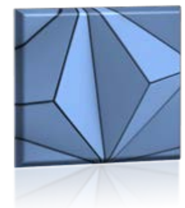
L.5 Operations on Sets and Compound Inequalities

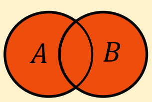


In the previous section, it was shown how to solve a three-part inequality. In this section, we will study how to solve systems of inequalities, often called **compound** inequalities, that consist of two linear inequalities joined by the words "*and*" or "*or*". Studying compound inequalities is an extension of studying three-part inequalities. For example, the three-part inequality, $2 < x \leq 5$, is in fact a system of two inequalities, $2 < x$ *and* $x \leq 5$. The solution set to this system of inequalities consists of all numbers that are larger than 2 *and* at the same time are smaller or equal to 5. However, notice that the system of the same two inequalities connected by the word "*or*", $2 < x$ *or* $x \leq 5$, is satisfied by any real number. This is because any real number is either larger than 2 *or* smaller than 5. Thus, to find solutions to compound inequalities, aside for solving each inequality individually, we need to pay attention to the joining words, "*and*" or "*or*". These words suggest particular operations on the sets of solutions.

Operations on Sets

Sets can be added, subtracted, or multiplied.

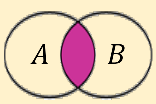
Definition 5.1 ► The result of the addition of two sets $\boldsymbol{A}$ and $\boldsymbol{B}$ is called a **union** (or sum), symbolized by $\boldsymbol{A \cup B}$, and defined as:



$$\boldsymbol{A \cup B = \{x \mid x \in A \text{ or } x \in B\}}$$

This is the set of all elements that belong to either $\boldsymbol{A}$ ***or*** $\boldsymbol{B}$.

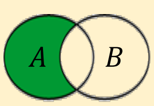
The result of the multiplication of two sets $\boldsymbol{A}$ and $\boldsymbol{B}$ is called an **intersection** (or product, or **common part**), symbolized by $\boldsymbol{A \cap B}$, and defined as:



$$\boldsymbol{A \cap B = \{x \mid x \in A \text{ and } x \in B\}}$$

This is the set of all elements that belong to both $\boldsymbol{A}$ ***and*** $\boldsymbol{B}$.

The result of the subtraction of two sets $\boldsymbol{A}$ and $\boldsymbol{B}$ is called a **difference**, symbolized by $\boldsymbol{A \setminus B}$, and defined as:



$$\boldsymbol{A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}}$$

This is the set of all elements that belong to $\boldsymbol{A}$ ***and*** do not belong to $\boldsymbol{B}$.

Example 1 ► **Performing Operations on Sets**

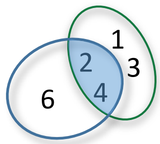
Suppose $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6\}$, and $C = \{6\}$. Find the following sets:

a. $A \cap B$ **b.** $A \cup B$

c. $A \setminus B$ **d.** $B \cup C$

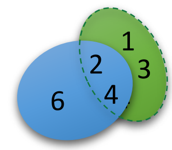
e. $B \cap C$ **f.** $A \cap C$

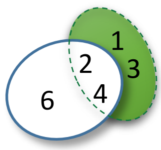
Solution ►



a. The intersection of sets $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$ consists of numbers that belong to both sets, so $\boldsymbol{A \cap B} = \{\mathbf{2}, \mathbf{4}\}$.

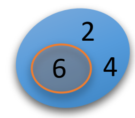
b. The union of sets $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$ consists of numbers that belong to at least one of the sets, so $\boldsymbol{A \cup B} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{6}\}$.



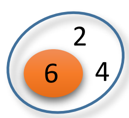


c. The difference $A \setminus B$ of sets $A = \{1, \cancel{2}, 3, \cancel{4}\}$ and $B = \{2, 4, 6\}$ consists of numbers that belong to the set A but do not belong to the set B, so $\boldsymbol{A \setminus B} = \{\mathbf{1}, \mathbf{3}\}$.

d. The union of sets $B = \{2, 4, 6\}$, and $C = \{6\}$ consists of numbers that belong to at least one of the sets, so $\boldsymbol{B \cup C} = \{\mathbf{2}, \mathbf{4}, \mathbf{6}\}$.



Notice that $B \cup C = B$. This is because C is a **subset** of B.

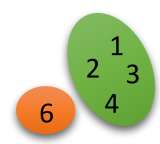


e. The intersection of sets $B = \{2, 4, 6\}$ and $C = \{6\}$ consists of numbers that belong to both sets, so $\boldsymbol{B \cap C} = \{\mathbf{6}\}$.

Notice that $B \cap C = C$. This is because C is a **subset** of B.

f. Since the sets $A = \{1, 2, 3, 4\}$ and $C = \{6\}$ do not have any common elements, then $\boldsymbol{A \cap C} = \emptyset$.

Recall: Two sets with no common part are called **disjoint**.
Thus, the sets A and C are disjoint.



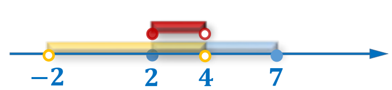
Example 2 ► **Finding Intersections and Unions of Intervals**

Write the result of each set operation as a single interval, if possible.

a. $(-2,4) \cap [2,7]$ **b.** $(-\infty, 1] \cap (-\infty, 3)$
c. $(-1,3) \cup (1,6]$ **d.** $(3,\infty) \cup [5,\infty)$
e. $(-\infty,3) \cup (4,\infty)$ **f.** $(-\infty,3) \cap (4,\infty)$
g. $(-\infty,5) \cup (4,\infty)$ **h.** $(-\infty,3] \cap [3,\infty)$

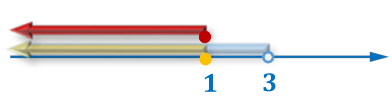
Solution ►

a. The interval of points that belong to both, the interval $(-2,4)$, in yellow, and the interval $[2,7]$, in blue, is marked in red in the graph below.



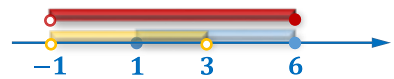
So, we write $(-2,4) \cap [2,7] = [2,4)$.

b. As in problem **a.**, the common part $(-\infty, 1] \cap (-\infty, 3)$ is illustrated in red on the graph below.



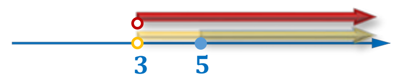
So, we write $(-\infty, 1] \cap (-\infty, 3) = (-\infty, 1]$.

c. This time, we take the union of the interval $(-1,3)$, in yellow, and the interval $(1,6]$, in blue. The result is illustrated in red on the graph below.



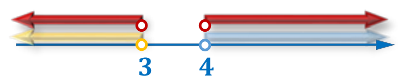
So, we write $(-1,3) \cup (1,6] = (-1,6]$.

d. The union $(3,\infty) \cup [5,\infty)$ is illustrated in red on the graph below.



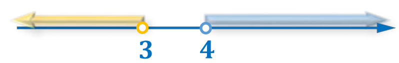
So, we write $(3,\infty) \cup [5,\infty) = (3,\infty)$.

e. As illustrated in the graph below, this time, the union $(-\infty,3) \cup (4,\infty)$ can't be written in the form of a single interval.



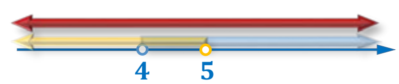
So, the expression $(-\infty,3) \cup (4,\infty)$ cannot be simplified.

f. As shown in the graph below, the interval $(-\infty,3)$ has no common part with the interval $(4,\infty)$.



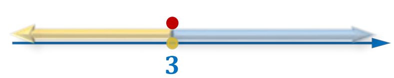
Therefore, $(-\infty,3) \cap (4,\infty) = \emptyset$

g. This time, the union $(-\infty,5) \cup (4,\infty)$ covers the entire number line.



Therefore, $(-\infty,5) \cup (4,\infty) = (-\infty,\infty)$

h. As shown in the graph below, there is only one point common to both intervals, $(-\infty,3]$ and $[3,\infty)$. This is the number 3.



Since a single number is not considered to be an interval, we should use a set rather than interval notation when recording the answer. So, $(-\infty,3] \cap [3,\infty) = \{3\}$.

Compound Inequalities

The solution set to a system of two inequalities joined by the word *and* is the intersection of solutions of each inequality in the system. For example, the solution set to the system

$$\begin{cases} x > 1 \\ x \leq 4 \end{cases}$$

is the intersection of the solution set for $x > 1$ and the solution set for $x \leq 4$. This means that the solution to the above system equals $(1, \infty) \cap (-\infty, 4] = (1, 4]$, as illustrated in the graph below.

The solution set to a system of two inequalities joined by the word *or* is the union of solutions of each inequality in the system. For example, the solution set to the system

$$x \leq 1 \text{ or } x > 4$$

is the union of the solution of $x \leq 1$ and, the solution of $x > 4$. This means that the solution to the above system equals $(-\infty, 1) \cup [4, \infty)$, as illustrated in the graph below.

Example 4 ► **Solving Compound Linear Inequalities**

Solve each compound inequality. Pay attention to the joining word *and* or *or* to find the overall solution set. Give the solution set in both interval and graph form.

a. $3x + 7 \geq 4 \quad and \quad 2x - 5 < -1$

b. $-2x - 5 \geq 1 \quad or \quad x - 5 \geq -3$

c. $3x - 11 < 4 \quad or \quad 4x + 9 \geq 1$

d. $-2 < 3 - \frac{1}{4}x < \frac{1}{2}$

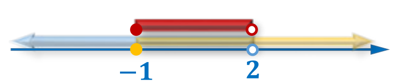
e. $\begin{cases} 4x - 7 < 1 \\ 7 - 3x > -8 \end{cases}$

f. $4x - 2 < -8 \quad or \quad 5x - 3 < 12$

Solution ► **a.** To solve this system of inequalities, first, we solve each individual inequality, keeping in mind the joining word *and*. So, we have

$$\begin{array}{cccccc} 3x + 7 \geq 4 & / -7 & and & 2x - 5 < -1 & / +5 \\ 3x \geq -3 & / \div 3 & and & 2x < 4 & / \div 2 \\ x \geq -1 & & and & x < 2 & \end{array}$$

The joining word *and* indicates that we look for the intersection of the obtained solutions. These solutions (in yellow and blue) and their intersection (in red) are shown in the graphed below.



Therefore, the system of inequalities is satisfied by all $x \in [\mathbf{-1}, \mathbf{2})$.

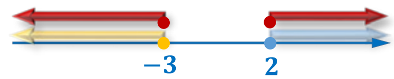
b. As in the previous example, first, we solve each individual inequality, except this time we keep in mind the joining word *or*. So, we have

$$\begin{array}{cccccc} -2x - 5 \geq 1 & / +5 & or & x - 5 \geq -3 & / +5 \end{array}$$

reverse the signs

$$\begin{array}{lll} -2x \geq 6 \quad {\color{orange}/ \div (-2)}\,! & \textit{or} & x \geq 2 \\ x \leq -3 & & \end{array}$$

The joining word *or* indicates that we look for the union of the obtained solutions. These solutions (in yellow and blue) and their union (in red) are indicated in the graph below.

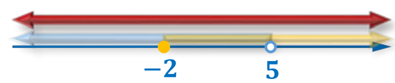


Therefore, the system of inequalities is satisfied by all $x \in (-\infty, -\mathbf{3}] \cup [\mathbf{2}, \infty)$.

c. As before, we solve each individual inequality, keeping in mind the joining word *or*. So, we have

$$\begin{array}{llllll} 3x - 11 < 4 & {\color{orange}/ +11} & \textit{or} & 4x + 9 \geq 1 & {\color{orange}/ -9} \\ 3x < 15 & {\color{orange}/ \div 3} & \textit{or} & 4x \geq -8 & {\color{orange}/ \div 4} \\ x < 5 & & \textit{or} & x \geq -2 & \end{array}$$

The joining word *or* indicates that we look for the union of the obtained solutions. These solutions (in yellow and blue) and their union (in red) are indicated in the graph below.



Therefore, the system of inequalities is satisfied by all real numbers. The solution set equals to $\mathbb{R}$.

d. Any three-part inequality is a system of inequalities with the joining word *and*. The system $-2 < 3 - \frac{1}{4}x < \frac{1}{2}$ could be written as

$$-2 < 3 - \frac{1}{4}x \quad \textit{and} \quad 3 - \frac{1}{4}x < \frac{1}{2}$$

and solved as in *Example 4a*. Alternatively, it could be solved in the three-part form, similarly as in *Section L4*, *Example 4*. Here is the three-part form solution.

$$\begin{aligned} -2 < 3 - \frac{1}{4}x &< \frac{1}{2} \qquad {\color{blue}/ -3} \\ -5 < -\frac{1}{4}x &< \frac{1}{2} - \frac{3 \cdot {\color{blue}2}}{{\color{blue}2}} \\ -5 < -\frac{1}{4}x &< -\frac{5}{2} \qquad {\color{blue}/ \cdot (-4)} \\ 20 > x &> \frac{5 \cdot 4}{2} \\ 20 > x &> 10 \end{aligned}$$

reverse the signs !

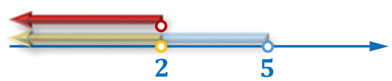
So the solution set is the interval $(10, 20)$, visualized in the graph below.

Remark: Solving a system of inequalities in three-part form has its benefits. First, the same operations are applied to all three parts, which eliminates the necessity of repeating the solving process for the second inequality. Second, the solving process of a three-part inequality produces the final interval of solutions rather than two intervals that need to be intersected to obtain the final solution set.

e. The system $\begin{cases} 4x - 7 < 1 \\ 7 - 3x > -8 \end{cases}$ consists of two inequalities joined by the word *and*. So, we solve it similarly as in *Example 4a*.

$$\begin{array}{ccccc} 4x - 7 < 1 & & and & 7 - 3x > -8 & \\ & / +7 & & & / -7 \\ 4x < 8 & & and & -3x > -15 & \\ & / \div 4 & & & / \div (-3) \\ x < 2 & & and & x < 5 & \end{array}$$

These solutions of each individual inequality (in yellow and blue) and the intersection of these solutions (in red) are indicated in the graph below.

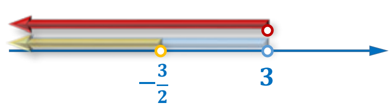


Therefore, the interval $(-\infty, 2)$ is the solution to the whole system.

f. As in *Example 4b* and *4c*, we solve each individual inequality, keeping in mind the joining word *or*. So, we have

$$\begin{array}{ccccc} 4x - 2 < -8 & & or & 5x - 3 < 12 & \\ & / +2 & & & / +3 \\ 4x \geq -6 & & or & 5x \geq 15 & \\ & / \div 4 & & & / \div 5 \\ x < -\frac{3}{2} & & or & x < 3 & \end{array}$$

The joining word *or* indicates that we look for the union of the obtained solutions. These solutions (in yellow and blue) and their union (in red) are indicated in the graph below.



Therefore, the interval $(-\infty, 3)$ is the solution to the whole system.

Compound Inequalities in Application Problems

Compound inequalities are often used to solve problems that ask for a range of values satisfying certain conditions.

Example 5 ► **Finding the Range of Values Satisfying Conditions of a Problem**

The equation $P = 1 + \frac{d}{33}$ gives the pressure P, in atmospheres (atm), at a depth of d feet in the ocean. Atlantic cod occupy waters with pressures between 1.6 and 7 atmospheres. To the nearest foot, what is the depth range at which Atlantic cod should be searched for?

Solution ► The pressure P suitable for Atlantic cod is between 1.6 to 7 atmospheres. We record this fact in the form of the three-part inequality $1.6 \leq P \leq 7$. To find the corresponding depth d, in feet, we substitute $P = 1 + \frac{d}{33}$ and solve the three-part inequality for d. So, we have

$$1.6 \leq 1 + \frac{d}{33} \leq 7 \qquad / -1$$

$$0.6 \leq \frac{d}{33} \leq 6 \qquad / \cdot 33$$

$$19.8 \leq d \leq 198$$

Thus, Atlantic cod should be searched for between 20 and 198 feet below the surface.

Example 6 ► **Using Set Operations to Solve Applied Problems Involving Compound Inequalities**

Given the information in the table,

Film	Admissions (in millions)	Lifetime Gross Income (in millions)
Gone With the Wind	226	3440
Star Wars: The Force Awakens	194	2825
The Sound of Music	156	2366
Titanic	128	2516
Avatar	78	3020

list the films that belong to each set.

a. The set of films with admissions greater than 180,000,000 *and* a lifetime gross income greater than 3,000,000,000.

b. The set of films with admissions greater than 150,000,000 *or* a lifetime gross income greater than 3,000,000,000.

c. The set of films with admissions smaller than 180,000,000 *and* a lifetime gross income greater than 2,500,000,000.

Solution ► **a.** The set of films with admissions greater than 180,000,000 consists of *Gone With the Wind* and *Star Wars: The Force Awakens.* The set of films with a lifetime gross income greater than 3,000,000,000 consists of *Gone With the Wind* and *Avatar*. Therefore, the set of films satisfying both of these properties contains just one film, which is ***Gone With the Wind***.

b. The set of films with admissions greater than 150,000,000 consists of *Gone With the Wind, Star Wars: The Force Awakens,* and *The Sound of Music*. The set of films with lifetime gross income greater than 3,000,000,000 includes *Gone With the Wind* and *Avatar*. Therefore, the set of films satisfying at least one of these properties consists of ***Gone With the Wind, Star Wars: The Force Awakens, The Sound of Music***, and ***Avatar***.

c. The set of films with admissions greater than 180,000,000 includes *Gone With the Wind* and *Star Wars: The Force Awakens.* The set of films with a lifetime gross income greater than 2,500,000,000 consists of *Gone With the Wind, Star Wars: The Force*

Awakens, *Titanic*, and *Avatar*. Therefore, the set of films satisfying both of these properties consists of ***Gone With the Wind*** *and* ***Star Wars: The Force Awakens***.

L.5 Exercises

Vocabulary Check *Complete each blank with the most appropriate term from the given list:* ***bounded, compound, intersection, unbounded, union.***

1. The _____________ of sets A and B is the set of all elements that are in both A and B.

2. The _____________ of sets A and B is the set of all elements that are either in A or in B.

3. Two inequalities joined by the words *and* or *or* is a _____________ inequality.

4. The solution set of a compound inequality involving the word *or* usually consists of two _____________ intervals.

5. The solution set of a compound inequality involving the word *and* usually consists of one _____________ interval.

Concept Check *Let* $A = \{1,2,3,4\}$, $B = \{1, 3, 5\}$, $C = \{5\}$. *Find each set.*

6. $A \cap B$ **7.** $A \cup B$ **8.** $B \cup C$ **9.** $A \setminus B$

10. $A \cap C$ **11.** $A \cup B \cup C$ **12.** $B \cap C$ **13.** $A \cup C$

Concept Check *Write the result of each set operation as a single interval, if possible.*

14. $(-7,3] \cap [1,6]$

15. $(-8,5] \cap (-1,13)$

16. $(0,3) \cup (1,7]$

17. $[-7,2] \cup (1,10)$

18. $(-\infty,13) \cup (1,\infty)$

19. $(-\infty,1) \cap (2,\infty)$

20. $(-\infty,1] \cap [1,\infty)$

21. $(-\infty,1] \cup [1,\infty)$

22. $(-2,\infty) \cup [3,\infty)$

23. $(-2,\infty) \cap [3,\infty)$

Concept Check *Solve each compound inequality. Give the solution set in both interval and graph form.*

24. $x + 1 > 6 \quad or \quad 1 - x > 3$

25. $-3x \geq -6 \quad and \quad -2x \leq 12$

26. $4x + 1 < 5 \quad and \quad 4x + 7 > -1$

27. $3y - 11 > 4 \quad or \quad 4y + 9 \leq 1$

28. $3x - 7 < -10 \quad and \quad 5x + 2 \leq 22$

29. $\frac{1}{4}y - 2 < -3 \quad or \quad 1 - \frac{3}{2}y \geq 4$

30. $\begin{cases} 1 - 7x \leq -41 \\ 3x + 1 \geq -8 \end{cases}$

31. $\begin{cases} 2(x+1) < 8 \\ -2(x-4) > -2 \end{cases}$

32. $-\frac{2}{3} \leq 3 - \frac{1}{2}a < \frac{2}{3}$

33. $-4 \leq \frac{7-3a}{5} \leq 4$

34. $5x + 12 > 2$ *or* $7x - 1 < 13$

35. $4x - 2 > 10$ *and* $8x + 2 \le -14$

36. $7t - 1 > -1$ *and* $2t - 5 \ge -10$

37. $7z - 6 > 0$ *or* $-\frac{1}{2}z \le 6$

38. $\frac{5x+4}{2} \ge 7$ *or* $\frac{7-2x}{3} \ge 2$

39. $\frac{2x-5}{-2} \ge 2$ *and* $\frac{2x+1}{3} \ge 0$

40. $13 - 3x > -8$ *and* $12x + 7 \ge -(1 - 10x)$

41. $1 \le -\frac{1}{3}(4b - 27) \le 3$

Discussion Point *Discuss how to solve the three-part inequalities in problem 42 and 43. Then, solve them.*

42. $-4x < 2x - 18 \le -x$

43. $7x - 5 \le 4x - 3 \le 8x - 3$

Analytic Skills *Solve each problem.*

44. On a cross-country move, a couple plans to drive between 450 and 600 miles per day. If they estimate that their average driving speed will be 60 mph, how many hours per day will they be driving?

45. Matter is in a liquid state between its melting point and its boiling point. The melting point of mercury is about $-38.8°C$ and its boiling point is about $356.7°C$. The formula $C = \frac{5}{9}(F - 32)$ is used to convert temperatures in degrees Fahrenheit ($°F$) to temperatures in degrees Celsius ($°C$). To the nearest tenth of a degree, determine the temperatures in $°F$ for which mercury is not in the liquid state.

46. The cost of a wedding reception is \$2500 plus \$50 for each guest. If a couple would like to keep the cost of the reception between \$7500 and \$10,000, how many guests can the couple invite?

47. A factory worker earns \$12 per hour plus an overtime rate of \$16 for every hour over 40 hr she works per week. How many hours of overtime must she work if she wants to make between \$600 and \$800 per week?

48. Average expenses for full-time resident college students at 4-year institutions during the 2007–2008 academic year are shown in the table.

Type of Expense	Public Schools	Private Schools
Tuition and fees	5950	21588
Board rates	3402	3993
Dormitory charges	4072	4812

List the elements of the following sets:

a. the set of expenses that are less than \$6000 for public schools *and* are greater than \$10,000 for private schools

b. the set of expenses that are greater than \$3000 for public schools *and* are less than \$4000 for private schools

c. the set of expenses that are less than \$5000 for public schools *or* are greater than \$10,000 for private schools

d. the set of expenses that are greater than \$4,000 for public schools *or* are between \$4000 and \$6000 for private schools