equation by 3. This suggests that to solve a linear equation, it is enough to write a sequence of simpler and simpler equations that preserve the solution set, and eventually result in an equation of the form:

$$x=constant \quad \text{or} \quad 0=constant.$$

If the resulting equation is of the form $x=constant$, the solution is this constant.
If the resulting equation is $\mathbf{0=0}$, then the original equation is an **identity**, as it is true for **all real values** $\boldsymbol{x}$.
If the resulting equation is $\mathbf{0}=\boldsymbol{constant\ other\ than\ zero}$, then the original equation is a **contradiction**, as there is **no real values** $\boldsymbol{x}$ that would make it true.

Definition 1.4 ► **Equivalent equations** are equations with the same solution set.

How can we transform an equation to obtain a simpler but equivalent one?

We can certainly simplify expressions on both sides of the equation, following properties of operations listed in *section R3*. Also, recall that an equation works like a scale in balance.

Therefore, adding (or subtracting) the same quantity to (from) both sides of the equation will preserve this balance. Similarly, multiplying (or dividing) both sides of the equation by a nonzero quantity will preserve the balance.

Suppose we work with an equation $A = B$, where A and B represent some algebraic expressions. In addition, suppose that C is a real number (or another expression).
Here is a summary of the basic equality operations that can be performed to produce equivalent equations:

Equality Operation	General Rule	Example
Simplification	Write each expression in a simpler but equivalent form	$2(x-3) = 1+3$ can be written as $2x-6=4$
Addition	$A + C = B + C$	$2x - 6 + 6 = 4 + 6$
Subtraction	$A - C = B - C$	$2x - 6 - 4 = 4 - 4$
Multiplication	$CA = CB, \quad if\ C \neq 0$	$\frac{1}{2} \cdot 2x = \frac{1}{2} \cdot 10$
Division	$\frac{A}{C} = \frac{B}{C}, \quad if\ C \neq 0$	$\frac{2x}{2} = \frac{10}{2}$

Example 4 ► **Using Equality Operations to Solve Linear Equations in One Variable**

Solve each equation.

a. $4x - 12 + 3x = 3 + 5x - 2x$ **b.** $2[3(x-6) - x] = 3x - 2(5 - x)$

Solution ► **a.** First, simplify each side of the equation and then isolate the linear terms (terms containing x) on one side of the equation. Here is a sequence of equivalent equations that leads to the solution:

$4x - 12 + 3x = 3 + 6x - 2x$	collect like terms	(1)
$7x - 12 = 3 + 4x$	move the x-terms to the left side by subtracting $4x$ and the constant terms to the right side by adding 12	(2)
$7x - \cancel{12} - 4x + \cancel{12} = 3 + \cancel{4x} - \cancel{4x} + 12$		(3)
$7x - 4x = 3 + 12$	collect like terms	(4)
$3x = 15$	isolate x by dividing both sides by 3	(5)
$\frac{\cancel{3}x}{\cancel{3}} = \frac{15}{3}$	Simplify each side	(6)
$x = \mathbf{5}$		(7)

Each equation is written underneath the previous one, with the **"=" symbol aligned in a column**

There is only **one "=" symbol** in each line

Let us analyze the relation between line (2) and (4).

$$7x - 12 = 3 + 4x \qquad (2)$$

$$7x - 4x = 3 + 12 \qquad (4)$$

By subtracting $4x$ from both sides of the equation (2), we actually 'moved' the term $+4x$ to the left side of equation (4) as $-4x$. Similarly, the addition of 12 to both sides of equation (2) caused the term -12 to 'move' to the other side as $+12$, in equation (4). This shows that the addition and subtraction property of equality allows us to change the position of a term from one side of an equation to another, by simply changing its sign. Although line (3) is helpful when explaining why we can move particular terms to another side by changing their signs, it is often cumbersome, especially when working with longer equations. So, in practice, we will avoid writing lines such as (3). Since it is important to indicate what operation is applied to the equation, we will record the operations performed in the right margin, after a slash symbol $(/)$. Here is how we could record the solution to equation (1) in a concise way.

$$4x - 12 + 3x = 3 + 6x - 2x$$

$$7x - 12 = 3 + 4x$$

$$7x - 4x = 3 + 12$$

$$3x = 15$$

$$x = \mathbf{5}$$

b. First, release all the brackets, starting from the inner-most brackets. If applicable, remember to collect like terms after releasing each bracket. Finally, isolate x by applying appropriate equality operations. Here is our solution:

$$2[3(x-6) - x] = 3x - 2(5 - x) \qquad \text{release red brackets}$$

$$2[3x - 18 - x] = 3x - 10 + 2x \qquad \text{collect like terms}$$

$$2[2x - 18] = 5x - 10 \qquad \text{release the blue bracket}$$

$$4x - 36 = 5x - 10 \qquad / \ -5x, +36$$

$$-x = 26 \qquad / \ \div(-1)$$

$$x = \mathbf{-26}$$

multiplication by -1 works as well

Note: *Notice that we could choose to collect x-terms on the right side of the equation as well. This would shorten the solution by one line and save us the division by -1. Here is the alternative ending of the above solution.*

$$4x - 36 = 5x - 10 \qquad / \ -4x, +10$$

$$-26 = x$$

Example 5 ► **Solving Linear Equations Involving Fractions**

Solve

$$\frac{x-4}{4}+\frac{2x+1}{6}=5.$$

Solution ► First, clear the fractions and then solve the resulting equation as in *Example 4*. To clear fractions, multiply both sides of the equation by the LCD of 4 and 6, which is 12.

$$\frac{x-4}{4}+\frac{2x+1}{6}=5 \qquad / \cdot 12 \qquad (8)$$

$$\overset{3}{\cancel{12}}\left(\frac{x-4}{\cancel{4}}\right)+\overset{2}{\cancel{12}}\left(\frac{2x+1}{\cancel{6}}\right)=12\cdot 5 \qquad (9)$$

$$3(x-4)+2(2x+1)=60 \qquad (10)$$

$$3x-12+4x+2=60 \qquad (11)$$

$$7x-10=60 \qquad (12)$$

$$7x=70 \qquad (13)$$

$$x=\frac{70}{7}=\mathbf{10} \qquad (14)$$

When multiplying each term by the $LCD = 12$, **simplify** it with the **denominator** before **multiplying** the result by the **numerator**.

So the solution to the given equation is $x = \mathbf{10}$.

Note: *Notice, that if the division of 12 by 4 and then by 6 can be performed fluently in our minds, writing equation* (9) *is not necessary. One could write equation* (10) *directly after the original equation* (8). *One could think: 12 divided by 4 is 3 so I multiply the resulting 3 by the numerator* $(x-4)$. *Similarly, 12 divided by 6 is 2 so I multiply the resulting 2 by the numerator* $(2x+1)$. *It is important though that each term, including the free term 5, gets multiplied by 12.*

Also, notice that the reason we multiply equations involving fractions by LCD's is to clear the denominators of those fractions. That means that if the multiplication by an appropriate LCD is performed correctly, the resulting equation should not involve any denominators!

Example 6 ► **Solving Linear Equations Involving Decimals**

Solve $0.07x - 0.03(15 - x) = 0.05(14)$.

Solution ► To solve this equation, it is convenient (although not necessary) to clear the decimals first. This is done by multiplying the given equation by 100.

Each **term** (product of numbers and variable expressions) needs to be multiplied by 100.

$$0.07x - 0.03(15-x) = 0.05(14) \qquad / \cdot 100$$

$$7x - 3(15-x) = 5(14)$$

$$7x - 45 + 3x = 70$$

$$10x = 70 + 45$$

$$x = \frac{115}{10} = \mathbf{11.5}$$

So the solution to the given equation is $x = \mathbf{11.5}$.

Note: *In general, if n is the highest number of decimal places to clear in an equation, we multiply it by 10^n.*

Attention: *To multiply a product AB by a number C, we multiply just one factor of this product, either A or B, but not both! For example,*

$$10 \cdot 0.3(0.5 - x) = (10 \cdot 0.3)(0.5 - x) = 3(0.5 - x) \quad ✓$$

or

$$10 \cdot 0.3(0.5 - x) = 0.3 \cdot [10(0.5 - x)] = 0.3(5 - 10x) \quad ✓$$

but

$$10 \cdot 0.3(0.5 - x) \neq (10 \cdot 0.3)[10(0.5 - x)] = 3(5 - 10x) \quad ✗$$

Summary of Solving a Linear Equation in One Variable

- **Clear fractions or decimals**. Eliminate fractions by multiplying each side by the least common denominator (LCD). Eliminate decimals by multiplying by a power of 10.
- **Clear brackets (**starting from the inner-most ones) by applying the distributive property of multiplication. **Simplify** each side of the equation by **combining like terms**, as needed.
- **Collect** and **combine variable terms** on one side and free terms on the other side of the equation. Use the addition property of equality to collect all variable terms on one side of the equation and all free terms (numbers) on the other side.
- **Isolate the variable** by dividing the equation by the linear coefficient (coefficient of the variable term).

L.1 Exercises

True or False? Justify your answer.

1. The equation $5x - 1 = 9$ is equivalent to $5x - 5 = 5$.

2. The equation $x + \sqrt{x} = -1 + \sqrt{x}$ is equivalent to $x = -1$.

3. The solution set to $12x = 0$ is $\emptyset$.

4. The equation $x - 0.3x = 0.97x$ is an identity.

5. To solve $-\frac{2}{3}x = \frac{3}{5}$, we could multiply each side by the reciprocal of $-\frac{2}{3}$.

6. If a and b are real numbers, then $ax + b = 0$ has a solution.

Decide whether each of the following is an ***equation to solve*** *or an* ***expression to simplify****.*

7. $3x + 2(x - 6) - 1$

8. $3x + 2(x - 6) = 1$

9. $-5x + 19 = 3x - 5$

10. $-5x + 19 - 3x + 5$

Determine whether or not the given equation is linear.

11. $4x + 2 = x - 3$

12. $12 = x^2 + x$

13. $x + \frac{1}{x} = 1$

14. $2 = 5$

15. $\sqrt{16} = x$

16. $\sqrt{x} = 9$

Determine whether the given value is a solution of the equation.

17. $2,\ 3x - 4 = 2$

18. $-2,\ \frac{1}{x} - \frac{1}{2} = -1$

19. $6,\ \sqrt{2x + 4} = -4$

20. $-4,\ (x - 1)^2 = 25$

Solve each equation. If applicable, tell whether the equation is an ***identity*** *or a* ***contradiction****.*

21. $6x - 5 = 0$

22. $-2x + 5 = 0$

23. $-3x + 6 = 12$

24. $5x - 3 = -13$

25. $3y - 5 = 4 + 12y$

26. $9y - 4 = 14 + 15y$

27. $2(2a - 3) - 7 = 4a - 13$

28. $3(4 - 2b) = 4 - (6b - 8)$

29. $-3t + 5 = 4 - 3t$

30. $5p - 3 = 11 + 4p + p$

31. $13 - 9(2n + 3) = 4(6n + 1) - 15n$

32. $5(5n - 7) + 40 = 2n - 3(8n + 5)$

33. $3[1 - (4x - 5)] = 5(6 - 2x)$

34. $-4(3x + 7) = 2[9 - (7x + 10)]$

35. $3[5 - 3(4 - t)] - 2 = 5[3(5t - 4) + 8] - 16$

36. $6[7 - 4(8 - t)] - 13 = -5[3(5t - 4) + 8]$

37. $\frac{2}{3}(9n - 6) - 5 = \frac{2}{5}(30n - 25) - 7n$

38. $\frac{1}{2}(18 - 6n) + 5n = 10 - \frac{1}{4}(16n + 20)$

39. $\frac{8x}{3} - \frac{5x}{4} = -17$

40. $\frac{7x}{2} - \frac{5x}{10} = 5$

41. $\frac{3x-1}{4} + \frac{x+3}{6} = 3$

42. $\frac{3x+2}{7} - \frac{x+4}{5} = 2$

43. $\frac{2}{3}\left(\frac{7}{8} + 4x\right) - \frac{5}{8} = \frac{3}{8}$

44. $\frac{3}{4}\left(3x - \frac{1}{2}\right) - \frac{2}{3} = \frac{1}{3}$

45. $x - 2.3 = 0.08x + 3.5$

46. $x + 1.6 = 0.02x - 3.6$

47. $0.05x + 0.03(5000 - x) = 0.04 \cdot 5000$

48. $0.02x + 0.04 \cdot 3000 = 0.03(x + 3000)$

L2 Formulas and Applications

In the previous section, we studied how to solve linear equations. Those skills are often helpful in problem solving. However, the process of solving an application problem has many components. One of them is the ability to construct a mathematical model of the problem. This is usually done by observing the relationship between the variable quantities in the problem and writing an equation that describes this relationship.

Definition 2.1 ► An equation that represents or models a relationship between two or more quantities is called a **formula**.

To model real situations, we often use well-known formulas, such as $R \cdot T = D$, or $a^2 + b^2 = c^2$. However, sometimes we need to construct our own models.

Data Modelling

Example 1 ► **Constructing a Formula to Model a Set of Data Following a Linear Pattern**

In Santa Barbara, CA, a passenger taking a taxicab for a d-mile-long ride pays the fare of F dollars as per the table below.

distance $\boldsymbol{d}$ (in miles)	1	2	3	4	5
fare $\boldsymbol{F}$ (in dollars)	5.50	8.50	11.50	14.50	17.50

a. Write a formula that calculates fare F, in dollars, when distance driven d, in miles, is known.
b. Find the fare for a 16-mile ride by this taxi.
c. How long was the ride of a passenger who paid the fare of \$29.50?

Solution ► **a.** Observe that the increase in fare when driving each additional mile after the first is constantly \$3.00. This is because

$$17.5 - 14.5 = 14.5 - 11.5 = 11.5 - 8.5 = 8.5 - 5.5 = 3$$

If d represents the number of miles driven, then the number of miles after the first can be represented by $(d - 1)$. The fare for driving n miles is the cost of driving the first mile plus the cost of driving the additional miles, after the first one. So, we can write

$$\boldsymbol{fare\ F} = \left(\begin{matrix} cost\ of\ the \\ \boldsymbol{first\ mile} \end{matrix}\right) + \left(\begin{matrix} \boldsymbol{cost\ increase} \\ per\ mile \end{matrix}\right) \cdot \left(\begin{matrix} \boldsymbol{number\ of} \\ additional\ \boldsymbol{miles} \end{matrix}\right)$$

or symbolically,

$$F = 5.5 + 3(d - 1)$$

The above equation can be simplified to

$$F = 5.5 + 3d - 3 = 3d + 2.5.$$

Therefore, $\boldsymbol{F = 3d + 2.5}$ is the formula that models the given data.

b. Since the number of driven miles is $d = 16$, we evaluate

$$F = 3 \cdot 16 + 2.5 = 50.5$$

Therefore, the fare for a 16-mile ride is **$50.50**.

c. This time, we are given the fare $F = 29.50$, and we are looking for the corresponding number of miles d. To find d, we substitute 31.7 for F in our formula $F = 3d + 2.5$ and then solve the resulting equation for d. We obtain

$$\begin{aligned} 29.5 &= 3d + 2.5 && /-2.5 \\ 27 &= 3d && /\div 3 \\ d &= 9 \end{aligned}$$

So, the ride was **9 miles** long.

Notice that in the solution to *Example 1c*, we could first solve the equation $F = 3d + 2.5$ for d:

$$\begin{aligned} F &= 3d + 2.5 && /-2.5 \\ F - 2.5 &= 3d && /\div 3 \end{aligned}$$

$$\boldsymbol{d = \frac{F - 2.5}{3}},$$

and then use the resulting formula to evaluate d at $F = 29.50$.

$$d = \frac{29.5 - 2.5}{3} = \frac{27}{3} = 9.$$

The advantage of solving the formula $F = 3d + 2.5$ for the variable d first is such that the resulting formula $d = \frac{F-2.5}{3}$ makes evaluations of d for various values of F easier. For example, to find the number of miles d driven for the fare of $35.5, we could evaluate directly using $d = \frac{35.5-2.5}{3} = \frac{33}{3} = 11$ rather than solving the equation $35.5 = 3d + 2.5$ again.

Solving Formulas for a Variable

If a formula is going to be used for repeated evaluation of a specific variable, it is convenient to rearrange this formula in such a way that the desired variable is **isolated on one side** of the equation and it does not appear on the other side. Such a formula may also be called a function.

Definition 2.2 ► A **function** is a rule for determining the value of one variable from the values of one or more other variables, in a unique way. We say that the first variable **is a function** of the other variable(s).

For example, consider the uniform motion relation between distance, rate, and time.

To evaluate rate when distance and time is given, we use the formula

$$\boldsymbol{R = \frac{D}{T}}.$$

This formula describes **rate as a function of distance and time**, as rate can be uniquely calculated for any possible input of distance and time.
To evaluate time when distance and rate is given, we use the formula

$$\boldsymbol{T = \frac{D}{R}}.$$

This formula describes **time as a function of distance and rate**, as time can be uniquely calculated for any possible input of distance and rate.
Finally, to evaluate distance when rate and time is given, we use the formula

$$\boldsymbol{D = R \cdot T}.$$

Here, the **distance is** presented as **a function of rate and time**, as it can be uniquely calculated for any possible input of rate and time.

To **solve a formula for a given variable** means to rearrange the formula so that the desired **variable equals to an expression that contains only <u>other</u> variables** but not the one that we solve for. This can be done the same way as when solving equations.

Here are some hints and guidelines to keep in mind when solving formulas:

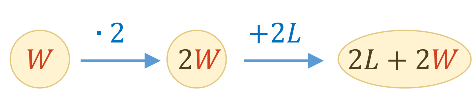
- **Highlight** the variable of interest and solve the equation as if the other variables were just numbers (think of easy numbers), without actually performing the given operations.

  Example: To solve $mx + b = c$ for m,

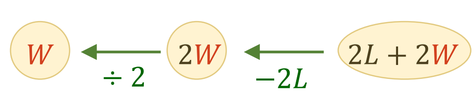
we pretend to solve, for example:		so we write:	
$m \cdot 2 + 3 = 1$	$/-3$	$mx + b = c$	$/-b$
$m \cdot 2 = 1 - 3$	$/\div 2$	$mx = c - b$	$/\div x$
$m = \frac{1-3}{2}$		$\boldsymbol{m = \frac{c-b}{x}}$	

- **Reverse** (undo) **operations** to isolate the desired variable.

  Example: To solve $2L + 2W = P$ for W, first, observe the operations applied to W:



  Then, reverse these operations, starting from the last one first.



  So, we solve the formula as follows:

  $$2L + 2W = P \qquad /-2L$$
  $$2W = P - 2L \qquad /\div 2$$

$$W = \frac{P - 2L}{2}$$

Notice that the last equation can also be written in the equivalent form $W = \frac{P}{2} - L$.

- **Keep the** desired **variable in the numerator**.

Example: To solve $R = \frac{D}{T}$ for T, we could take the reciprocal of each side of the equation to keep T in the numerator,

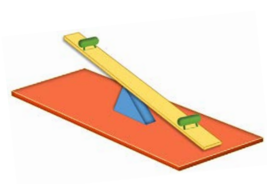
$$\frac{T}{D} = \frac{1}{R},$$

and then multiply by D to "undo" the division. Therefore, $T = \frac{D}{R}$.

Observation: *Another way of solving $R = \frac{D}{T}$ for T is by multiplying both sides by T and dividing by R.*

$$\frac{T}{R} \cdot R = \frac{D}{T} \cdot \frac{T}{R}$$

This would also result in $T = \frac{D}{R}$. Observe, that no matter how we solve this formula for T, the result differs from the original formula by interchanging (swapping) the variables T and R.



Note: *When working only with multiplication and division, by applying inverse operations, any factor of the numerator can be moved to the other side into the denominator, and likewise, any factor of the denominator can be moved to the other side into the numerator. Sometimes it helps to think of this movement of variables as the movement of a "teeter-totter".*

For example, the formula $\frac{bh}{2} = A$ can be solved for h by dividing by b and multiplying by 2. So, we can write directly $h = \frac{2A}{b}$.

2 was down so now goes up
and
b was up so now goes down

- **Keep the** desired **variable in one place.**

Example: To solve $A = P + Prt$ for P, we can factor P out,

$$A = P(1 + rt)$$

and then divide by the bracket. Thus,

$$P = \frac{A}{1 + rt}.$$

Example 2 ► **Solving Formulas for a Variable**

Solve each formula for the indicated variable.

a. $a_n = a_1 + (n - 1)d$ for n

b. $\frac{PV}{T} = \frac{P_0 V_0}{T_0}$ for T

Solution ► **a.** To solve $a_n = a_1 + (n-1)d$ for n we use the reverse operations strategy, starting with reversing the addition, then multiplication, and finally the subtraction.

$$a_n = a_1 + (n-1)d \qquad /-a_1$$

$$a_n - a_1 = (n-1)d \qquad /\div d$$

$$\frac{a_n - a_1}{d} = n - 1 \qquad /+1$$

$$\boldsymbol{n = \frac{a_n - a_1}{d} + 1}$$

The last equation can also be written in the equivalent form $\boldsymbol{n = \frac{a_n - a_1 + d}{d}}$.

b. To solve $\frac{PV}{T} = \frac{P_0V_0}{T_0}$ for T, first, we can take the reciprocal of each side of the equation to keep T in the numerator,

$$\frac{T}{PV} = \frac{T_0}{P_0V_0},$$

and then multiply by PV to "undo" the division. So,

$$\boldsymbol{T = \frac{T_0PV}{P_0V_0}}.$$

Attention: *Taking reciprocal of each side of an equation is a good strategy only if both sides are in the form of a* ***single fraction****. For example, to use the reciprocal property when solving* $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$ *for a, first, we perform the addition to create a single fraction,* $\frac{1}{a} = \frac{c+b}{bc}$*. Then, taking reciprocals of both sides will give us an instant result of* $a = \frac{bc}{c+b}$*.*

Warning! *The reciprocal of* $\frac{1}{b} + \frac{1}{c}$ *is* ***not*** *equal to* $b + c$*.*

Example 3 ► **Using Formulas in Application Problems**

To determine a healthy weight for a person's height, we can use the body mass index I given by the formula $\boldsymbol{I} = \frac{\boldsymbol{W}}{\boldsymbol{H^2}}$, where W represents weight, in kilograms, and H represents height, in meters. Weight is considered healthy when the index is in the range 18.5–24.9.

a. Adam is 182 cm tall and weighs 89 kg. What is his body index?

b. Barb has a body mass index of 24.5 and a height of 1.7 meters. What is her weight?

Solution ► **a.** Since the formula calls for height H is in meters, first, we convert 182 cm to 1.82 meters, and then we substitute $H = 1.82$ and $W = 89$ into the formula. So,

$$\boldsymbol{I} = \frac{89}{1.82^2} \approx \mathbf{26.9}$$

When rounded to one decimal place. Thus Adam is overweighted.

b. To find Barb's weight, we may want to solve the given formula for W first, and then plug in the given data. So, Barb's weight is

$$W = I H^2 = 24.5 \cdot 1.7^2 \approx \mathbf{70.8\ kg}$$

Direct and Joined Variation

When two quantities vary proportionally, we say that there is a **direct variation** between them. For example, such a situation can be observed in the relation between time T and distance D covered by a car moving at a constant speed R. In particular, if $R = 60$ kph, we have

$$D = 60T$$

This relation tells us that the distance is 60 times larger than the time. Observe though that when the time doubles, the distance doubles as well. When the time triples, the distance also triples. So, the distance increases proportionally to the increase of the time. Such a **linear relation** between the two quantities is called a **direct variation**.

Definition 2.3 ► Two quantities, $\boldsymbol{x}$ and $\boldsymbol{y}$, are **directly proportional** to each other (there is a **direct variation** between them) iff there is a real constant $\boldsymbol{k \neq 0}$, such that

$$\boldsymbol{y = kx}.$$

We say that $\boldsymbol{y}$ **varies directly** as $\boldsymbol{x}$ with the **variation constant** $\boldsymbol{k}$.
(or equivalently: $\boldsymbol{y}$ is **directly proportional to** $\boldsymbol{x}$ with the **proportionality constant** $\boldsymbol{k}$.)

Example 4 ► **Solving Direct Variation Problems**

In scale drawing, the actual distance between two objects is directly proportional to the distance between the drawn objects. Suppose a kitchen room that is 4.6 meters long appears on a drawing as 2 cm long.

a. Find the direct variation equation that relates the actual distance D and the corresponding distance S on the drawing.

b. Find the actual dimensions of a 2.6 cm by 3.7 cm room on this drawing?

Solution ►

a. To find the direct variation equation that relates D and S, we need to find the variation constant k first. This can be done by substituting $D = 4.6$ and $S = 2$ into the equation $D = kS$. So, we obtain

$$4.6 = k \cdot 2 \qquad / \div 2$$

$$k = \frac{4.6}{2} = 2.3.$$

Therefore, the direct variation equation is $\boldsymbol{D = 2.3S}$.

b. To find the actual dimensions of a room that is drawn as 2.6 cm by 3.7 cm rectangle, we substitute these S-values into the above equation. This gives us

$$D = 2.3 \cdot 2.6 = 5.98 \quad \text{and} \quad D = 2.3 \cdot 3.7 = 8.51.$$

So, the actual dimensions of the room are $\mathbf{5.98}$ by $\mathbf{8.51}$ meters.

Sometimes a quantity varies directly as the n-th power of another quantity. For example, the formula $A = \pi r^2$ describes the direct variation between the area A of a circle and the square of the radius r of this circle. Here, the proportionality constant is π, while $n = 2$.

Extension: Generally, the fact that $\boldsymbol{y}$ **varies directly** as the $\boldsymbol{n}$**-th power** of $\boldsymbol{x}$ tells us that

$$\boldsymbol{y = kx^n},$$

for some nonzero constant $\boldsymbol{k}$.

Example 5 ► **Solving a Direct Variation Problem Involving the Square of a Variable**

Disregarding air resistance, the distance a body falls from rest is directly proportional to the square of the elapsed time. If a skydiver falls 24 meters in the first 2 seconds, how far will he fall in 5 seconds?

Solution ► Let d represents the distance the skydiver falls and t the time elapsed during this fall. Since d varies directly as t^2, we set the equation

$$d = kt^2$$

After substituting the data given in the problem, we find the value of k:

$$24 = k \cdot 2^2 \qquad / \div 4$$

$$k = \frac{24}{4} = 6$$

So, the direct variation equation is $d = 6t^2$. Hence, during 5 seconds the skydiver falls the distance $d = 6 \cdot 5^2 = 6 \cdot 25 = \mathbf{150}$ **meters**.

If one variable varies directly as the product of several other variables (possibly raised to some powers), we say that the first variable varies **jointly** as the other variables. For example, a joint variation can be observed in the formula for the area of a triangle, $A = \frac{1}{2}bh$, where the area A varies directly as the base b and directly as the height h of this triangle. We say that A varies jointly as b and h.

Definition 2.4 ► Variable $\boldsymbol{z}$ is **jointly proportional** to a set of variables (possibly raised to some powers) iff $\boldsymbol{z}$ is **directly proportional** to each of these variables (or equivalently: $\boldsymbol{z}$ is **directly proportional** to the product of these variables including their powers.)

Example:
z varies jointly as $\boldsymbol{x}$ and a **cube** of $\boldsymbol{y}$ iff $\boldsymbol{y = kxy^3}$, for some real constant $\boldsymbol{k \neq 0}$.

Example 6 ► **Solving Joint Variation Problems**

a. Kinetic energy is jointly proportional to the mass and the square of the velocity. Suppose a mass of 5 kilograms moving at a velocity of 4 meters per second has a kinetic energy of 40 joules.

b. Find the kinetic energy of a 3-kilogram ball moving at 6 meters per second.

c. Find the mass of an object that has 50 joules of kinetic energy when moving at 5 meters per second.

Solution ► **a.** Let E, m, and v represent respectively kinetic energy, mass, and velocity. Since E is jointly proportional to m and v^2, we set the equation

$$E = kmv^2.$$

Substituting the data given in the problem, we have

$$40 = k \cdot 5 \cdot 4^2, \qquad / \div 80$$

which gives us

$$k = \frac{40}{80} = \frac{1}{2}.$$

So, the joint variation equation is $E = \frac{1}{2}mv^2$. Hence, the kinetic energy of the 3-kilogram ball moving at 6 meters per second is $E = \frac{1}{2} \cdot 3 \cdot 6^2 = 3 \cdot 18 =$ **54 joules**.

b. First, we may want to solve the equation $E = \frac{1}{2}mv^2$ for m and then evaluate it using substitutions $E = 50$, and $v = 5$. So, we have

$$E = \frac{1}{2}mv^2 \qquad / \cdot 2$$

$$2E = mv^2 \qquad / \div v^2$$

reduce by 25

$$m = \frac{2E}{v^2} = \frac{2 \cdot 50}{5^2} = 4$$

The mass of an object with the required parameters is **4 kilograms**.

L.2 Exercises

1. *When solving a formula for a particular variable, the answer can often be stated in various forms. Which of the following formulas are correct answers when solving* $A = \frac{a+b}{2}h$ *for* b*?*

   A. $b = \frac{2A-a}{h}$ **B.** $b = \frac{2A}{h} - a$ **C.** $b = \frac{2A-ah}{h}$ **D.** $b = \frac{A-ah}{\frac{1}{2}h}$

2. *Which of the following formulas are* ***not*** *correct answers when solving* $A = P + Prt$ *for* P*? Jastify your answer.*

   A. $P = \frac{A}{rt}$ **B.** $P = \frac{A}{1+rt}$ **C.** $P = A - Prt$ **D.** $P = \frac{A-P}{rt}$

Solve each formula for the specified variable.

3. $I = Prt$ for r (*simple interest*)

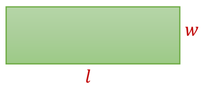
4. $C = 2\pi r$ for r (*circumference of a circle*)

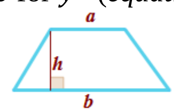
5. $E = mc^2$ for m (*mass-energy relation*)

6. $F = \frac{mv^2}{r}$ for m (*force in a circular motion*)

7. $A = \frac{(a+b)}{2}$ for b (*average*)

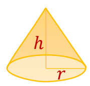
8. $Ax + By = c$ for y (*equation of a line*)

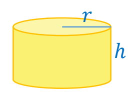




9. $P = 2l + 2w$ for l (*perimeter of a rectangle*)

10. $A = \frac{h}{2}(a + b)$ for a (*area of a trapezoid*)





11. $S = \pi rs + \pi r^2$ for π (*surface area of a cone*)

12. $S = 2\pi rh + 2\pi r^2$ for h (*surface area of a cylinder*)

13. $F = \frac{9}{5}C + 32$ for C (*Celsius to Fahrenheit*)

14. $C = \frac{9}{5}(F - 32)$ for F (*Fahrenheit to Celsius*)

15. $Q = \frac{p-q}{2}$ for p

16. $Q = \frac{p-q}{2}$ for q

17. $T = B + Bqt$ for q

18. $d = R - Rst$ for t

19. $d = R - Rst$ for R

20. $T = B + Bqt$ for B

Solve each problem.

21. On average, a passenger who drives n-kilometers in a taxicab in Abbotsford, BC, is charged C dollars as per the table below.

distance $\boldsymbol{n}$ (in kilometers)	1	2	3	4	5
cost $\boldsymbol{C}$ (in dollars)	5.10	7.00	8.90	10.80	12.70

a. Write a formula that calculates the cost C, in dollars, of driving a distance of n kilometers.
b. Find the cost for a 10-kilometer ride by this taxi.
c. If a passenger paid \$31.70, how far did he drive by this taxi?

22. On average, a passenger who drives n-kilometers in a taxicab in Vancouver, BC, is charged C dollars as per the table below.

distance $\boldsymbol{n}$ (in kilometers)	1	2	3	4	5
cost $\boldsymbol{C}$ (in dollars)	5.35	7.20	9.05	10.90	12.75

a. Write a formula that calculates the cost C, in dollars, of driving a distance of n kilometers.
b. Find the cost for a 20-kilometer ride by this taxi.
c. If a passenger paid \$22.00, how far did he drive by this taxi?

23. 

Assume that the amount of a medicine dosage for a child can be determined by the formula

$$c = \frac{ad}{a + 12},$$

where a represents the child's age, in years, and d represents the usual adult dosage, in milliliters.

a. If the adult dosage of a certain medication is 25 ml, what is the corresponding dosage for a three-year-old child?
b. Solve the formula for d.
c. Find the corresponding adult dosage, if a six-year-old child uses 5 ml of a certain medication.

24. The number of "full-time-equivalent" students, F, is often determined by the formula

$$F = \frac{n}{15},$$

where n represents the total number of credits taken by all students in a semester.

a. Suppose that in a particular institution students register for a total of 39,315 credits in one semester. What is the number of full-time-equivalent students in this institution?
b. Solve the formula for n.
c. Find the total number of credits students enroll in a semester if the number of full-time-equivalent students in this semester is 3254.

25. Suppose a cyclist can burn 530 calories in a 45-minute cycling session.

a. Write a formula that determines the number of calories C burned during two 45-minute sessions of cycling per day for d days.
b. According to this formula, how many calories would the cyclist burn in a week of cycling two 45-minute sessions per day?

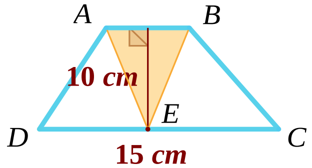
26. *Refer to information given in problem 25.*
a. On average, a person loses 1 kilogram for every 7000 calories burned. Write a formula that calculates the number of kilograms K lost in d days of cycling two 45-minute sessions per day.
b. How many kilograms could the cyclist lose in 30 days? *Round the answer to the nearest half of a kilogram.*

27. Express the width L of a rectangle in terms of its perimeter P and length W. *Here "in terms of P and W" means using an expression that involves only variables P and W.*

28. Express the area A of a circle in terms of its diameter d. *Here "in terms of d" means using an expression that involves only variable d.*

29. a. Solve the formula $I = Prt$ for t.
b. Using the formula from (**a**) determine how long it will take a deposit of \$125 to make the interest of \$15 when invested at 4% simple interest.

30. *Refer to information given in the accompanying figure.*
Find the area of the trapezoid $ABCD$ if its area is three times as large as the area of the shaded triangle ABE.



31. The number N of plastic bottles used each year is directly proportional to the number of people P using them.
a. Assuming that 150 people use 18,000 bottles in one year, find the variation constant and state the direct variation equation.
b. How many bottles are used each year in Vancouver, BC, which has a population of 631,490?

32. Under certain conditions, the volume V of a fixed amount of gas varies directly as its temperature T, in Kelvin degrees.
a. Assuming the gas in a hot-air balloon occupies 120 m^3 at 200 K, find the direct variation equation.
b. If the pressure of the gas remains constant, what would the volume of the gas be at 250 Kelvin degrees?

33. The recommended daily intake of fat varies directly as the number of calories consumed per day. Alice is on a 1200 diet and her healthy intake of fat is about 40 grams per day. Christopher needs about 2000 calories per day. To the nearest gram, what is his recommended daily intake of fat?

34. The distance covered by a falling object varies directly as the square of the elapsed time of the fall. A person flying in a hot-air balloon accidentally dropped a camera. Suppose the camera fell 16 meters during the first 2 seconds of the fall. If the camera hits the ground 5 seconds after it was dropped, how high was the balloon?

35. The air distance between Vancouver and Warsaw is 8,225 kilometers. The two cities are 45.2 centimeters apart on a desk globe. The air distance between Paris and Warsaw is 1367 kilometers. To the nearest millimeter, how far is Paris from Warsaw on this globe?

36. The stopping distance for a car varies directly as the square of its speed. If a car travelling 50 kilometers per hour requires 40 meters to stop, what would be the stopping distance for a car travelling 80 kilometers per hour?

37. The simple interest I varies jointly as the interest rate r and the principal P. Mother and daughter invested some money in simple interest accounts for the same period of time t. The daughter earns \$130 interest on the investment of \$2000 at 3.25%. What was the amount that the mother invested at 3.75% if she earns \$225 interest?

38. The lateral surface area (*surface area excluding the bases*) of a cylinder is jointly proportional to the height and radius of the cylinder. If a cylinder with radius 5 cm and height 8 cm has a lateral surface area of approximately 250 cm^2, what is the approximate lateral surface area of a can with a diameter of 6 cm and height of 12 cm?

39. The area of a triangle is jointly proportional to the height and the base of the triangle. If the base is increased by 50% and the height is decreased by 50%, how would the area of the triangle change?

40. The volume V of wood in a tree is directly proportional to the height h and the square of the **girth** (*circumference around the trunk*), g. Suppose the volume of a 20 meters tall tree with the girth of 1 meter is 64 cubic meters. To the nearest meter, find the height of a tree with a volume of 250 cubic meters and girth of 1.8 meters?

41. The number of barrels of oil used by a ship travelling at a constant speed is jointly proportional to the distance traveled and the square of the speed. If the ship uses 180 barrels of oil when travelling 200 miles at 40 miles per hour, find the number of barrels of oil needed for a ship that travels 300 miles at 25 miles per hour. *Round the answer to the nearest barrel.*

L3 Applications of Linear Equations

In this section, we study some strategies for solving problems with the use of linear equations, or well-known formulas. While there are many approaches to problem-solving, the following steps prove to be helpful.

Five Steps for Problem Solving
1. **Familiarize** yourself with the problem.
2. **Translate** the problem to a symbolic representation (usually an **equation** or an **inequality**).
3. **Solve** the equation(s) or the inequality(s).
4. **Check** if the answer makes sense in the original problem.
5. **State the answer** to the original problem clearly.

Here are some hints of how to **familiarize** yourself with the problem:

- **Read** the problem carefully a few times. In the first reading focus on the general setting of the problem. See if you can identify this problem as one of a motion, investment, geometry, age, mixture or solution, work, or a number problem, and draw from your experiences with these types of problems. During the second reading, focus on the specific information given in the problem, skipping unnecessary words, if possible.
- **List the information** given, including **units**, and check **what the problem asks for**.
- If applicable, **make a diagram** and label it with the given information.
- **Introduce a variable**(s) for the unknown quantity(ies). Make sure that the variable(s) is/are clearly defined (including units) by writing a "let" statement or labeling appropriate part(s) of the diagram. Choose descriptive letters for the variable(s). For example, let l be the length in centimeters, let t be the time in hours, etc.
- Express **other unknown values** in terms of the already introduced variable(s).
- Write applicable **formulas**.
- **Organize your data** in a meaningful way, for example by filling in a table associated with the applicable formula, inserting the data into an appropriate diagram, or listing them with respect to an observed pattern or rule.
- **Guess** a possible answer and check your guess. Observe the way in which the guess is checked. This may help you translate the problem into an equation.

Translation of English Phrases or Sentences to Expressions or Equations

One of the important phases of problem-solving is **translating** English words into a **symbolic representation**.

Here are the most commonly used **key words** suggesting a particular operation:

ADDITION (+)	SUBTRACTION (−)	MULTIPLICATION (·)	DIVISION (÷)
sum	**difference**	**product**	**quotient**
plus	minus	multiply	divide
add	**subtract from**	times	ratio
total	**less than**	**of**	**out of**
more than	less	half of	per
increase by	decrease by	half as much as	shared
together	diminished	twice, triple	cut into
perimeter	shorter	area	

Example 1 ► **Translating English Words to an Algebraic Expression or Equation**

Translate the word description into an algebraic expression or equation.

a. The sum of half of a number and two
b. The square of a difference of two numbers
c. Triple a number, increased by five, is one less than twice the number.
d. The quotient of a number and seven diminished by the number
e. The quotient of a number and seven, diminished by the number
f. The perimeter of a rectangle is four less than its area.
g. In a package of 12 eggs, the ratio of white to brown eggs is one out of three.
h. Five percent of the area of a triangle whose base is one unit shorter than the height

Solution ► **a.** Let x represents "*a number*". Then

*The **sum of** **half of** a number **and** two* translates to $\frac{1}{2}x + 2$

Notice that the word "***sum***" indicates addition sign at the position of the word "***and***". Since addition is a binary operation (needs two inputs), we reserve space for "*half of a number*" on one side and "*two*" on the other side of the addition sign.

b. Suppose x and y are the "*two numbers*". Then

The square of *a **difference of** two numbers* translates to $(x - y)^2$

Notice that we are squaring everything that comes after "*the square of*".

c. Let x represents "*a number*". Then

Triple *a number, increased by five, **is** one **less than** **twice** the number.*

translates to the equation: $3x + 5 = 2x - 1$

This time, we translated a sentence that results in an equation rather than expression. Notice that the "equal" sign is used in place of the word "is". Also, remember that phrases "less than" or "subtracted from" work "backwards". For example, *A* **less than** *B* or *A* **subtracted from** *B* translates to ***B − A***. However, the word "less" is used in the usual direction, from left to right. For example, *A* **less** *B* translates to ***A − B***.

d. Let x represent "*a number*". Then

The quotient of *a number **and** seven **diminished** by <u>the</u> number* translates to $\frac{x}{7-x}$

Notice that "**<u>the</u>** number" refers to the same number x.

e. Let x represent "*a number*". Then

The quotient of *a number **and** seven**,** **diminished** by <u>the</u> number* translates to $\frac{x}{7} - x$

Here, the **comma** indicates the end of the "**quotient section**". So, we diminish the quotient rather than diminishing the seven, as in *Example 1d*.

f. Let l and w represent the length and the width of a rectangle. Then

*The **perimeter** of a rectangle **is** four **less than** its **area**.*

translates to the equation: $2l + 2w = lw - 4$

Here, we use a formula for the perimeter ($2l + 2w$) and for the area (lw) of a rectangle.

g. Let w represent the number of white eggs in a package of 12 eggs. Then $(12 - w)$ represents the number of brown eggs in this package. Therefore,

*In a package of 12 eggs, **the ratio of the number of white eggs to the number of brown eggs is the same as two to three**.*

translates to the equation: $\frac{w}{12-w} = \frac{2}{3}$

Here, we expressed the unknown number of brown eggs $(12 - w)$ in terms of the number w of white eggs. Also, notice that the order of listing terms in a proportion is essential. Here, the first terms of the two ratios are written in the numerators (in blue) and the second terms (in brown) are written in the denominators.

h. Let h represent the height of a triangle. Since the base is *one unit shorter than the height*, we express it as $(h - 1)$. Using the formula $\frac{1}{2}bh$ for the area of a triangle, we translate

five percent of the area of a triangle** whose **base is one unit shorter than the height

to the expression: $0.05 \cdot \frac{1}{2}(h - 1)h$

Here, we convert *five percent* to the number 0.05, as *per-cent* means *per hundred*, which tells us to divide 5 by a hundred.
Also, observe that the above word description is not a sentence, even though it contains the word "*is*". Therefore, the resulting symbolic form is an expression, not an equation. The word "*is*" relates the base and the height, which in turn allows us to substitute $(h - 1)$ in place of b, and obtain an expression in one variable.

So far, we provided some hints of how to familiarize ourself with a problem, we worked through some examples of how to translate word descriptions to a symbolic form, and we reviewed the process of solving linear equations (*see Section L1*). In the rest of this section, we will show various methods of solving commonly occurring types of problems, using representative examples.

Number Relation Problems

In number relation type of problems, we look for relations between quantities. Typically, we introduce a variable for one quantity and express the other quantities in terms of this variable following the relations given in the problem.

Example 2 ► **Solving a Number Relation Problem with Three Numbers**

The sum of three numbers is thirty-four. The second number is twice the first number, and the third number is one less than the second number. Find the three numbers.

Solution ► There are three unknown numbers such that their sum is thirty-four. This information allows us to write the equation

$$1^{st} \text{ number} + 2^{nd} \text{ number} + 3^{rd} \text{ number} = 34.$$

To solve such an equation, we wish to express all three unknown numbers in terms of one variable. Since the second number refers to the first, and the third number refers to the second, which in turn refers to the first, it is convenient to introduce a variable for the first number.

So, let n represent the first number.

The second number is twice the first, so $2n$ represents the second number.

The third number is one less than the second number, so $2n - 1$ represents the third number.

Therefore, our equation turns out to be

$$\begin{aligned} n + 2n + (2n - 1) &= 34 \\ 5n - 1 &= 34 \quad\quad / +1 \\ 5n &= 35 \quad\quad / \div 5 \\ n &= 7. \end{aligned}$$

Hence, the first number is **7**, the second number is $2n = 2 \cdot 7 = \mathbf{14}$, and the third number is $2n - 1 = 14 - 1 = \mathbf{13}$.

Consecutive Numbers Problems

Since **consecutive numbers** differ by one, we can represent them as $\boldsymbol{n, n+1, n+2}$, and so on.

Consecutive even or **consecutive odd** numbers differ by two, so both types of numbers can be represented by $\boldsymbol{n, n+2, n+4}$, and so on.

Notice that if the first number n is even, then $n + 2, n + 4, \ldots$ are also even; however, if the first number n is odd then $n + 2, n + 4, \ldots$ are also odd.

Example 3 ► **Solving a Consecutive Odd Integers Problem**

Find three consecutive odd integers such that three times the middle integer is five less than double the sum of the first and the third integer.

Solution ► Let the three consecutive odd numbers be called $n, n + 2$, and $n + 4$. We translate *three times the middle integer is five less than double the sum of the first and the third integer* into the equation

$$3(n + 2) = 2[n + (n + 4)] - 5$$

which gives

$$3n + 6 = 4n + 3 \qquad / \; -3, -3n$$

$$n = 3$$

Hence, the first number is **3**, the second number is $n + 2 = \mathbf{5}$, and the third number is $n + 4 = \mathbf{7}$.

Percent Problems

Rules to remember when solving percent problems:

$$1 = 100\% \quad \text{and} \quad \frac{\textit{is a part}}{\textit{of a whole}} = \frac{\%}{100}$$

Also, remember that

$$\textit{percent increase}(\textit{decrease}) = \frac{\textit{last} - \textit{first}}{\textit{first}} \cdot 100\%$$

Example 4 ► **Finding the Amount of Tax**

Kristin bought a new fridge for \$1712.48, including 12% of PST and GST tax. How much tax did she pay?

Solution ► Suppose the fridge costs p dollars. Then the tax paid for this fridge is 12% of p dollars, which can be represented by the expression $0.12p$. Since the total cost of the fridge including tax is \$1712.48, we set up the equation

$$p + 0.12p = 1712.48$$

which gives us

$$1.12p = 1712.48 \qquad / \; \div 1.12$$

$$p = 1529$$

The question calls for the amount of tax, so we calculate $0.12p = 0.12 \cdot 1529 = 183.48$.

Kristin paid \$183.48 of tax for the fridge.

Example 5 ► **Solving a Percent Increase Problem**

Susan got her hourly salary raised from \$11.50 per hour to \$12.75 per hour. To the nearest tenths of a percent, what was the percent increase in her hourly wage?

Solution ► We calculate the percent increase by following the rule $\frac{\textit{last}-\textit{first}}{\textit{first}} \cdot 100\%$.

So, Susan's hourly wage was increased by $\frac{12.75-11.50}{11.50} \cdot 100\% \approx \mathbf{10.9}\%$.

Investment Problems

When working with investment problems we often use the simple interest formula $\boldsymbol{I = Prt}$, where I represents the amount of interest, P represents the principal (amount of money invested), r represents the interest rate, and t stands for the time in years.

Also, it is helpful to organize data in a diagram like this:

total amount

account ***I*** account ***II***

$$\textbf{\textit{interest}}\ \textit{from account}\ \boldsymbol{I} + \textbf{\textit{interest}}\ \textit{from account}\ \boldsymbol{II} = \textbf{\textit{total interest}}$$

or

$$\textbf{\textit{interest}}\ \textit{from account}\ \boldsymbol{I} = \textbf{\textit{interest}}\ \textit{from account}\ \boldsymbol{II}$$

Example 6 ► **Solving an Investment Problem**

A student took out two student loans for a total of \$6000. One loan is at 5% annual interest and the other at 4.5% annual interest. If the total interest paid in a year is \$281, find the amount of each loan.

Solution ► To solve this problem in one equation, we would like to introduce only one variable. Suppose $\boldsymbol{x}$ is the amount of the first loan. Then the amount of the second loan is the remaining portion of the \$6000. So, it is $(6000 - \boldsymbol{x})$.

Using the simple interest formula $I = Prt$, for $t = 1$, we calculate the interest obtained from the 5% to be $0.05 \cdot \boldsymbol{x}$ and from the 4.5% account to be $0.045(6000 - \boldsymbol{x})$. Since the total interest equals to \$281, we set the equation as indicated in the diagram below.

\$6000

$\boldsymbol{x}$ at 5% $(6000 - \boldsymbol{x})$ at 4.5%

$$0.05 \cdot \boldsymbol{x} + 0.045(6000 - \boldsymbol{x}) = 281$$

For easier calculations, we may want to clear decimals by multiplying this equation by 1000.

This gives us

$$50x + 45(6000 - x) = 281000$$
$$50x + 270000 - 45x = 281000 \quad / -270000$$
$$5x = 11000 \quad / \div 5$$

and finally

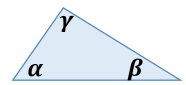
$$x = \$2200$$

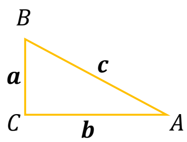
Thus, the first loan is **\$2200** and the second loan is $6000 - x = 6000 - 2200 = \$\mathbf{3800}$.

Geometry Problems

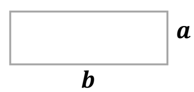
In geometry problems, we often use well-known formulas or facts that pertain to geometric figures. Here is a list of facts and formulas that are handy to know when solving various problems.

- The **sum of angles** in a triangle equals **180°**.



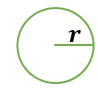


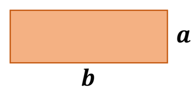
- The lengths of sides in a right-angle triangle ABC satisfy the **Pythagorean equation** $\boldsymbol{a^2 + b^2 = c^2}$, where c is the hypotenuse of the triangle.



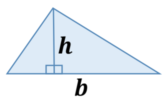
- The **perimeter of a rectangle** with sides a and b is given by the formula $\mathbf{2a + 2b}$.

- The **circumference** of a circle with radius r is given by the formula $\mathbf{2\pi r}$.

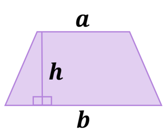




- The **area of** a **rectangle** or a **parallelogram** with base b and height h is given by the formula $\boldsymbol{bh}$.

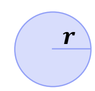


- The **area of** a **triangle** with base b and height h is given by the formula $\boldsymbol{\frac{1}{2}bh}$.



- The **area of** a **trapezoid** with bases a and b, and height h is given by the formula $\boldsymbol{\frac{1}{2}(a+b)h}$.

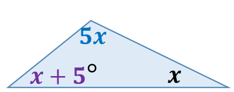
- The **area of** a **circle** with radius r is given by the formula πr^2.



Example 7 ► **Finding the Measure of Angles in a Triangle**

A cross section of a roof has a shape of a triangle. The largest angle of this triangle is five times as large as the smallest angle. The remaining angle is 5° greater than the smallest angle. Find the measure of each angle.

Solution ► Observe that the size of the largest and the remaining angle is compared to the size of the smallest angle. Therefore, it is convenient to introduce a variable, $\boldsymbol{x}$, for the measure of the smallest angle. Then, the expression for the measure of the largest angle, which is *five times as large as the smallest one*, is $\mathbf{5x}$ and the expression for the measure of the remaining angle, which is *5° greater than the smallest onet*, is $\boldsymbol{x+5°}$. To visualize the situation, it might be helpful to draw a triangle and label the three angles.



Since the sum of angles in any triangle is equal to 180°, we set up the equation

$$x + 5x + x + 5° = 180°$$

This gives us

$$7x + 5° = 180° \qquad / -5°$$

$$7x = 175° \qquad / \div 5$$

$$x = 25°$$

So, the measure of the smallest angle is $\mathbf{25}°$,
the measure of the largest angle is $5x = 5 \cdot 25° = \mathbf{125}°$, and
the measure of the remaining angle is $x + 5° = 25° + 5° = \mathbf{30}°$.

Total Value Problems

When solving total value types of problems, it is helpful to organize the data in a table that compares the number of items and the value of these items. For example:

	item ***A***	item ***B***	total
number of items			
value of items			

Example 8 ► **Solving a Coin Problem**

The value of twenty-four coins consisting of dimes and quarters is \$3.75. How many quarters are in the collection of coins?

Solution ► Suppose the number of quarters is n. Since the whole collection contains 24 coins, then the number of dimes can be represented by $24 - n$. Also, in cents, the value of n quarters is $25n$, while the value of $24 - n$ dimes is $10(24 - n)$. We can organize this information as in the table below.

	dimes	quarters	Total
number of coins	$24 - n$	n	24
value of coins (in cents)	$10(24 - n)$	$25n$	375

The value is written in cents!

Using the last row of this table, we set up the equation

$$10(24 - n) + 25n = 375$$

and then solve it for n.

$$240 - 10n + 25n = 375 \qquad / -240$$
$$15n = 135 \qquad / \div 15$$
$$n = 9$$

So, there are **9** quarters in the collection of coins.

Mixture-Solution Problems

When solving total mixture or solution problems, it is helpful to organize the data in a table that follows one of the formulas

$unit\ price \cdot number\ of\ units = total\ value$ or $percent \cdot volume = content$

	unit price ·	# of units	= value
type I			
type II			
mix			

	% ·	volume	= content
type I			
type II			
solution			

Example 9 ► **Solving a Mixture Problem**

Dark chocolate kisses costing \$13.50 per kilogram are going to be mixed with white chocolate kisses costing \$7.00 per kilogram. How many kilograms of each type of chocolate kisses should be used to obtain 30 kilograms of a mixture that costs \$10.90 per kilogram?

Solution ► In this problem, we mix two types of chocolate kisses: dark and white. Let x represent the number of kilograms of dark chocolate kisses. Since there are 30 kilograms of the mixture, we will express the number of kilograms of the white chocolate kisses as $30 - x$.

The information given in the problem can be organized as in the following table.

	unit price ·	# of units	= value (in \$)
dark kisses	13.50	x	$13.5x$
white kisses	7.00	$30 - x$	$7(30 - x)$
mix	10.90	30	327

To complete the last column, multiply the first two columns.

Using the last column of this table, we set up the equation

$$13.5x + 7(30 - x) = 327$$

and then solve it for x.

$$13.5x + 210 - 7x = 327 \qquad / -210$$
$$6.5x = 117 \qquad / \div 6.5$$
$$x = 18$$

So, the mixture should consist of **18** kilograms of dark chocolate kisses and $30 - x = 30 - 18 = \mathbf{12}$ kilograms of white chocolate kisses.

Example 10 ► **Solving a Solution Problem**

How many milliliters of pure alcohol should be added to 80 ml of a 20% alcohol solution to make a 50% alcohol solution?

Solution ► Let x represent the volume of pure alcohol, in milliliters. The 50% solution is made by combining x ml of the pure alcohol with 80 ml of a 20% alcohol solution. So, the volume of the 50% solution can be expressed as $x + 80$.

Now, let us organize this information in the table below.

	% ·	volume	= acid
pure alcohol	1	x	x
20% solution	0.2	80	16
50% solution	0.5	$x + 80$	$0.5(x + 80)$

To complete the last column, multiply the first two columns.

Using the last column of this table, we set up the equation

$$x + 16 = 0.5(x + 80)$$

and then solve it for x.

$$x + 16 = 0.5x + 40 \qquad / -0.5x, -16$$
$$0.5x = 24 \qquad / \div 0.8$$
$$x = 12$$

So, there should be added **12** milliliters of pure alcohol.

Motion Problems

When solving motion problems, refer to the formula

$$\boldsymbol{Rate \cdot Time = Distance}$$

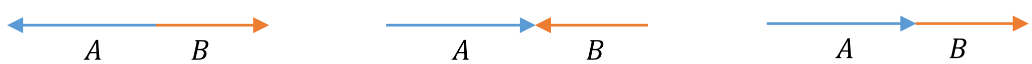
and organize date in a table like this:

	R ·	***T***	= ***D***
motion I			
motion II			
total			

Some boxes in the "total" row are often left empty. For example, in motion problems, we usually do not add rates. Sometimes, the "total" row may not be used at all.

If two moving object (or two components of a motion) are analyzed, we usually encounter the following situations:

1. The two objects A and B move apart, approach each other, or move successively in the same direction (see the diagram below). In these cases, it is likely we are interested in the **total distance** covered. So, the last row in the above table will be useful to record the total values.



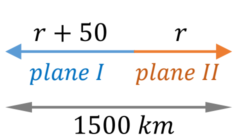
2. Both objects follow the same pathway. Then the **distances** corresponding to the two motions are the same and we may want to **equal** them. I such cases, there may not be any total values to consider, so the last row in the above table may not be used at all.

Example 11 ► **Solving a Motion Problem where Distances Add**

Two private planes take off from the same town and fly in opposite directions. The first plane is flying 50 km/h faster than the second one. In 3 hours, the planes are 1500 kilometers apart. Find the rate of each plane.

Solution ►

The rates of both planes are unknown. However, since the rate of the first plane is 50 km/h faster than the rate of the second plane, we can introduce only one variable. For example, suppose r represents the rate of the second plane. Then the rate of the first plane is represented by the expression $r + 50$.



In addition, notice that 1500 kilometers is the **total distance** covered by both planes, and 3 hours is the flight time of each plane.

Now, we can complete a table following the formula $R \cdot T = D$.

	R $\cdot$	T	$=$ D
plane I	$r+50$	3	$3(r+50)$
plane II	r	3	$3r$
total			1500

Notice that neither the total rate nor the total time was included here. This is because these values are not relevant to this particular problem. The equation that relates distances comes from the last column:

$$3(r+50)+3r=1500$$

After solving it for r,

$$3r+150+3r=1500 \qquad / -150$$

$$6r=1350 \qquad / \div 6$$

we obtain

$$r=225$$

Therefore, the speed of the first plane is $r+50=225+50=$ **275 km/h** and the speed of the second plane is **225 km/h**.

Example 12 ► **Solving a Motion Problem where Distances are the Same**

A police officer spotted a speeding car moving at 120 km/h. Ten seconds later, the police officer starts chasing the car, travelling on a motorcycle at 140 km/h. How long does it take the police officer to catch the car?

Solution ► Let t represent the time, in minutes, needed for the police officer to catch the car. The time that the speeding car drives is 10 seconds longer than the time that the police officer drives. To match the denominations, we convert 10 seconds to $\frac{10}{60}=\frac{1}{6}$ of a minute. So, the time used by the car is $t+\frac{1}{6}$.

In addition, the rates are given in kilometers per hour, but we need to have them in kilometers per minute. So, we convert $\frac{120\ km}{1\ h}=\frac{120\ km}{60\ min}=2\frac{km}{min}$, and similarly $\frac{140\ km}{1\ h}=\frac{140\ km}{60\ min}=\frac{7}{3}\frac{km}{min}$.

Now, we can complete a table that follows the formula $R \cdot T = D$.

	R $\cdot$	T	$=$ D
car	2	$t+\frac{1}{6}$	$2(t+\frac{1}{6})$
police	$\frac{7}{3}$	t	$\frac{7}{3}t$

Notice that this time there is no need for the "total" row.

Since distances covered by the car and the police officer are the same, we set up the equation

$$2\left(t+\frac{1}{6}\right)=\frac{7}{3}t \qquad / \cdot 3$$

To solve it for t, we may want to clear some fractions first. After multiplying by 3, we obtain

$$6\left(t+\frac{1}{6}\right)=7t$$

which becomes

$$6t+1=7t \quad / -6t$$

and finally

$$1=t$$

So, the police officer needs one minute to catch this car.

Even though the above examples show a lot of ideas and methods used in solving specific types of problems, we should keep in mind that the best way to learn problem-solving is to **solve a lot of problems**. This is because every problem might present slightly different challenges than the ones that we have seen before. The more problems we solve, the more experience we gain, and with time, problem-solving becomes easier.

L.3 Exercises

Translate each word description into an algebraic expression or equation.

1. A number less seven

2. A number less than seven

3. Half of the sum of two numbers

4. Two out of all apples in the bag

5. The difference of squares of two numbers

6. The product of two consecutive numbers

7. The sum of three consecutive integers is 30.

8. Five more than a number is double the number.

9. The quotient of three times a number and 10

10. Three percent of a number decreased by a hundred

11. Three percent of a number, decreased by a hundred

12. The product of 8 more than a number and 5 less than the number

13. A number subtracted from the square of the number

14. The product of six and a number increased by twelve

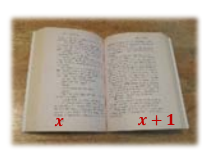
Solve each problem.

15. When the quotient of a number and 4 is added to twice the number, the result is 10 more than the number. Find the number.

16. When 25% of a number is added to 9, the result is 3 more than the number. Find the number.

17. The numbers on two adjacent safety deposit boxes add to 477. What are the numbers?

18. The sum of page numbers on two consecutive pages of a book is 543. What are the page numbers?



19. The number of international students at UFV, Abbotsford, BC, increased from 914 in the school year 2013/14 to 1708 in 2017/18. To the nearest tenths of a percent, what was the percent increase in enrollment during this time?

20. The number of domestic students at UFV, Abbotsford, BC, declined from 13762 in the school year 2013/14 to 12864 in 2017/18. To the nearest tenths of a percent, what was the percent decrease in enrollment during this time?

21. Find three consecutive odd integers such that the sum of the first, three times the second, and two times the third is 80.

22. Find three consecutive even integers such that the sum of the first, two times the second, and five times the third is 120.

23. Twice the sum of three consecutive odd integers is 210. Find the three integers.

24. Stephano paid $30,495 for a new Honda Civic. If this amount includes 7% of the sales tax, what is the cost of this car before tax?

25. After a 4% raise, the new monthly salary of a factory worker is $1924. What was the old monthly salary of this worker?

26. Jason bought a discounted fridge. The regular price of the fridge was $790.00, but he only paid $671.50. What was the percent discount?

27. The U.S. government issued about 156,000 patents in 2015. This was a decrease of about 1.7% from the number of patents issued in 2014. To the nearest hundred, how many patents were issued in 2014?

28. An investor has some funds at a 4% simple interest account and some at a 5% simple interest account. If the overall investment of $25000 gains $1134.00 of interest in one year, find the amount invested at each rate.

29. Jessie has $51,000 to invest. She plans to invest part of the money in an account paying 3% simple interest and the rest of the money into bonds paying 6.5% simple interest. How much should she invest at each rate to gain $3000 interest in a year?

30. Jan invested some money at 2.5% simple interest and twice this amount at 3.25%. Her total annual interest was $405. How much was invested at each rate?

31. Peter invested some money at 4.5% simple interest, and $2000 more than this amount at 5.25%. His total annual interest was $690. How much was invested at each rate?

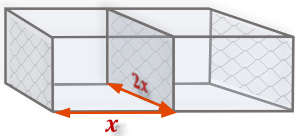
32. Daria invested $15,000 in bonds paying 6.5%. If she had some additional funds, she could invest in a saving account paying 2.5% simple interest. How much money would have to be invested at 2.5% for the average return on the two investments to be 5%?

33. Jack received a bonus payment of $12,000 and invested it in bonds paying 4.5% simple interest. If he had some additional funds, he could invest in a saving account paying 2.75% simple interest. How much additional money should he deposit in the 2.75% account so that his return on the two investments will be 4%?

34. A 126 cm long wire is cut into two pieces. Each piece is bent to form an equilateral triangle. If one triangle is twice as large as the other, how long are the sides of the triangles?

35. The measure of the smallest angle in a triangle is half the measure of the largest angle. The third angle is 15° less than the largest angle. Find the measure of each angle.

36. 35 ft of molding was used to trim a garage door. If the longer side of the door was 3 ft longer than twice the length of each of the shorter sides, then what are the dimensions of the door?



37. Billy plans to construct two adjacent rectangular outdoor cages for his rabbits. The cages would have open tops and bottoms, and share their longer side, as on the accompanying diagram. Each cage is planned to be twice as long as it is wide. If Billy has 80 ft of fencing, how large can the cages be?

38. The perimeter of a tennis court is 76 meters. The width of the court is 14 meters less than the length. Find the dimensions of the court.

39. Teresa inserted 16 coins into a vending machine to purchase a chocolate bar for $1.25. If she used only dimes and nickels, how many of each type of coins did she use?

40. Robert used 12 coins consisting of dimes, nickels, and quarters to buy the *Vancouver Sun* for $1.50. If he had twice as many dimes as nickels and the same many nickels as quarters, how many of each type of coins did he use?

41. A 30-kilogram mixture at $25.28 per kilogram consists of pecans at $27.50 per kilogram and cashews at $23.80 per kilogram. How many kilograms of each were used to make the mixture?

42. A store owner bought 15 kilograms of peanuts for $72. He wants to mix these peanuts with raisins costing $7.50 per kilogram to get a mixture costing $6 per kilogram. How many kilograms of raisins should he use?

43. Tickets to a movie theatre cost $8.50 for an adult and $3.50 for a child. If $1253 were collected for selling a total of 178 tickets, how many of each type of tickets were sold?

44. Find the unit cost of a sunscreen made from 160 milliliters of lotion that cost $1.49 per milliliter and 90 milliliters of lotion that cost $2.49 per milliliter.

45. A tea mixture was prepared by mixing 20 kg of tea costing $10.80 per kilogram with 30 kg of tea costing $6.50 per pound. Find the unit cost of the tea mixture.

46. A pharmacist has 150 milliliters of a solution that contains 80% of a particular medication. How much pure water should he add to change the concentration of the medication to 25%?

47. How many grams of a 50% gold alloy must be mixed with 100 grams of an 80% gold alloy to make a 75% gold alloy?

48. A jeweller mixed 40 g of an 80% silver alloy with 60 g of a 25% silver alloy. What percent of silver contains the resulting alloy?

49. How many milliliters of water must be added to 250 ml of a 7% hydrogen peroxide solution to make a 3% hydrogen peroxide solution?

50. A car radiator contains 9 liters of a 50% antifreeze solution. How many liters need to be replaced with pure antifreeze to bring the antifreeze concentration to 80%?

51. Jessica is working with sulphuric acid solutions in a lab. She needs to dilute 50 milliliters of a 70% sulphuric acid solution to a 50% solution by mixing it with a 25% sulphuric acid solution. How many milliliters of a 25% solution should she use?

52. Two planes fly towards each other starting from two cities that are 4200 km apart. If one plane is travelling 150 km/h faster than the other and they pass each other after 2.5 hours, what is the speed of each plane?

53. A plane flies at 630 km/h in still air. To the nearest minute, how long will it take the plane to travel 1000 kilometers

a. into a 90-km/h headwind?

b. with a 90-km/h tailwind?

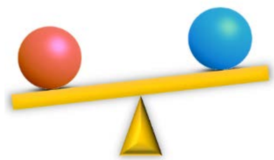
54. At 7:00 am, Jacob left his house jogging at 10 km/h to a nearby park for his routine morning exercises. Six minutes later, his brother Andrew followed him using the same route. Running at 15 km/h, in how many minutes will Andrew catch up with his brother?

55. Tina walked at a rate of 8 km/h from home to a bike shop. She bought a bike there and rode it back home at a rate of 24 km/h. If the total time spent travelling was one hour, how far from Tina's home was the bike shop?

56. A jogger and a cyclist went to a park for their morning exercise. They start moving on a trail loop from the same point and in the same direction. On average, the cyclist travels three times as fast as the jogger. If after 20 minutes the cyclist has finished his first loop and the jogger has still 6 km to complete the loop, how long is the trail loop?

57. A 2 km long freight train is moving at 35 km/h on a straight segment of a train-track. Suppose a car is moving at 65 km/h on a parallel road, in the opposite direction. How long would it take the car to pass from the front till the end of the train?

L4 Linear Inequalities and Interval Notation



Mathematical inequalities are often used in everyday life situations. We observe speed limits on highways, minimum payments on credit card bills, maximum smartphone data usage per month, the amount of time we need to get from home to school, etc. When we think about these situations, we often refer to limits, such as "a speed limit of 100 kilometers per hour" or "a limit of 1 GB of data per month." However, we don't have to travel at exactly 100 kilometers per hour on the highway or use exactly 1 GB of data per month. The limit only establishes a boundary for what is allowable. For example, a driver travelling x kilometers per hour is obeying the speed limit of 100 kilometers per hour if $x \leq 100$ and breaking the speed limit if $x > 100$. A speed of $x = 100$ represents the boundary between obeying the speed limit and breaking it. Solving linear inequalities is closely related to solving linear equations because equality is the boundary between *greater than* and *less than*. In this section, we discuss techniques needed to solve linear inequalities and ways of presenting these solutions.

Linear Inequalities

Definition 4.1 ► A **linear inequality** is an inequality with only **constant** or **linear terms**.
A linear inequality in one variable can be written in one of the following forms:

$$\boldsymbol{Ax + B > 0}, \quad \boldsymbol{Ax + B \geq 0}, \quad \boldsymbol{Ax + B < 0}, \quad \boldsymbol{Ax + B \leq 0}, \quad \boldsymbol{Ax + B \neq 0},$$

for some real numbers A and B, and a variable x.

A variable value that makes an inequality true is called a **solution** to this inequality. We say that such variable value **satisfies** the inequality.

Example 1 ► **Determining if a Given Number is a Solution of an Inequality**

Determine whether each of the given values is a solution of the inequality.

a. $3x - 7 > -2$; 2, 1

b. $\frac{y}{2} - 6 \geq -3$; 8, 6

Solution ►

a. To check if 2 is a solution of $3x - 7 > -2$, replace x by 2 and determine whether the resulting inequality $3 \cdot 2 - 7 > -2$ is a true statement. Since $6 - 7 = -1$ is indeed larger than -2, then 2 satisfies the inequality. So 2 is a solution of $3x - 7 > -2$.

After replacing x by 1, we obtain $3 \cdot 1 - 7 > -2$, which simplifies to the false statement $-4 > -2$. This shows that 1 is not a solution of the given inequality.

b. To check if 8 is a solution of $\frac{y}{2} - 6 \geq -3$, substitute $y = 8$. The inequality becomes $\frac{8}{2} - 6 \geq -3$, which simplifies to $-2 \geq -3$. Since this is a true statement, 8 is a solution of the given inequality.

Similarly, after substituting $y = 6$, we obtain a true statement $\frac{6}{2} - 6 \geq -3$, as the left side of this inequality equals to -3. This shows that -3 is also a solution to the original inequality.

Usually, an inequality has an infinite number of solutions. For example, one can check that the inequality

$$2x - 10 < 0$$

is satisfied by -5, 0, 1, 3, 4, 4.99, and generally by any number that is smaller than 5. So in the above example, the set of all solutions, called the **solution set**, is infinite. Generally, the solution set to a linear inequality in one variable can be stated either using **set-builder notation**, or **interval notation**. Particularly, the solution set of the above inequality could be stated as $\{x | x > 5\}$, or as $(-\infty, 5)$.

In addition, it is often beneficial to visualize solution sets of inequalities in one variable as graphs on a number line. The solution set to the above example would look like this:

For more information about presenting solution sets of inequalities in the form of a graph or interval notation, refer to *Example 3* and the subsection on "*Interval Notation*" in *Section R2* of the *Review* chapter.

To solve an inequality means to find all the variable values that satisfy the inequality, which in turn means to find its solution set. Similarly as in the case of equations, we find these solutions by producing a sequence of simpler and simpler inequalities preserving the solution set, which eventually result in an inequality of one of the following forms:

$$x > constant, \quad x \geq constant, \quad x < constant, \quad x \leq constant, \quad x \neq constant.$$

Definition 4.2 ► **Equivalent inequalities** are inequalities with the same solution set.

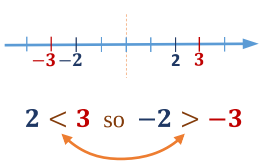
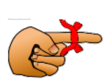


Figure 1

Generally, we create equivalent inequalities in the same way as we create equivalent equations, **except for *multiplying* or *dividing* an inequality by a *negative* number**. Then, we ***reverse the inequality symbol***, as illustrated in *Figure 1*.

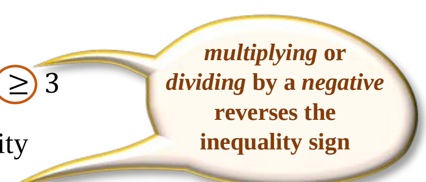


So, if we multiply (or divide) the inequality

$$-x \geq 3$$

by -1, then we obtain an equivalent inequality

$$x \leq -3.$$



We encourage the reader to confirm that the solution set to both of the above inequalities is $(-\infty, -3]$.

Multiplying or dividing an inequality by a positive number, leaves the inequality sign unchanged.

The table below summarizes the basic inequality operations that can be performed to produce equivalent inequalities, starting with $A < B$, where A and B are any algebraic expressions. Suppose C is a real number or another algebraic expression. Then, we have:

Inequality operation	General Rule	Example
Simplification	Write each expression in a simpler but equivalent form	$2(x-3) < 1+3$ can be written as $2x-6 < 4$
Addition	if $A < B$ then $A + C < B + C$	if $2x - 6 < 4$ then $2x - 6 + 6 < 4 + 6$
Subtraction	if $A < B$ then $A - C < B - C$	if $2x < x + 4$ $2x - x < x + 4 - x$
Multiplication **when multiplying by a negative value, reverse the inequality sign**	if $C > 0$ and $A < B$ then $CA < CB$ if $C < 0$ and $A < B$ then $CA > CB$	if $2x < 10$ then $\frac{1}{2} \cdot 2x < \frac{1}{2} \cdot 10$ if $-x < -5$ then $x > 5$
Division **when dividing by a negative value, reverse the inequality sign**	if $C > 0$ and $A < B$ then $\frac{A}{C} < \frac{B}{C}$ if $C < 0$ and $A < B$ then $\frac{A}{C} > \frac{B}{C}$	if $2x < 10$ then $\frac{2x}{2} < \frac{10}{2}$ if $-2x < 10$ then $x > -5$

Example 2 ► **Using Inequality Operations to Solve Linear Inequalities in One Variable**

Solve the inequalities. Graph the solution set on a number line and state the answer in interval notation.

a. $\frac{3}{4}x + 3 > 15$ **b.** $-2(x+3) > 10$

c. $\frac{1}{2}x - 3 \leq \frac{1}{4}x + 2$ **d.** $-\frac{2}{3}(2x-3) - \frac{1}{2} \geq \frac{1}{2}(5-x)$

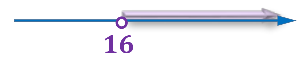
Solution ► **a.** To isolate x, we apply inverse operations in reverse order. So, first we subtract the 3, and then we multiply the inequality by the reciprocal of the leading coefficient. Thus,

$$\frac{3}{4}x + 3 > 15 \quad / -3$$

$$\frac{3}{4}x > 12 \quad / \cdot \frac{4}{3}$$

$$x > \frac{\overset{4}{\cancel{12}} \cdot 4}{\cancel{3}} = 16$$

To visualize the solution set of the inequality $x > 16$ on a number line, we graph the interval of all real numbers that are greater than 16.



Finally, we give the answer in interval notation by stating $\boldsymbol{x} \in (\mathbf{16}, \infty)$. This tells us that any x-value greater than 16 satisfies the original inequality.

Note: The answer can be stated as $x \in (16, \infty)$, or simply as $(16, \infty)$. Both forms are correct.

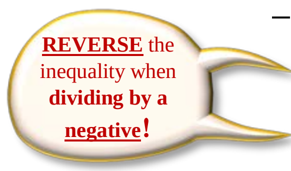
b. Here, we will first simplify the left-hand side expression by expanding the bracket and then follow the steps as in *Example 2a*. Thus,

$$-2(x+3) > 10$$

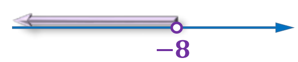
$$-2x - 6 > 10 \qquad / \; -3$$

$$-2x > 16 \qquad / \div (-2)$$

$$x < -8$$



The corresponding graph looks like this:



The solution set in interval notation is $(-\infty, -\mathbf{8})$.

c. To solve this inequality, we will collect and combine linear terms on the left-hand side and free terms on the right-hand side of the inequality.

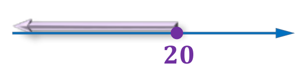
$$\frac{1}{2}x - 3 \le \frac{1}{4}x + 2 \qquad / -\frac{1}{4}x, +3$$

$$\frac{1}{2}x - \frac{1}{4}x \le 5$$

$$\frac{1}{4}x \le 5 \qquad / \cdot 4$$

$$x \le 20$$

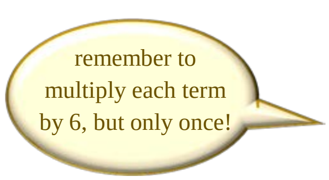
This can be graphed as



and stated in interval notation as $(-\infty, \mathbf{20}]$.

d. To solve this inequality, it would be beneficial to clear the fractions first. So, we will multiply the inequality by the LCD of 3 and 2, which is 6.

$$-\frac{2}{3}(2x-3) - \frac{1}{2} \le \frac{1}{2}(5-x) \qquad / \cdot 6$$



$$-\frac{2 \cdot \overset{2}{\cancel{6}}}{\cancel{3}}(2x-3) - \frac{1 \cdot \overset{3}{\cancel{6}}}{\cancel{2}} \le \frac{1 \cdot \overset{3}{\cancel{6}}}{\cancel{2}}(5-x)$$

$$-4(2x-3) - 3 \le 3(5-x)$$

$$-8x + 12 - 3 \leq 15 - 3x$$

$$-8x + 9 \leq 15 - 3x \qquad / \; +8x, -15$$

At this point, we could collect linear terms on the left or on the right-hand side of the inequality. Since it is easier to work with a positive coefficient by the x-term, let us move the linear terms to the right-hand side this time.

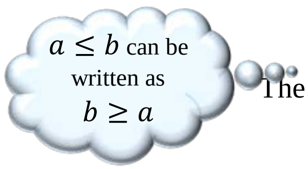
So, we obtain

$$-6 \leq 5x \qquad / \; \div 5$$

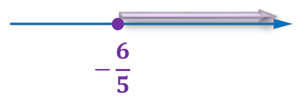
$$\frac{-6}{5} \leq x,$$

which after writing the inequality from right to left, gives us the final result

$$x \geq -\frac{6}{5}$$



The solution set can be graphed as



which means that all real numbers $x \in \left[-\frac{6}{5}, \infty\right)$ satisfy the original inequality.

Example 3 ► **Solving Special Cases of Linear Inequalities**

Solve each inequality.

a. $-2(x - 3) > 5 - 2x$ **b.** $-12 + 2(3 + 4x) < 3(x - 6) + 5x$

Solution ► **a.** Solving the inequality

$$-2(x - 3) > 5 - 2x$$

$$-2x + 6 > 5 - 2x \qquad / \; +2x$$

$$6 > 5,$$

leads us to a true statement that does not depend on the variable x. This means that any real number x satisfies the inequality. Therefore, the solution set of the original inequality is equal to all real numbers $\mathbb{R}$. This could also be stated in interval notation as $(-\infty, \infty)$.

b. Solving the inequality

$$-12 + 2(3 + 4x) < 3(x - 6) + 5x$$

$$-12 + 6 + 8x < 3x - 18 + 5x$$

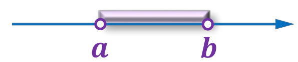
$$-6 + 8x < 8x - 18 \qquad / \; -8x$$

$$-6 < -18$$

leads us to a false statement that does not depend on the variable x. This means that no real number x would satisfy the inequality. Therefore, the solution set of the original inequality is an empty set $\emptyset$. We say that the inequality has **no solution**.

Three-Part Inequalities

The fact that an unknown quantity x lies between two given quantities a and b, where $a < b$, can be recorded with the use of the three-part inequality $a < x < b$. We say that x is enclosed by the values (or oscillates between the values) a and b. For example, the systolic high blood pressure p oscillates between 120 and 140 mm Hg. It is convenient to record this fact using the three-part inequality $120 < p < 140$, rather than saying that p < 140 and at the same time p > 120. The solution set of the three-part inequality $a < x < b$ or $b > x > a$ is a **bounded** interval (a, b) that can be graphed as



The hollow (open) dots indicate that thy endpoints do not belong to the solution set. Such interval is called **open**.

If the inequality symbol includes equation ($\leq$ or $\geq$), the corresponding endpoint of the interval is included in the solution set. On a graph, this is reflected as a solid (closed) dot. For example, the solution set of the three-part inequality $a \leq x < b$ is the interval $[a, b)$, which is graphed as

Such interval is called **half-open** or **half-closed**.

An interval with both endpoints included is referred to as **closed** interval. For example, $[a, b]$ is a closed interval and its graph looks like this

Any three-part inequality of the form

$$\textit{constant } \boldsymbol{a} < (\leq) \textit{ one variable linear } \boldsymbol{expression} < (\leq) \textit{ constant } \boldsymbol{b},$$

where $a \leq b$ can be solved similarly as a single inequality, by applying inequality operations to all of the three parts. When solving such inequality, the goal is to isolate the variable in the middle part by moving all constants to the outside parts.

Example 4 ► **Solving Three-Part Inequalities**

Solve each three-part inequality. Graph the solution set on a number line and state the answer in interval notation.

a. $-2 \leq 1 - 3x \leq 3$ **b.** $-3 < \frac{2x-3}{4} \leq 6$

Solution ► **a.** To isolate x from the expression $1 - 3x$, subtract 1 first, and then divide by -3. These operations must be applied to all three parts of the inequality. So, we have

Remember to **reverse** both inequality symbols when **dividing by a negative** number!

$$-2 \le 1 - 3x \le 3 \qquad / -1$$

$$-3 \le -3x \le 2 \qquad / \div(-3)$$

$$1 \ge x \ge -\frac{2}{3}$$

The result can be graphed as

The inequality is satisfied by all $x \in \left[-\frac{2}{3}, 1\right]$.

b. To isolate x from the expression $\frac{2x-3}{4}$, we first multiply by 4, then add 3, and finally divide by 2.

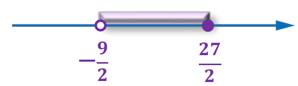
$$-3 < \frac{2x-3}{4} \le 6 \qquad / \cdot 4$$

$$-12 < 2x - 3 \le 24 \qquad / +3$$

$$-9 < 2x \le 27 \qquad / \div 2$$

$$-\frac{9}{2} < x \le \frac{27}{2}$$

The result can be graphed as



The inequality is satisfied by all $x \in \left(-\frac{9}{2}, \frac{27}{2}\right]$.

Inequalities in Application Problems

Linear inequalities are often used to solve problems in areas such as business, construction, design, science, or linear programming. The solution to a problem involving an inequality is generally an interval of real numbers. We often ask for the range of values that solve the problem.

Below is a list of common words and phrases indicating the use of particular inequality symbols.

Word expression	Interpretation
a is less (smaller) than ***b***	$a < b$
a is less than or equal to ***b***	$a \le b$
a is greater (more, bigger) than ***b***	$a > b$
a is greater than or equal to ***b***	$a \ge b$
a is at least ***b***	$a \ge b$
a is at most ***b***	$a \le b$
a is no less than ***b***	$a \ge b$
a is no more than ***b***	$a \le b$
a is exceeds ***b***	$a > b$
a is different than ***b***	$a \ne b$
x is between ***a*** and ***b***	$a < x < b$
x is between ***a*** and ***b*** inclusive	$a \le x \le b$

Example 5 ► **Translating English Words to an Inequality**

Translate the word description into an inequality and then solve it.

a. Twice a number, increased by 3 is at most 9.
b. Two diminished by five times a number is between -4 and 7

Solution ► **a.** Twice a number, increased by 3 translates to $2x + 3$. Since "at most" corresponds to the symbol "$\leq$", the inequality to solve is

$$2x + 3 \leq 9 \qquad / -3$$

$$2x \leq 6 \qquad / \div 2$$

$$x \leq 3$$

So, all $\boldsymbol{x \in (-\infty, 3]}$ satisfy the condition of the problem.

b. Two more than five times a number translates to $2 - 5x$. The phrase "between -4 and 7" tells us that the expression $2 - 5x$ is enclosed by the numbers -4 and 7, but not equal to these numbers. So, the inequality to solve is

$$-4 < 2 - 5x < 7 \qquad / -2$$

$$-6 < -5x < 5 \qquad / \div (-5)$$

$$\frac{6}{5} > x > -1$$

Therefore, the solution set to this problem is the interval of numbers $\boldsymbol{\left(-1, \frac{6}{5}\right)}$.

Remember: To record an interval, list its endpoints in ***increasing order*** (from the smaller to the larger number.)

Example 6 ► **Using a Linear Inequality to Compare Cellphone Charges**

A cellphone company advertises two pricing plans for day-time minutes. The first plan costs \$14.99 per month with 30 free day-time minutes and \$0.36 per minute after that. The second plan costs \$24.99 per month with 20 free day-time minutes and \$0.25 per minute after that. A customer figured that he will pay less by choosing the first plan. What could be the maximum number of day-time minutes that he predicts to use per month?

Solution ► Let n represent the number of cellphone minutes used per month. In the first plan, since the first 30 minutes are free, the number of paid-minutes can be represented by $n - 30$. Hence, the total charge according to the first plan is $14.99 + 0.36(n - 30)$. Similarly, in the second plan, the number of paid-minutes can be represented by $n - 20$. Therefore the total charge according to the second plan is $24.99 + 0.25(n - 20)$.

Since the first plan is to be cheaper, than the inequality to solve is

$$14.99 + 0.36(n - 30) < 24.99 + 0.25(n - 20).$$

To work with 'nicer' numbers, such as integers, we may want to eliminate the decimals by multiplying the above inequality by 100 first. Then, after removing the brackets via distribution, we obtain

$$1499 + 36n - 1080 < 2499 + 25n - 500 \qquad / -25n$$

$$419 + 11n < 1999 \qquad / -419$$

$$11n < 1580 \qquad / \div 11$$

$$n < \frac{1580}{11} \approx 143.6$$

So, the maximum number of day-time minutes for the first plan to be cheaper is **143**.

Example 7 ► **Finding the Test Score Range of the Missing Test**

Arek obtained 71% on a midterm test. If he wishes to bring his course mark to a B, the average of his midterm and final exam marks must be between 73% and 76%, inclusive. What range of scores on the final exam would guarantee Arek a mark of B in this course?

Solution ► Let n represent Arek's score on his final exam. Then, the average of his midterm and final exam is represented by the expression

$$\frac{71 + n}{2}.$$

Since this average must be between 73% and 76% inclusive, we need to solve the three-part inequality

$$73 \le \frac{71 + n}{2} \le 76 \qquad / \cdot 2$$

$$146 \le 71 + n \le 152 \qquad / -74$$

$$75 \le n \le 81$$

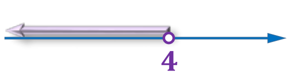
To attain a final grade of a B, Arek's score on his final exam should fall **between 75% and 81%**, inclusive.

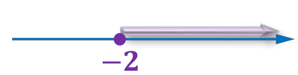
L.4 Exercises

Using interval notation, record the set of numbers represented by each graph. (Refer to the part "Interval Notation" in Section R2 of the Review chapter, if needed.)

1. 

2. 

3. 

4. 

***Graph** each solution set. For each interval write the corresponding **inequality** (or inequalities), and for each inequality, write the solution set in **interval notation**. (Refer to the part "Interval Notation" in Section R2 of the Review chapter, if needed.)*

5. $(3,\infty)$ **6.** $(-\infty,2]$ **7.** $[-7,5]$ **8.** $[-1,4)$

9. $x \geq -5$ **10.** $x > 6$ **11.** $x < -2$ **12.** $x \leq 0$

13. $-4 < x < 1$ **14.** $3 \leq x < 7$ **15.** $-5 < x \leq -2$ **16.** $0 \leq x \leq 1$

Determine whether or not the given value is a solution to the inequality.

17. $4n + 15 > 6n + 20;\ \ -5$

18. $16 - 5a > 2a + 9;\ \ 1$

19. $\frac{x}{4} + 7 \geq 5;\ \ -8$

20. $6y - 7 \leq 2 - y;\ \ \frac{2}{3}$

***Solve** each inequality. **Graph** the solution set and write the solution using **interval notation**.*

21. $2 - 3x \geq -4$

22. $4x - 6 > 12 - 10x$

23. $\frac{3}{5}x > 9$

24. $-\frac{2}{3}x \leq 12$

25. $5(x+3) - 2(x-4) \geq 2(x+7)$

26. $5(y+3) + 9 < 3(y-2) + 6$

27. $2(3x-4) - 4x \leq 2x + 3$

28. $7(4-x) + 5x > 2(16-x)$

29. $\frac{4}{5}(7x+6) > 40$

30. $\frac{2}{3}(4x-3) \leq 30$

31. $\frac{5}{2}(2a-3) < \frac{1}{3}(6-2a)$

32. $\frac{2}{3}(3x-1) \geq \frac{3}{2}(2x-3)$

33. $\frac{5-2x}{2} \geq \frac{2x+1}{4}$

34. $\frac{3x-2}{-2} \geq \frac{x-4}{-5}$

35. $0.05 + 0.08x < 0.01x - 0.04(3-3x)$

36. $-0.2(5x+2) > 0.4 + 1.5x$

37. $-\frac{1}{4}(p+6) + \frac{3}{2}(2p-5) \leq 10$

38. $\frac{3}{5}(t-2) - \frac{1}{4}(2t-7) \leq 3$

39. $-6 \leq 5x - 7 \leq 4$

40. $-10 < 3b - 5 < -1$

41. $2 \leq -3m - 7 \leq 4$

42. $4 < -9x + 5 < 8$

43. $-\frac{1}{2} < \frac{1}{4}x - 3 < \frac{1}{2}$

44. $-\frac{2}{3} \leq 4 - \frac{1}{4}x \leq \frac{2}{3}$

45. $-3 \leq \frac{7-3x}{2} < 5$

46. $-7 < \frac{3-2x}{3} \leq -2$

Using interval notation, state the set of numbers satisfying each description.

47. The sum of a number and 5 exceeds 12.

48. 5 times a number, decreased by 6, is smaller than -16.

49. 2 more than three times a number is at least 8.

50. Triple a number, subtracted from 5, is at most 7.

51. Half of a number increased by 3 is no more than 12.

52. Twice a number increased by 1 is different than 14.

53. Double a number is between -6 and 8.

54. Half a number, decreased by 3, is between 1 and 12.

Solve each problem.

55. There are three major tests in the algebra course that Nicole takes. She already wrote the first two tests and received 79% and 89% respectively. What score must she aim for when writing her third test to keep an average test mark of 85% or higher?

56. To receive a B in a university course, the average mark needs to be between 73 and 76, inclusive. Suppose a final grade in a particular course is calculated by taking average of the four major tests, including the final exam. On the first three tests, a student obtained the following scores: 59, 71, and 86. What range of scores on the final exam will guarantee the student a B in this course?

57. A marketing company has a budget of $1400 to run an advertisement on a particular website. The website charges $10.50 per day to display the add and $220 set up fee. Maximally, how many days the ad can be posted on this site?

58. Ken plans to paint a room with 340 square feet of wall area. He needs to buy some masking tape, drop sheets, and paint brushes for a total of $32. A gallon of paint covers 100 square feet of area, and the paint is sold only in gallons. If Ken plans to stay within $150 for the whole job what is the maximum cost per gallon of paint that he can afford?

59. Last week, the temperature in Banff, BC, ranged between 23°F and 68°F. Using the formula $F = \frac{9}{5}C + 32$, find the temperature range in degrees Celsius.

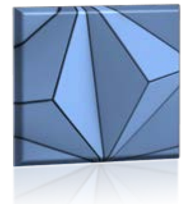
60. One day, the temperature range in Whistler, BC, was between -2°C and 18°C. Using the formula $C = \frac{5}{9}(F - 32)$, find the temperature range in degrees Fahrenheit.

61. Adon makes $1600 a month with an additional commission of 7% of his sales. This month, Adon's earnings were higher than $3700. How high must have been his sales?

62. Suppose a particular bank offers two chequing accounts. The first account charges $5 per month and $0.75 per cheque after the first 10 cheques. The second account charges $12.50 per month with unlimited cheque writing. What is the maximum number of cheques processed for a customer who chooses the first account as the better option?

63. The Toronto Dominion Bank offers a chequing account that charges $10.95 per month plus $1.25 per cheque after the first 25 cheques. A competitor bank is offering an account for $7.95 per month plus $1.30 per cheque after the first 25 cheques. If a business chooses the first account as the cheaper option, what is the minimum number of cheques that the business predicts to write monthly?

L5 Operations on Sets and Compound Inequalities

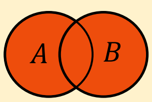


In the previous section, it was shown how to solve a three-part inequality. In this section, we will study how to solve systems of inequalities, often called **compound** inequalities, that consist of two linear inequalities joined by the words "*and*" or "*or*". Studying compound inequalities is an extension of studying three-part inequalities. For example, the three-part inequality, $2 < x \leq 5$, is in fact a system of two inequalities, $2 < x$ *and* $x \leq 5$. The solution set to this system of inequalities consists of all numbers that are larger than 2 *and* at the same time are smaller or equal to 5. However, notice that the system of the same two inequalities connected by the word "*or*", $2 < x$ *or* $x \leq 5$, is satisfied by any real number. This is because any real number is either larger than 2 *or* smaller than 5. Thus, to find solutions to compound inequalities, aside for solving each inequality individually, we need to pay attention to the joining words, "*and*" or "*or*". These words suggest particular operations on the sets of solutions.

Operations on Sets

Sets can be added, subtracted, or multiplied.

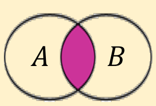
Definition 5.1 ► The result of the addition of two sets $\boldsymbol{A}$ and $\boldsymbol{B}$ is called a **union** (or sum), symbolized by $\boldsymbol{A} \cup \boldsymbol{B}$, and defined as:



$$\boldsymbol{A} \cup \boldsymbol{B} = \{\boldsymbol{x} | \boldsymbol{x} \in \boldsymbol{A} \ \textbf{or} \ \boldsymbol{x} \in \boldsymbol{B}\}$$

This is the set of all elements that belong to either $\boldsymbol{A}$ ***or*** $\boldsymbol{B}$.

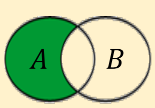
The result of the multiplication of two sets $\boldsymbol{A}$ and $\boldsymbol{B}$ is called an **intersection** (or product, or **common part**), symbolized by $\boldsymbol{A} \cap \boldsymbol{B}$, and defined as:



$$\boldsymbol{A} \cap \boldsymbol{B} = \{\boldsymbol{x} | \boldsymbol{x} \in \boldsymbol{A} \ \textbf{and} \ \boldsymbol{x} \in \boldsymbol{B}\}$$

This is the set of all elements that belong to both $\boldsymbol{A}$ ***and*** $\boldsymbol{B}$.

The result of the subtraction of two sets $\boldsymbol{A}$ and $\boldsymbol{B}$ is called a **difference**, symbolized by $\boldsymbol{A} \setminus \boldsymbol{B}$, and defined as:



$$\boldsymbol{A} \setminus \boldsymbol{B} = \{\boldsymbol{x} | \boldsymbol{x} \in \boldsymbol{A} \ \textbf{and} \ \boldsymbol{x} \notin \boldsymbol{B}\}$$

This is the set of all elements that belong to $\boldsymbol{A}$ ***and*** do not belong to $\boldsymbol{B}$.

Example 1 ► **Performing Operations on Sets**

Suppose $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6\}$, and $C = \{6\}$. Find the following sets:

a. $A \cap B$

b. $A \cup B$

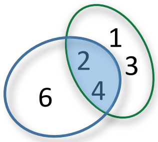
c. $A \setminus B$

d. $B \cup C$

e. $B \cap C$

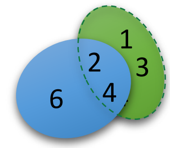
f. $A \cap C$

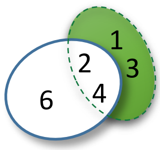
Solution ►



a. The intersection of sets $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$ consists of numbers that belong to both sets, so $\boldsymbol{A \cap B = \{2, 4\}}$.

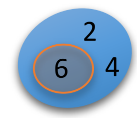
b. The union of sets $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$ consists of numbers that belong to at least one of the sets, so $\boldsymbol{A \cup B = \{1, 2, 3, 4, 6\}}$.



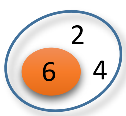


c. The difference $A \setminus B$ of sets $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$ consists of numbers that belong to the set A but do not belong to the set B, so $\boldsymbol{A \setminus B = \{1, 3\}}$.

d. The union of sets $B = \{2, 4, 6\}$, and $C = \{6\}$ consists of numbers that belong to at least one of the sets, so $\boldsymbol{B \cup C = \{2, 4, 6\}}$.



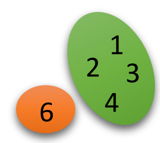
Notice that $B \cup C = B$. This is because C is a **subset** of B.



e. The intersection of sets $B = \{2, 4, 6\}$ and $C = \{6\}$ consists of numbers that belong to both sets, so $\boldsymbol{B \cap C = \{6\}}$.

Notice that $B \cap C = C$. This is because C is a **subset** of B.

f. Since the sets $A = \{1, 2, 3, 4\}$ and $C = \{6\}$ do not have any common elements, then $\boldsymbol{A \cap C = \emptyset}$.



Recall: Two sets with no common part are called **disjoint**.
Thus, the sets A and C are disjoint.

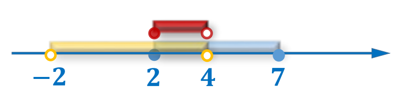
Example 2 ► **Finding Intersections and Unions of Intervals**

Write the result of each set operation as a single interval, if possible.

a. $(-2,4) \cap [2,7]$
b. $(-\infty, 1] \cap (-\infty, 3)$
c. $(-1,3) \cup (1,6]$
d. $(3,\infty) \cup [5,\infty)$
e. $(-\infty,3) \cup (4,\infty)$
f. $(-\infty,3) \cap (4,\infty)$
g. $(-\infty,5) \cup (4,\infty)$
h. $(-\infty,3] \cap [3,\infty)$

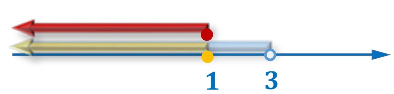
Solution ►

a. The interval of points that belong to both, the interval $(-2,4)$, in yellow, and the interval $[2,7]$, in blue, is marked in red in the graph below.



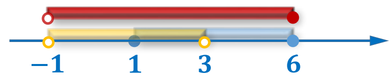
So, we write $(-2,4) \cap [2,7] = [2, 4)$.

b. As in problem **a.**, the common part $(-\infty, 1] \cap (-\infty, 3)$ is illustrated in red on the graph below.



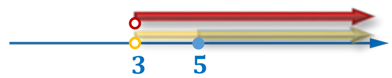
So, we write $(-\infty, 1] \cap (-\infty, 3) = (-\infty, 1]$.

c. This time, we take the union of the interval $(-1,3)$, in yellow, and the interval $(1,6]$, in blue. The result is illustrated in red on the graph below.



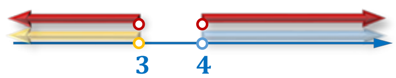
So, we write $(-1,3) \cup (1,6] = (-1,6]$.

d. The union $(3,\infty) \cup [5,\infty)$ is illustrated in red on the graph below.



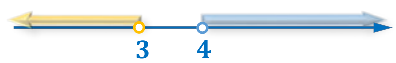
So, we write $(3,\infty) \cup [5,\infty) = (3,\infty)$.

e. As illustrated in the graph below, this time, the union $(-\infty,3) \cup (4,\infty)$ can't be written in the form of a single interval.



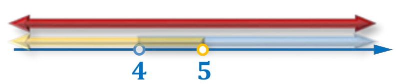
So, the expression $(-\infty,3) \cup (4,\infty)$ cannot be simplified.

f. As shown in the graph below, the interval $(-\infty,3)$ has no common part with the interval $(4,\infty)$.



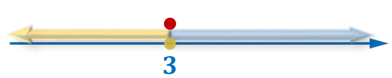
Therefore, $(-\infty,3) \cap (4,\infty) = \emptyset$

g. This time, the union $(-\infty,5) \cup (4,\infty)$ covers the entire number line.



Therefore, $(-\infty,5) \cup (4,\infty) = (-\infty,\infty)$

h. As shown in the graph below, there is only one point common to both intervals, $(-\infty,3]$ and $[3,\infty)$. This is the number 3.



Since a single number is not considered to be an interval, we should use a set rather than interval notation when recording the answer. So, $(-\infty,3] \cap [3,\infty) = \{3\}$.

Compound Inequalities

The solution set to a system of two inequalities joined by the word *and* is the intersection of solutions of each inequality in the system. For example, the solution set to the system

$$\begin{cases} x > 1 \\ x \le 4 \end{cases}$$

is the intersection of the solution set for $x > 1$ and the solution set for $x \le 4$. This means that the solution to the above system equals $(1, \infty) \cap (-\infty, 4] = (1, 4]$, as illustrated in the graph below.

The solution set to a system of two inequalities joined by the word *or* is the union of solutions of each inequality in the system. For example, the solution set to the system

$$x \le 1 \text{ or } x > 4$$

is the union of the solution of $x \le 1$ and, the solution of $x > 4$. This means that the solution to the above system equals $(-\infty, 1) \cup [4, \infty)$, as illustrated in the graph below.

Example 4 ► **Solving Compound Linear Inequalities**

Solve each compound inequality. Pay attention to the joining word *and* or *or* to find the overall solution set. Give the solution set in both interval and graph form.

a. $3x + 7 \ge 4 \quad and \quad 2x - 5 < -1$

b. $-2x - 5 \ge 1 \quad or \quad x - 5 \ge -3$

c. $3x - 11 < 4 \quad or \quad 4x + 9 \ge 1$

d. $-2 < 3 - \frac{1}{4}x < \frac{1}{2}$

e. $\begin{cases} 4x - 7 < 1 \\ 7 - 3x > -8 \end{cases}$

f. $4x - 2 < -8 \quad or \quad 5x - 3 < 12$

Solution ►

a. To solve this system of inequalities, first, we solve each individual inequality, keeping in mind the joining word *and*. So, we have

$$\begin{array}{cccc} 3x + 7 \ge 4 & /-7 & and & 2x - 5 < -1 & /+5 \\ 3x \ge -3 & /\div 3 & and & 2x < 4 & /\div 2 \\ x \ge -1 & & and & x < 2 & \end{array}$$

The joining word *and* indicates that we look for the intersection of the obtained solutions. These solutions (in yellow and blue) and their intersection (in red) are shown in the graphed below.

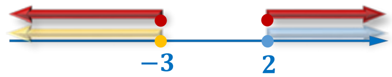
Therefore, the system of inequalities is satisfied by all $x \in [\mathbf{-1}, \mathbf{2})$.

b. As in the previous example, first, we solve each individual inequality, except this time we keep in mind the joining word *or*. So, we have

$$-2x - 5 \ge 1 \quad /+5 \qquad or \qquad x - 5 \ge -3 \quad /+5$$

$$-2x \geq 6 \quad / \div (-2)! \qquad or \qquad x \geq 2$$
$$x \leq -3$$

The joining word *or* indicates that we look for the union of the obtained solutions. These solutions (in yellow and blue) and their union (in red) are indicated in the graph below.

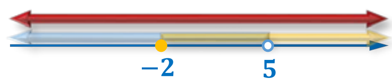


Therefore, the system of inequalities is satisfied by all $x \in (-\infty, -\mathbf{3}] \cup [\mathbf{2}, \infty)$.

c. As before, we solve each individual inequality, keeping in mind the joining word *or*. So, we have

$$3x - 11 < 4 \quad / +11 \qquad or \qquad 4x + 9 \geq 1 \quad / -9$$
$$3x < 15 \quad / \div 3 \qquad or \qquad 4x \geq -8 \quad / \div 4$$
$$x < 5 \qquad or \qquad x \geq -2$$

The joining word *or* indicates that we look for the union of the obtained solutions. These solutions (in yellow and blue) and their union (in red) are indicated in the graph below.



Therefore, the system of inequalities is satisfied by all real numbers. The solution set equals to $\mathbb{R}$.

d. Any three-part inequality is a system of inequalities with the joining word *and*. The system $-2 < 3 - \frac{1}{4}x < \frac{1}{2}$ could be written as

$$-2 < 3 - \frac{1}{4}x \quad and \quad 3 - \frac{1}{4}x < \frac{1}{2}$$

and solved as in *Example 4a*. Alternatively, it could be solved in the three-part form, similarly as in *Section L4*, *Example 4*. Here is the three-part form solution.

$$-2 < 3 - \frac{1}{4}x < \frac{1}{2} \qquad / -3$$
$$-5 < -\frac{1}{4}x < \frac{1}{2} - \frac{3 \cdot 2}{2}$$
$$-5 < -\frac{1}{4}x < -\frac{5}{2} \qquad / \cdot (-4)$$

$$20 > x > \frac{5 \cdot 4}{2}$$
$$20 > x > 10$$

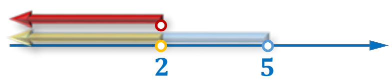
So the solution set is the interval $(10, 20)$, visualized in the graph below.

Remark: Solving a system of inequalities in three-part form has its benefits. First, the same operations are applied to all three parts, which eliminates the necessity of repeating the solving process for the second inequality. Second, the solving process of a three-part inequality produces the final interval of solutions rather than two intervals that need to be intersected to obtain the final solution set.

e. The system $\begin{cases} 4x - 7 < 1 \\ 7 - 3x > -8 \end{cases}$ consists of two inequalities joined by the word *and*. So, we solve it similarly as in *Example 4a*.

$4x - 7 < 1$	$/ +7$	*and*	$7 - 3x > -8$	$/ -7$
$4x < 8$	$/ \div 4$	*and*	$-3x > -15$	$/ \div (-3)$
$x < 2$		*and*	$x < 5$	

These solutions of each individual inequality (in yellow and blue) and the intersection of these solutions (in red) are indicated in the graph below.

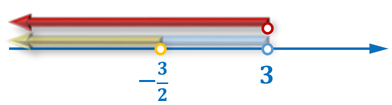


Therefore, the interval $(-\infty, 2)$ is the solution to the whole system.

f. As in *Example 4b* and *4c*, we solve each individual inequality, keeping in mind the joining word *or*. So, we have

$4x - 2 < -8$	$/ +2$	*or*	$5x - 3 < 12$	$/ +3$
$4x \geq -6$	$/ \div 4$	*or*	$5x \geq 15$	$/ \div 5$
$x < -\frac{3}{2}$		*or*	$x < 3$	

The joining word *or* indicates that we look for the union of the obtained solutions. These solutions (in yellow and blue) and their union (in red) are indicated in the graph below.



Therefore, the interval $(-\infty, 3)$ is the solution to the whole system.

Compound Inequalities in Application Problems

Compound inequalities are often used to solve problems that ask for a range of values satisfying certain conditions.

Example 5 ► **Finding the Range of Values Satisfying Conditions of a Problem**

The equation $P = 1 + \frac{d}{11}$ gives the pressure P, in atmospheres (atm), at a depth of d meters in the ocean. Atlantic cod occupy waters with pressures between 1.6 and 7 atmospheres. To the nearest meter, what is the depth range at which Atlantic cod should be searched for?

Solution ► The pressure P suitable for Atlantic cod is between 1.6 to 7 atmospheres. We record this fact in the form of the three-part inequality $1.6 \le P \le 7$. To find the corresponding depth d, in meters, we substitute $P = 1 + \frac{d}{11}$ and solve the three-part inequality for d. So, we have

$$1.6 \le 1 + \frac{d}{11} \le 7 \qquad / -1$$

$$0.6 \le \frac{d}{11} \le 6 \qquad / \cdot 11$$

$$6.6 \le d \le 66$$

Thus, Atlantic cod should be searched for between 7 and 66 meters below the surface.

Example 6 ► **Using Set Operations to Solve Applied Problems Involving Compound Inequalities**

Given the information in the table,

Film	Admissions (in millions)	Adjusted Gross Income (in millions of dollars)
Gone With the Wind	202	1825
Star Wars	178	1608
The Sound of Music	142	1286
Titanic	136	1224
Avatar	97	878

list the films that belong to each set.

a. The set of films with admissions greater than 150,000,000 *and* the adjusted gross income greater than \$1,000,000,000.

b. The set of films with admissions greater than 150,000,000 *or* the adjusted gross income greater than \$1,000,000,000.

c. The set of films with admissions smaller than 150,000,000 *and* the adjusted gross income greater than \$1,000,000,000.

Solution ►

a. The set of films with admissions greater than 150,000,000 consists of *Gone With the Wind* and *Star Wars*. The set of films with the adjusted gross income greater than 1,000,000,000 consists of *Gone With the Wind*, *Star Wars*, *The Sound of Music*, and *Titanic*. Therefore, the set of films satisfying both of these properties contains of ***Gone With the Wind*** and ***Star Wars***.

b. The set of films with admissions greater than 150,000,000 consists of *Gone With the Wind* and *Star Wars*. The set of films with the adjusted gross income greater than 1,000,000,000 consists of *Gone With the Wind*, *Star Wars*, *The Sound of Music*, and *Titanic*. Therefore, the set of films satisfying at least one of these properties consists of ***Gone With the Wind, Star Wars, The Sound of Music***, and ***Titanic***.

c. The set of films with admissions smaller than 180,000,000 includes *Star Wars*, *The Sound of Music*, *Titanic*, and *Avatar*. The set of films with the adjusted gross income greater than 1,000,000,000 consists of *Gone With the Wind*, *Star Wars*, *The Sound of*

Music, and *Titanic*. Therefore, the set of films satisfying both of these properties consists of ***The Sound of Music***, and ***Titanic***.

L.5 Exercises

Let $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 5\}$, $C = \{5\}$. *Find each set.*

1. $A \cap B$ **2.** $A \cup B$ **3.** $B \cup C$ **4.** $A \setminus B$

5. $A \cap C$ **6.** $A \cup B \cup C$ **7.** $B \cap C$ **8.** $A \cup C$

Write the result of each set operation as a single interval, if possible.

9. $(-7, 3] \cap [1,6]$

10. $(-8, 5] \cap (-1, 13)$

11. $(0,3) \cup (1, 7]$

12. $[-7, 2] \cup (1, 10)$

13. $(-\infty, 13) \cup (1, \infty)$

14. $(-\infty, 1) \cap (2, \infty)$

15. $(-\infty, 1] \cap [1, \infty)$

16. $(-\infty, -1] \cup [1, \infty)$

17. $(-2, \infty) \cup [3, \infty)$

18. $(-2, \infty) \cap [3, \infty)$

Solve each compound inequality. Give the solution set in both interval and graph form.

19. $x + 1 > 6 \quad or \quad 1 - x > 3$

20. $-3x \geq -6 \quad and \quad -2x \leq 12$

21. $4x + 1 < 5 \quad and \quad 4x + 7 > -1$

22. $3y - 11 > 4 \quad or \quad 4y + 9 \leq 1$

23. $3x - 7 < -10 \quad and \quad 5x + 2 \leq 22$

24. $\frac{1}{4}y - 2 < -3 \quad or \quad 1 - \frac{3}{2}y \geq 4$

25. $\begin{cases} 1 - 7x \leq -41 \\ 3x + 1 \geq -8 \end{cases}$

26. $\begin{cases} 2(x + 1) < 8 \\ -2(x - 4) > -2 \end{cases}$

27. $-\frac{2}{3} \leq 3 - \frac{1}{2}a < \frac{2}{3}$

28. $-4 \leq \frac{7 - 3a}{5} \leq 4$

29. $5x + 12 > 2 \quad or \quad 7x - 1 < 13$

30. $4x - 2 > 10 \quad and \quad 8x + 2 \leq -14$

31. $7t - 1 > -1 \quad and \quad 2t - 5 \geq -10$

32. $7z - 6 > 0 \quad or \quad -\frac{1}{2}z \leq 6$

33. $\frac{5x+4}{2} \geq 7 \quad or \quad \frac{7-2x}{3} \geq 2$

34. $\frac{2x-5}{-2} \geq 2 \quad and \quad \frac{2x+1}{3} \geq 0$

35. $13 - 3x > -8 \quad and \quad 12x + 7 \geq -(1 - 10x)$

36. $1 \leq -\frac{1}{3}(4b - 27) \leq 3$

Solve each problem.

37. Two friends plan to drive between 680 and 920 kilometers per day. If they estimate that their average driving speed will be 80 km/h, how many hours per day will they be driving?

38. A substance is in a liquid state if its temperature is between its melting point and its boiling point. The melting point of phosphorus is $44°C$ and its boiling point is $280.5°C$. Using the conversion formula $C = \frac{5}{9}(F - 32)$, determine the rage of temperatures in $°F$ for which phosphorus assumes a liquid state.

39. Kevin's birthday party will cost $400 to rent a banquet hall and an additional $25 for every guest. If Kevin wants to keep the cost of the party between $750 and $1000, how many guests could he invite?

40. Michael works for $15 per hour plus $18 per every overtime hour after the first 40 hours per week. How many hours of overtime must he work to earn between $800 and $1000 per week?

41. The table below shows average tuition fees for full-time international undergraduate and graduate students during the 20018/19 academic year, by field of study.

Field of Study	Undergraduate	Graduate
Education	$19,461	$15,236
Humanities	$26,175	$13,520
Business	$26,395	$22,442
Mathematics	$30,187	$15,553
Dentistry	$55,802	$21,635
Nursing	$20,354	$13,713
Veterinary Medicine	$60,458	$9,088

Which fields of study belong to the following sets:

a. the set of the fields of study that cost less than $25000 for undergraduates *or* less than $15,000 for graduates

b. the set of the fields of study that cost less than $25000 for undergraduates *and* less than $15,000 for graduates

c. the set of the fields of study that cost more than $25000 for undergraduates *or* more than $15,000 for graduates

d. the set of the fields of study that cost more than $25000 for undergraduates *and* more than $15,000 for graduates

L6 Absolute Value Equations and Inequalities

The concept of **absolute value** (also called **numerical value**) was introduced in *Section R2*. Recall that when using geometrical visualisation of real numbers on a number line, the absolute value of a number x, denoted $|x|$, can be interpreted as the distance of the point x from zero. Since distance cannot be negative, the result of absolute value is always nonnegative. In addition, the distance between points x and a can be recorded as $|x-a|$ (see *Definition 2.2* in *Section R2*), which represents the nonnegative difference between the two quantities. In this section, we will take a closer look at absolute value properties, and then apply them to solve absolute value equations and inequalities.

Properties of Absolute Value

The formal definition of absolute value

$$|x| \stackrel{def}{=} \begin{cases} x, & if \quad x \geq 0 \\ -x, & if \quad x < 0 \end{cases}$$

tells us that, when x is nonnegative, the absolute value of x is the same as x, and when x is negative, the absolute value of it is the **opposite** of x.

So, $|2| = 2$ and $|-2| = -(-2) = 2$. Observe that this complies with the notion of a distance from zero on a number line. Both numbers, 2 and -2 are at a distance of 2 units from zero. They are both solutions to the equation $|x| = 2$.

Since $|x|$ represents the distance of the number x from 0, which is never negative, we can claim the first absolute value property:

$|x| \geq 0$, for any real x

Here are several other absolute value properties that allow us to simplify algebraic expressions.

Let x and y are any real numbers. Then

$|x| = 0$ if and only if $x = 0$

Only zero is at the distance zero from zero.

$|-x| = |x|$

The distance of opposite numbers from zero is the same.

$|xy| = |x||y|$

Absolute value of a product is the product of absolute values.

$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ for $y \neq 0$

Absolute value of a quotient is the quotient of absolute values.

Attention: *Absolute value doesn't 'split' over addition or subtraction! That means*

$$|\boldsymbol{x} \pm \boldsymbol{y}| \neq |\boldsymbol{x}| \pm |\boldsymbol{y}|$$

For example, $|2 + (-3)| = 1 \neq 5 = |2| + |-3|$.

Example 1 ► **Simplifying Absolute Value Expressions**

Simplify, leaving as little as possible inside each absolute value sign.

a. $|-2x|$

b. $|3x^2y|$

c. $\left|\frac{-a^2}{2b}\right|$

d. $\left|\frac{-1+x}{4}\right|$

Solution ►

a. Since absolute value can 'split' over multiplication, we have

$$|-2x| = |-2||x| = \boldsymbol{2|x|}$$

b. Using the multiplication property of absolute value and the fact that x^2 is never negative, we have

$$|3x^2y| = |3||x^2||y| = \boldsymbol{3x^2|y|}$$

c. Using properties of absolute value, we have

$$\left|\frac{-a^2}{2b}\right| = \frac{|-1||a^2|}{|2||b|} = \boldsymbol{\frac{a^2}{2|b|}}$$

c. Since absolute value does not 'split' over addition, the only simplification we can perform here is to take 4 outside of the absolute value sign. So, we have

$$\left|\frac{-1+x}{4}\right| = \boldsymbol{\frac{|x-1|}{4}} \text{ or equivalently } \boldsymbol{\frac{1}{4}|x-1|}$$

Remark: *Convince yourself that* $|x - 1|$ *is not equivalent to* $x + 1$ *by evaluating both expressions at, for example,* $x = 1$.

Absolute Value Equations

The formal definition of absolute value (see *Definition 2.1* in *Section R2*) applies not only to a single number or a variable x but also to any algebraic expression. Generally, we have

$$|expr.| \overset{def}{=} \begin{cases} expr., & \text{if } expr. \geq 0 \\ -(expr.), & \text{if } expr. < 0 \end{cases}$$

This tells us that, when an *expression* is nonnegative, the absolute value of the *expression* is the **same** as the ***expression***, and when the *expression* is negative, the absolute value of the *expression* is the **opposite** of the ***expression***.

For example, to evaluate $|x-1|$, we consider when the expression $x-1$ is nonnegative and when it is negative. Since $x-1 \geq 0$ for $x \geq 1$, we have

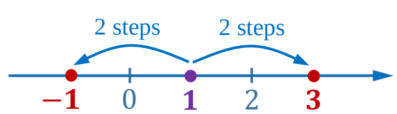
$$|x-1| = \begin{cases} x-1, & for\ x \geq 1 \\ -(x-1), & for\ x < 1 \end{cases}$$

Notice that both expressions, $x-1\ for\ x \geq 1$ and $-(x-1)\ for\ x<1$ produce nonnegative values that represent the distance of a number x from 0.

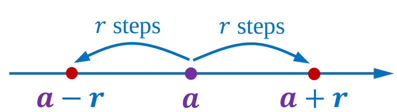
In particular,

if $x=3$, then $|\boldsymbol{x}-\mathbf{1}| = x-1 = 3-1 = \mathbf{2}$,
and
if $x=-1$, then $|\boldsymbol{x}-\mathbf{1}| = -(x-1) = -(-1-1) = -(-2) = \mathbf{2}$.

As illustrated on the number line below, both numbers, **3** and **−1** are at the distance of **2** units from **1**.



Generally, the equation $|\boldsymbol{x}-\boldsymbol{a}| = \boldsymbol{r}$ tells us that the distance between $\boldsymbol{x}$ and $\boldsymbol{a}$ is equal to $\boldsymbol{r}$. This means that $\boldsymbol{x}$ is $\boldsymbol{r}$ units away from number $\boldsymbol{a}$, in either direction.



Therefore, $\boldsymbol{x} = \boldsymbol{a}-\boldsymbol{r}$ and $\boldsymbol{x} = \boldsymbol{a}+\boldsymbol{r}$ are the solutions of the equation $|\boldsymbol{x}-\boldsymbol{a}| = \boldsymbol{r}$.

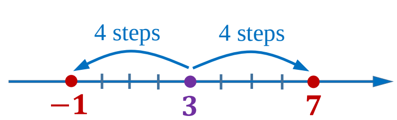
Example 1 ► **Solving Absolute Value Equations Geometrically**

For each equation, state its geometric interpretation, illustrate the situation on a number line, and then find its solution set.

a. $|x-3| = 4$ **b.** $|x+5| = 3$

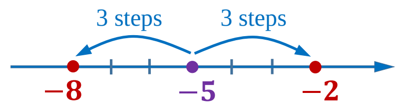
Solution ►

a. Geometrically, $|x-3|$ represents the distance between x and 3. Thus, in $|x-3| = 4$, x is a number whose distance from 3 is 4. So, $x = 3 \pm 4$, which equals either -1 or 7.



Therefore, the solution set is $\{-\mathbf{1}, \mathbf{7}\}$.

b. By rewriting $|x+5|$ as $|x-(-5)|$, we can interpret this expression as the distance between x and -5. Thus, in $|x+5| = 3$, x is a number whose distance from -5 is 3. Thus, $x = -5 \pm 3$, which results in -8 or -2.



Therefore, the solution set is $\{\mathbf{-8}, \mathbf{-2}\}$.

Although the geometric interpretation of absolute value proves to be very useful in solving some of the equations, it can be handy to have an algebraic method that will allow us to solve any type of absolute value equation.

Suppose we wish to solve an equation of the form

$$|\boldsymbol{expr.}| = \boldsymbol{r}, \quad \text{where } r > 0$$

We have two possibilities. Either the $\boldsymbol{expression}$ inside the absolute value bars is nonnegative, or it is negative. By definition of absolute value, if the $\boldsymbol{expression}$ is nonnegative, our equation becomes

$$\boldsymbol{expr.} = \boldsymbol{r}$$

If the $\boldsymbol{expression}$ is negative, then to remove the absolute value bar, we must change the sign of the $\boldsymbol{expression}$. So, our equation becomes

$$-\boldsymbol{expr.} = \boldsymbol{r},$$

which is equivalent to

$$\boldsymbol{expr.} = -\boldsymbol{r}$$

In summary, for $\boldsymbol{r} > 0$, the equation

$$|\boldsymbol{expr.}| = \boldsymbol{r}$$

is equivalent to the system of equations with the connecting word *or*.

$$\boldsymbol{expr.} = \boldsymbol{r} \quad \text{or} \quad \boldsymbol{expr.} = -\boldsymbol{r},$$

If $\boldsymbol{r} = 0$, then $|\boldsymbol{expr.}| = \mathbf{0}$ is equivalent to the equation $\boldsymbol{expr.} = \mathbf{0}$ with no absolute value.

If $\boldsymbol{r} < 0$, then $|\boldsymbol{expr.}| = \boldsymbol{r}$ has **NO SOLUTION**, as an absolute value is never negative.

Now, suppose we wish to solve an equation of the form

$$|\boldsymbol{expr.A}| = |\boldsymbol{expr.B}|$$

Since both expressions, $\boldsymbol{A}$ and $\boldsymbol{B}$, can be either nonnegative or negative, when removing absolute value bars, we have four possibilities:

$$\boldsymbol{expr.A} = \boldsymbol{expr.B} \quad \text{or} \quad \boldsymbol{expr.A} = -\boldsymbol{expr.B}$$
$$-\boldsymbol{expr.A} = -\boldsymbol{expr.B} \quad \text{or} \quad -\boldsymbol{expr.A} = \boldsymbol{expr.B}$$

However, observe that the equations in blue are equivalent. Also, the equations in green are equivalent. So, in fact, it is enough to consider just the first two possibilities.

Therefore, the equation

$$|expr.A| = |expr.B|$$

is equivalent to the system of equations with the connecting word *or*.

$$expr.A = expr.B \text{ or } expr.A = -(expr.B),$$

Example 2 ► **Solving Absolute Value Equations Algebraically**

Solve the following equations.

a. $|2 - 3x| = 7$ **b.** $5|x| - 3 = 12$

c. $\left|\frac{1-x}{4}\right| = 0$ **d.** $|6x + 5| = -4$

e. $|2x - 3| = |x + 5|$ **f.** $|x - 3| = |3 - x|$

Solution ►

a. To solve $|2 - 3x| = 7$, we remove the absolute value bars by changing the equation into the corresponding system of equations with no absolute value anymore. Then, we solve the resulting linear equations. So, we have

$$|2 - 3x| = 7$$

$$2 - 3x = 7 \quad \text{or} \quad 2 - 3x = -7$$

$$2 - 7 = 3x \quad \text{or} \quad 2 + 7 = 3x$$

$$x = \frac{-5}{3} \quad \text{or} \quad x = \frac{9}{3} = 3$$

Therefore, the solution set of this equation is $\left\{-\frac{5}{3}, 3\right\}$.

b. To solve $5|x| - 3 = 12$, first, we **isolate the absolute value**, and then replace the equation by the corresponding system of two linear equations.

$$5|x| - 3 = 12$$

$$5|x| = 15$$

$$|x| = 3$$

$$x = 3 \quad \text{or} \quad x = -3$$

So, the solution set of the given equation is $\{-3, 3\}$.

c. By properties of absolute value, $\left|\frac{1-x}{4}\right| = 0$ if and only if $\frac{1-x}{4} = 0$, which happens when the numerator $1 - x = 0$. So, the only solution to the given equation is $x = \mathbf{1}$.

d. Since an absolute value is never negative, the equation $|6x + 5| = -4$ does not have any solution.

e. To solve $|2x-3| = |x+5|$, we remove the absolute value symbols by changing the equation into the corresponding system of linear equations with no absolute value. Then, we solve the resulting equations. So, we have

$$|2x-3| = |x+5|$$

$$\begin{aligned} 2x-3 &= x+5 \quad &&\text{or} \quad 2x-3 = -(x+5) \\ 2x-x &= 5+3 \quad &&\text{or} \quad 2x-3 = -x-5 \\ x &= 8 \quad &&\text{or} \quad 3x = -2 \\ & &&\phantom{\text{or}} \quad x = \frac{-2}{3} \end{aligned}$$

Therefore, the solution set of this equation is $\left\{-\frac{2}{3}, 8\right\}$.

f. We solve $|x-3| = |3-x|$ as in *Example 2e*.

$$|x-3| = |3-x|$$

$$\begin{aligned} x-3 &= 3-x \quad &&\text{or} \quad x-3 = -(3-x) \\ 2x &= 6 \quad &&\text{or} \quad x-3 = -3+x \\ x &= 3 \quad &&\text{or} \quad 0 = 0 \end{aligned}$$

Since the equation $0 = 0$ is always true, any real x-value satisfies the original equation $|x-3| = |3-x|$. So, the solution set to the original equation is $\mathbb{R}$.

Remark: *Without solving the equation in Example 2f, one could observe that the expressions $x-3$ and $3-x$ are opposite to each other and as such, they have the same absolute value. Therefore, the equation is always true.*

Summary of Solving Absolute Value Equations

Step 1 **Isolate the absolute value** expression on one side of the equation.

Step 2 **Check for special cases**, such as

$$|A| = 0 \leftrightarrow A = 0$$
$$|A| = -r \rightarrow \text{No solution}$$

Step 2 **Remove the absolute value symbol** by replacing the equation with the corresponding system of equations with the joining word *or*,

$$|A| = r \quad (r > 0) \qquad\qquad |A| = |B|$$

$$A = r \ \text{ or } \ A = -r \qquad\qquad A = B \ \text{ or } \ A = -B$$

Step 3 **Solve** the resulting equations.

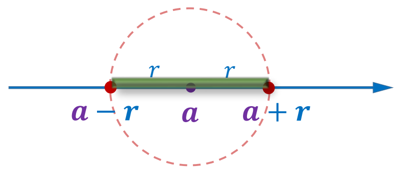
Step 4 **State the solution set** as a union of the solutions of each equation in the system.

Absolute Value Inequalities with One Absolute Value Symbol

GEOMETRIC VISUALIZATION

Suppose we wish to solve inequalities of the form $|x-a|<r$ or $|x-a|>r$, where r is a positive real number. Similarly as in the case of absolute value equations, we can either use a geometric interpretation with the aid of a number line, or we can rely on an algebraic procedure.

Using the geometrical visualization of $|x-a|$ as the distance between x and a on a number line, the inequality $|x-a|<r$ tells us that the number x is less than r units from number a. One could think of drawing a circle centered at a, with radius r. Then, the solutions of the inequality $|x-a|<r$ are all the points on a number line that lie inside such a circle (*see the green segment below*).



Therefore, the solution set is the interval $(a-r, a+r)$.

This result can be achieved algebraically by rewriting the absolute value inequality

$$|x-a|<r$$

in an equivalent three-part inequality form

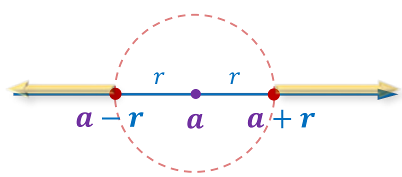
$$-r<x-a<r,$$

and then solving it for x

$$a-r<x<a+r,$$

which confirms that the solution set is indeed $(a-r, a+r)$.

Similarly, the inequality $|x-a|>r$ tells us that the number x is more than r units from number a. As illustrated in the diagram below, the solutions of this inequality are all points on a number line that lie outside of the circle centered at a, with radius r.



Therefore, the solution set is the union $(-\infty, a-r) \cup (a+r, \infty)$.

As before, this result can be achieved algebraically by rewriting the absolute value inequality

$$|x-a|>r$$

in an equivalent system of two inequalities joined by the word *or*

$$x-a<-r \quad \text{or} \quad r<x-a,$$

and then solving it for x

$$x < a - r \quad or \quad a + r < x,$$

which confirms that the solution set is $(-\infty, a - r) \cup (a + r, \infty)$.

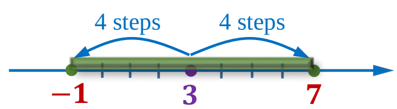
Example 3 ► **Solving Absolute Value Inequalities Geometrically**

For each inequality, state its geometric interpretation, illustrate the situation on a number line, and then find its solution set.

a. $|x - 3| \leq 4$ **b.** $|x + 5| > 3$

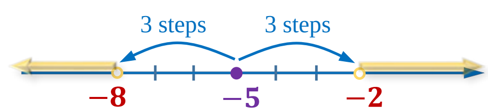
Solution ►

a. Geometrically, $|x - 3|$ represents the distance between x and 3. Thus, in $|x - 3| \leq 4$, x is a number whose distance from 3 is at most 4, in either direction.
So, $3 - 4 \leq x \leq 3 + 4$, which is equivalent to $-1 \leq x \leq 7$.



Therefore, the solution set is $[-\mathbf{1}, \mathbf{7}]$.

b. By rewriting $|x + 5|$ as $|x - (-5)|$, we can interpret this expression as the distance between x and -5. Thus, in $|x + 5| > 3$, x is a number whose distance from -5 is more than 3, in either direction. Thus, $x < -5 - 3$ or $-5 + 3 < x$, which results in $x < -8$ or $x > -2$.



Therefore, the solution set equals $(-\infty, -\mathbf{8}) \cup (-\mathbf{2}, \infty)$.

The algebraic strategy can be applied to any inequality of the form

$$|\boldsymbol{expr.}| < (\leq) \boldsymbol{r}, \quad \text{or} \quad |\boldsymbol{expr.}| > (\geq) \boldsymbol{r}, \quad \text{as long as } \boldsymbol{r} > \mathbf{0}.$$

Depending on the type of inequality, we follow these rules:

$|\boldsymbol{expr.}| < \boldsymbol{r}$ → $-\boldsymbol{r} < \boldsymbol{expr.} < \boldsymbol{r}$

or

$|\boldsymbol{expr.}| > \boldsymbol{r}$ → $\boldsymbol{expr.} < -\boldsymbol{r}$ or $\boldsymbol{r} < \boldsymbol{expr.}$

These rules also apply to weak inequalities, such as $\leq$ or $\geq$.

In the above rules, we assume that $\boldsymbol{r} > \mathbf{0}$. **What if** $\boldsymbol{r} = \mathbf{0}$ **?**

Observe that, the inequality $|\boldsymbol{expr.}| < \mathbf{0}$ is never true, so this inequality doesn't have any solution. Since $|\boldsymbol{expr.}| < \mathbf{0}$ is never true, the inequality $|\boldsymbol{expr.}| \leq \mathbf{0}$ is equivalent to the equation $|\boldsymbol{expr.}| = \mathbf{0}$.

On the other hand, $|\boldsymbol{expr.}| \geq \mathbf{0}$ is always true, so the solution set equals to $\mathbb{R}$. However, since $|\boldsymbol{expr.}|$ is either positive or zero, the solution to $|\boldsymbol{expr.}| > \mathbf{0}$ consists of all real numbers except for the solutions of the equation $\boldsymbol{expr.} = \mathbf{0}$.

What if $r < 0$?

Observe that both inequalities $|\boldsymbol{expr.}| > \boldsymbol{negative}$ and $|\boldsymbol{expr.}| \geq \boldsymbol{negative}$ are always true, so the solution set of such inequalities is equal to $\mathbb{R}$.

On the other hand, both inequalities $|\boldsymbol{expr.}| < \boldsymbol{negative}$ and $|\boldsymbol{expr.}| \leq \boldsymbol{negative}$ are never true, so such inequalities result in **NO SOLUTION**.

Example 4 ► **Solving Absolute Value Inequalities with One Absolute Value Symbol**

Solve each inequality. Give the solution set in both interval and graph form.

a. $|5x + 9| \leq 4$ **b.** $|-2x - 5| > 1$

e. $16 \leq |2x - 3| + 9$ **f.** $1 - 2|4x - 7| > -5$

Solution ► **a.** To solve $|5x + 9| \leq 4$, first, we remove the absolute value symbol by rewriting the inequality in the three-part inequality, as below.

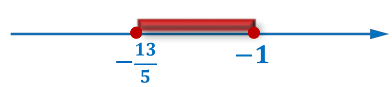
$$|5x + 9| \leq 4$$

$$-4 \leq 5x + 9 \leq 4 \qquad / -9$$

$$-13 \leq 5x \leq -5 \qquad / \div 5$$

$$-\frac{13}{5} \leq x \leq -1$$

The solution is shown in the graph below.



The inequality is satisfied by all $x \in \left[-\frac{13}{5}, -\mathbf{1}\right]$.

b. As in the previous example, first, we remove the absolute value symbol by replacing the inequality with the corresponding system of inequalities, joined by the word *or*. So, we have

$$|-2x - 5| > 1$$

$$-2x - 5 < -1 \quad \text{or} \quad 1 < -2x - 5 \qquad / +5$$

$$-2x < 4 \quad \text{or} \quad 6 < -2x \qquad / \div(-2)$$

$$x > -2 \quad \text{or} \quad -3 > x$$

The joining word *or* indicates that we look for the union of the obtained solutions. This union is shown in the graph below.

The inequality is satisfied by all $x \in (-\infty, -\mathbf{3}) \cup (-\mathbf{2}, \infty)$.

c. To solve $16 \leq |2x-3|+9$, first, we **isolate the absolute value**, and then replace the inequality with the corresponding system of two linear equations. So, we have

$$16 \leq |2x-3|+9 \qquad / -9$$

$$7 \leq |2x-3|$$

$$2x-3 \leq -7 \quad \textit{or} \quad 7 \leq 2x-3 \qquad / +3$$

$$2x \leq -4 \quad \textit{or} \quad 2x \geq 10 \qquad / \div 2$$

$$x \leq -2 \quad \textit{or} \quad x \geq 5$$

The joining word *or* indicates that we look for the union of the obtained solutions. This union is shown in the graph below.

So, the inequality is satisfied by all $x \in (-\infty, -\mathbf{2}) \cup (\mathbf{5}, \infty)$.

d. As in the previous example, first, we **isolate the absolute value**, and then replace the inequality with the corresponding system of two inequalities.

$$1-2|4x-7| > -5 \qquad / -1$$

$$-2|4x-7| > -6 \qquad / \div(-2)$$

! **reverse the signs**

$$|4x-7| < 3$$

$$-3 < 4x-7 < 3 \qquad / +7$$

$$4 < 4x < 10 \qquad / \div 4$$

$$1 < x < \frac{10}{4} = \frac{5}{2}$$

So the solution set is the interval $\left(\mathbf{1}, \frac{\mathbf{5}}{\mathbf{2}}\right)$, visualized in the graph below.

Example 5 ► **Solving Absolute Value Inequalities in Special Cases**

Solve each inequality.

a. $\left|\frac{1}{2}x+\frac{5}{3}\right| \geq -3$ **b.** $|4x-7| \leq 0$

c. $|3-4x| > 0$ **d.** $1-2\left|\frac{3}{2}x-5\right| > 3$

Solution ►

a. Since an absolute value is always bigger or equal to zero, the inequality $\left|\frac{1}{2}x+\frac{5}{3}\right| \geq -3$ is always true. Thus, it is satisfied by **any real number**. So the solution set is $\mathbb{R}$.

b. Since $|4x - 7|$ is never negative, the inequality $|4x - 7| \le 0$ is satisfied only by solutions to the equation $|4x - 7| = 0$. So, we solve

$$|4x - 7| = 0$$

$$4x - 7 = 0 \qquad / +7$$

$$4x = 7 \qquad / \div 4$$

$$x = \frac{7}{4}$$

Therefore, the inequality is satisfied only by $\boldsymbol{x = \frac{7}{4}}$.

c. Inequality $|3 - 4x| > 0$ is satisfied by all real x-values except for the solution to the equation $3 - 4x = 0$. Since

$$3 - 4x = 0 \qquad / +4x$$

$$3 = 4x \qquad / \div 4$$

$$\frac{3}{4} = x,$$

then the solution to the original inequality is $\boldsymbol{\left(-\infty, \frac{3}{4}\right) \cup \left(\frac{3}{4}, \infty\right)}$.

d. To solve $1 - 2\left|\frac{3}{2}x - 5\right| > 3$, first, we **isolate the absolute value**. So, we have

$$1 - 2\left|\frac{3}{2}x - 5\right| > 3 \qquad / +2\left|\frac{3}{2}x - 5\right|, -3$$

$$-2 > 2\left|\frac{3}{2}x - 5\right| \qquad / \div 2$$

$$-1 > \left|\frac{3}{2}x - 5\right|$$

Since $\left|\frac{3}{2}x - 5\right|$ is never negative, it can't be less than -1. So, there is **no solution** to the original inequality.

Summary of Solving Absolute Value Inequalities with One Absolute Value Symbol

Let $\boldsymbol{r}$ be a positive real number. To solve absolute value inequalities with one absolute value symbol, follow the steps:

- **Isolate the absolute value** expression on one side of the inequality.
- **Check for special cases**, such as

$|A| < 0$ → **No solution**
$|A| \le 0$ ↔ $\boldsymbol{A = 0}$
$|A| \ge 0$ → **All real numbers**
$|A| > 0$ → **All real numbers except for solutions of** $\boldsymbol{A = 0}$
$|A| > (\ge) -r$ → **All real numbers**
$|A| < (\le) -r$ → **No solution**

- **Remove the absolute value symbol** by replacing the equation with the corresponding system of equations as below:

$$|A| < r \quad\Rightarrow\quad -r < A < r$$

$$|A| > r \quad\Rightarrow\quad A < -r \text{ or } r < A$$

This also applies to weak inequalities, such as $\leq$ or $\geq$.

- **Solve** the resulting equations.
- **State the solution set** as a union of the solutions of each equation in the system.

Applications of Absolute Value Inequalities

One of the typical applications of absolute value inequalities is in error calculations. When discussing errors in measurements, we refer to the *absolute error* or the *relative error*. For example, if M is the actual measurement of an object and x is the approximated measurement, then the ***absolute error*** is given by the formula $|x - M|$ and the ***relative error*** is calculated according to the rule $\frac{|x-M|}{M}$.

In quality control situations, the relative error often must be less than some predetermined amount. For example, suppose a machine that fills two-litre milk cartons is set for a relative error no greater than 1%. We might be interested in how much milk a two-litre carton can actually contain? What is the absolute error that this machine is allowed to make?

Since $M = 2$ litres and the relative error = 1% $= 0.01$, we are looking for all x-values that would satisfy the inequality

$$\frac{|x-2|}{2} < 0.01.$$

This is the **relative error**.

This is equivalent to

$$|x-2| < 0.02$$

This is the **absolute error**.

$$-0.02 < x - 2 < 0.02$$

$$1.98 < x < 2.02,$$

so, a two-litre carton of milk can contain any amount of milk between 1.98 and 2.02 litres. The absolute error in this situation is $0.02\ l = 20\ ml$.

Example 6 ► **Solving Absolute Value Application Problems**

A scale is considered to be accurate if it measures the weight of an object up to 0.1% of its actual weight. Suppose x represents the reading on the scale when weighing a 20 kg object. Find the set of all possible x-readings.

Solution ► The difference (*error*) between the scale reading and the actual weight can be expressed as $|x - 20|$. Since the allowable error in the scale reading with respect to the actual weight (the ***relative error***) is smaller than 0.1% $= 0.001$, then x must satisfy the inequality

$$\frac{|x-20|}{20} < 0.001$$

After solving it for x,

$$|x-20| < 0.02$$

$$-0.02 < x - 20 < 0.02$$

$$19.98 < x < 20.02$$

So, the scale readings when weighing a 20 kg object can range between **19.98** and **20.02** kg. This tells us that the scale may err up to 0.02 kg = 2 dkg when weighing this object.

L.6 Exercises

Simplify, if possible, leaving as little as possible inside the absolute value symbol.

1. $|-2x^2|$ **2.** $|3x|$ **3.** $\left|\frac{-5}{y}\right|$ **4.** $\left|\frac{3}{-y}\right|$

5. $|7x^4y^3|$ **6.** $|-3x^5y^4|$ **7.** $\left|\frac{x^2}{y}\right|$ **8.** $\left|\frac{-4x}{y^2}\right|$

9. $\left|\frac{-3x^3}{6x}\right|$ **10.** $\left|\frac{5x^2}{-25x}\right|$ **11.** $|(x-1)^2|$ **12.** $|x^2-1|$

13. *In each situation, find the* ***number of solutions*** *for the equation* $|\boldsymbol{ax} + \boldsymbol{b}| = \boldsymbol{k}$.

a. $k < 0$ **b.** $k = 0$ **c.** $k > 0$

14. *Match each absolute value equation or inequality in Column I with the graph of its solution set in Column II.*

Column I	Column II
a. $|x| = 3$	**A.** 
b. $|x| > 3$	**B.** 
c. $|x| < 3$	**C.** 
d. $|x| \geq 3$	**D.** 
e. $|x| \leq 3$	**E.** 

Solve each equation.

15. $|-x| = 4$

16. $|5x| = 20$

17. $|y-3| = 8$

18. $|2y+5| = 9$

19. $7|3x-5| = 35$

20. $-3|2x-7| = -12$

21. $\left|\frac{1}{2}x+3\right| = 11$

22. $\left|\frac{2}{3}x-1\right| = 5$

23. $|2x-5| = -1$

24. $|7x+11| = 0$

25. $2+3|a| = 8$

26. $10-|2a-1| = 4$

27. $\left|\frac{2x-1}{3}\right| = 5$

28. $\left|\frac{3-5x}{6}\right| = 3$

29. $|2p+4| = |3p-1|$

30. $|5-q| = |q+7|$

31. $\left|\frac{1}{2}x+3\right| = \left|\frac{1}{5}x-1\right|$

32. $\left|\frac{2}{3}x-8\right| = \left|\frac{1}{6}x+3\right|$

33. $\left|\frac{3x-6}{2}\right| = \left|\frac{5+x}{5}\right|$

34. $\left|\frac{6-5x}{4}\right| = \left|\frac{7+3x}{3}\right|$

Solve *each inequality. Give the solution set in both* ***interval*** *and* ***graph*** *form.*

35. $|x+4| < 3$

36. $|x-5| > 7$

37. $|x-12| \geq 5$

38. $|x+14| \leq 5$

39. $|5x+3| \leq 8$

40. $|3x-2| \geq 10$

41. $|7-2x| > 5$

42. $|-5x+4| < 3$

43. $\left|\frac{1}{4}y-6\right| \leq 24$

44. $\left|\frac{2}{5}x+3\right| > 5$

45. $\left|\frac{3x-2}{4}\right| \geq 10$

46. $\left|\frac{2x+3}{3}\right| < 10$

47. $|-2x+4|-8 \geq -5$

48. $|6x-2|+3 < 9$

49. $7-2|x+4| \geq 5$

50. $9-3|x-2| < 3$

Solve each inequality.

51. $\left|\frac{2}{3}x+4\right| \leq 0$

52. $\left|-2x+\frac{4}{5}\right| > 0$

53. $\left|\frac{6x-2}{5}\right| < -3$

54. $|-3x+5| > -3$

55. $|-x+4|+5 \geq 4$

56. $|4x+1|-2 < -5$

Solve each problem.

57. The recommended dosage of daily intake of magnesium for a healthy adult is 370 mg with a tolerance of up to 50 mg.

a. Write an absolute value inequality that describes the recommended intake of magnesium, M, in milligrams per day.

b. Using the inequality from part (**a**), find the range of the recommended number of milligrams of magnesium intake per day.

58. The average income of an employee at a tire shop is \$36,000 per year. Patrik works in this shop, but his earnings are not within \$8000 of this average.

a. Write an absolute value inequality that describes Patrik's income, I, in this situation.

b. Solve the inequality created in part (**a**) to find the possible amounts for Patrik's annual earnings. *Assume that his earnings are positive.*

59. The recommended daily intake of calcium for females over 50 years old is 1200 mg with a tolerance of up to 100 mg. Use absolute value inequality to describe the recommended daily calcium intake C for this group of people. Then, solve the inequality to find the range of values for this daily calcium intake.

60. A police radar has a maximum allowable error of 2 km/h.

a. Use absolute value inequality to record the range of values of the actual speed s of the car if its radar reading is 63 km/h.

b. Solve the inequality in part (a) to find the minimum and maximum possible speeds for this car.

61. The speed limit on a freeway is 110 km/h. If cars are required to travel at least 60 km/h to use a freeway, what absolute value inequality describes the permitted rates r of mooving on this freeway?

62. The body temperature T of a healthy adult person is expected to be between 36.4°C and 37.6°C. Record this fact using an absolute value inequality.

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