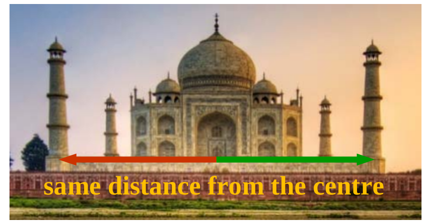
L.6 Absolute Value Equations and Inequalities



The concept of **absolute value** (also called **numerical value**) was introduced in *Section R2*. Recall that when using geometrical visualisation of real numbers on a number line, the absolute value of a number x, denoted $|x|$, can be interpreted as the distance of the point x from zero. Since distance cannot be negative, the result of absolute value is always nonnegative. In addition, the distance between points x and a can be recorded as $|x-a|$ (see *Definition 2.2* in *Section R2*), which represents the nonnegative difference between the two quantities. In this section, we will take a closer look at absolute value properties, and then apply them to solve absolute value equations and inequalities.

Properties of Absolute Value

The formal definition of absolute value

$$|\boldsymbol{x}| \overset{\boldsymbol{def}}{=} \begin{cases} \boldsymbol{x}, & \boldsymbol{if} \quad \boldsymbol{x} \ge \boldsymbol{0} \\ -\boldsymbol{x}, & \boldsymbol{if} \quad \boldsymbol{x} < \boldsymbol{0} \end{cases}$$

tells us that, when $\boldsymbol{x}$ is nonnegative, the absolute value of $\boldsymbol{x}$ is the same as $\boldsymbol{x}$, and when $\boldsymbol{x}$ is negative, the absolute value of it is the **opposite** of $\boldsymbol{x}$.

So, $|2| = 2$ and $|-2| = -(-2) = 2$. Observe that this complies with the notion of a distance from zero on a number line. Both numbers, 2 and -2 are at a distance of 2 units from zero. They are both solutions to the equation $|x| = 2$.

Since $|x|$ represents the distance of the number x from 0, which is never negative, we can claim the first absolute value property:

$$|\boldsymbol{x}| \ge \boldsymbol{0}, \text{ for any real } x$$

Here are several other absolute value properties that allow us to simplify algebraic expressions.

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ are any real numbers. Then

$$|\boldsymbol{x}| = \boldsymbol{0} \text{ if and only if } \boldsymbol{x} = \boldsymbol{0}$$

Only zero is at the distance zero from zero.

$$|-\boldsymbol{x}| = |\boldsymbol{x}|$$

The distance of opposite numbers from zero is the same.

$$|\boldsymbol{xy}| = |\boldsymbol{x}||\boldsymbol{y}|$$

Absolute value of a product is the product of absolute values.

$$\left|\frac{\boldsymbol{x}}{\boldsymbol{y}}\right| = \frac{|\boldsymbol{x}|}{|\boldsymbol{y}|} \text{ for } y \ne 0$$

Absolute value of a quotient is the quotient of absolute values.

Attention: *Absolute value doesn't 'split' over addition or subtraction! That means*

$$|\boldsymbol{x} \pm \boldsymbol{y}| \neq |\boldsymbol{x}| \pm |\boldsymbol{y}|$$

For example, $|2 + (-3)| = 1 \neq 5 = |2| + |-3|$.

Example 1 ► **Simplifying Absolute Value Expressions**

Simplify, leaving as little as possible inside each absolute value sign.

a. $|-2x|$ **b.** $|3x^2y|$

c. $\left|\frac{-a^2}{2b}\right|$ **d.** $\left|\frac{-1+x}{4}\right|$

Solution ► **a.** Since absolute value can 'split' over multiplication, we have

$$|-2x| = |-2||x| = \mathbf{2}|\boldsymbol{x}|$$

b. Using the multiplication property of absolute value and the fact that x^2 is never negative, we have

$$|3x^2y| = |3||x^2||y| = \mathbf{3}\boldsymbol{x}^{\mathbf{2}}|\boldsymbol{y}|$$

c. Using properties of absolute value, we have

$$\left|\frac{-a^2}{2b}\right| = \frac{|-1||a^2|}{|2||b|} = \frac{\boldsymbol{a}^{\mathbf{2}}}{\mathbf{2}|\boldsymbol{b}|}$$

c. Since absolute value does not 'split' over addition, the only simplification we can perform here is to take 4 outside of the absolute value sign. So, we have

$$\left|\frac{-1+x}{4}\right| = \frac{|\boldsymbol{x}-\mathbf{1}|}{\mathbf{4}} \text{ or equivalently } \frac{\mathbf{1}}{\mathbf{4}}|\boldsymbol{x}-\mathbf{1}|$$

Remark: *Convince yourself that* $|x-1|$ *is not equivalent to* $x+1$ *by evaluating both expressions at, for example,* $x = 1$.

Absolute Value Equations

The formal definition of absolute value (see *Definition 2.1* in *Section R2*) applies not only to a single number or a variable x but also to any algebraic expression. Generally, we have

$$|\boldsymbol{expr.}| \stackrel{\boldsymbol{def}}{=} \begin{cases} \boldsymbol{expr.}, & \boldsymbol{if} \quad \boldsymbol{expr.} \geq \mathbf{0} \\ -(\boldsymbol{expr.}), & \boldsymbol{if} \quad \boldsymbol{expr.} < \mathbf{0} \end{cases}$$

This tells us that, when an $expression$ is nonnegative, the absolute value of the $expression$ is the **same** as the $\boldsymbol{expression}$, and when the $expression$ is negative, the absolute value of the $expression$ is the **opposite** of the $\boldsymbol{expression}$.

For example, to evaluate $|x-1|$, we consider when the expression $x-1$ is nonnegative and when it is negative. Since $x-1 \geq 0$ for $x \geq 1$, we have

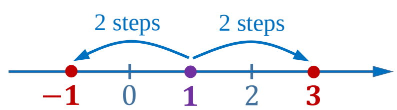
$$|x-1| = \begin{cases} x-1, & for\ x \geq 1 \\ -(x-1), & for\ x < 1 \end{cases}$$

Notice that both expressions, $x-1\ for\ x \geq 1$ and $-(x-1)\ for\ x < 1$ produce nonnegative values that represent the distance of a number x from 0.

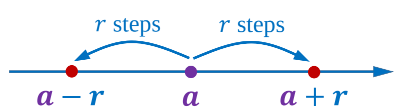
In particular,

if $x = 3$, then $|\boldsymbol{x} - \mathbf{1}| = x - 1 = 3 - 1 = \mathbf{2}$,
and
if $x = -1$, then $|\boldsymbol{x} - \mathbf{1}| = -(x-1) = -(-1-1) = -(-2) = \mathbf{2}$.

As illustrated on the number line below, both numbers, **3** and **−1** are at the distance of **2** units from **1**.



Generally, the equation $|\boldsymbol{x} - \boldsymbol{a}| = \boldsymbol{r}$ tells us that the distance between $\boldsymbol{x}$ and $\boldsymbol{a}$ is equal to $\boldsymbol{r}$. This means that $\boldsymbol{x}$ is $\boldsymbol{r}$ units away from number $\boldsymbol{a}$, in either direction.



Therefore, $\boldsymbol{x} = \boldsymbol{a} - \boldsymbol{r}$ and $\boldsymbol{x} = \boldsymbol{a} + \boldsymbol{r}$ are the solutions of the equation $|\boldsymbol{x} - \boldsymbol{a}| = \boldsymbol{r}$.

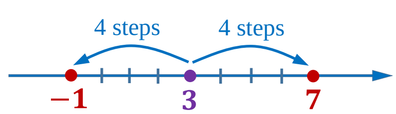
Example 1 ► **Solving Absolute Value Equations Geometrically**

For each equation, state its geometric interpretation, illustrate the situation on a number line, and then find its solution set.

a. $|x-3| = 4$ **b.** $|x+5| = 3$

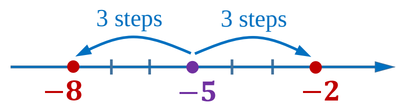
Solution ►

a. Geometrically, $|x-3|$ represents the distance between x and 3. Thus, in $|x-3| = 4$, x is a number whose distance from 3 is 4. So, $x = 3 \pm 4$, which equals either -1 or 7.



Therefore, the solution set is $\{\mathbf{-1}, \mathbf{7}\}$.

b. By rewriting $|x+5|$ as $|x-(-5)|$, we can interpret this expression as the distance between x and -5. Thus, in $|x+5| = 3$, x is a number whose distance from -5 is 3. Thus, $x = -5 \pm 3$, which results in -8 or -2.



Therefore, the solution set is $\{\boldsymbol{-8}, \boldsymbol{-2}\}$.

Although the geometric interpretation of absolute value proves to be very useful in solving some of the equations, it can be handy to have an algebraic method that will allow us to solve any type of absolute value equation.

Suppose we wish to solve an equation of the form

$$|\boldsymbol{expr.}| = \boldsymbol{r}, \quad \text{where } r > 0$$

We have two possibilities. Either the $\boldsymbol{expression}$ inside the absolute value bars is nonnegative, or it is negative. By definition of absolute value, if the $\boldsymbol{expression}$ is nonnegative, our equation becomes

$$\boldsymbol{expr.} = \boldsymbol{r}$$

If the $\boldsymbol{expression}$ is negative, then to remove the absolute value bar, we must change the sign of the $\boldsymbol{expression}$. So, our equation becomes

$$-\boldsymbol{expr.} = \boldsymbol{r},$$

which is equivalent to

$$\boldsymbol{expr.} = -\boldsymbol{r}$$

In summary, for $\boldsymbol{r} > 0$, the equation

$$|\boldsymbol{expr.}| = \boldsymbol{r}$$

$$\boldsymbol{expr.} = \boldsymbol{r} \quad \text{or} \quad \boldsymbol{expr.} = -\boldsymbol{r},$$

is equivalent to the system of equations with the connecting word *or*.

If $\boldsymbol{r} = 0$, then $|\boldsymbol{expr.}| = \boldsymbol{0}$ is equivalent to the equation $\boldsymbol{expr.} = \boldsymbol{0}$ with no absolute value.

If $\boldsymbol{r} < 0$, then $|\boldsymbol{expr.}| = \boldsymbol{r}$ has **NO SOLUTION**, as an absolute value is never negative.

Now, suppose we wish to solve an equation of the form

$$|\boldsymbol{expr.A}| = |\boldsymbol{expr.B}|$$

Since both expressions, $\boldsymbol{A}$ and $\boldsymbol{B}$, can be either nonnegative or negative, when removing absolute value bars, we have four possibilities:

$$\boldsymbol{expr.A} = \boldsymbol{expr.B} \quad \text{or} \quad \boldsymbol{expr.A} = -\boldsymbol{expr.B}$$
$$-\boldsymbol{expr.A} = -\boldsymbol{expr.B} \quad \text{or} \quad -\boldsymbol{expr.A} = \boldsymbol{expr.B}$$

However, observe that the equations in blue are equivalent. Also, the equations in green are equivalent. So, in fact, it is enough to consider just the first two possibilities.

Therefore, the equation

$$|\boldsymbol{expr.A}| = |\boldsymbol{expr.B}|$$

is equivalent to the system of equations with the connecting word *or*.

$$\boldsymbol{expr.A} = \boldsymbol{expr.B} \text{ or } \boldsymbol{expr.A} = -(\boldsymbol{expr.B}),$$

Example 2 **Solving Absolute Value Equations Algebraically**

Solve the following equations.

a. $|2 - 3x| = 7$ **b.** $5|x| - 3 = 12$

c. $\left|\frac{1-x}{4}\right| = 0$ **d.** $|6x + 5| = -4$

e. $|2x - 3| = |x + 5|$ **f.** $|x - 3| = |3 - x|$

Solution

a. To solve $|2 - 3x| = 7$, we remove the absolute value bars by changing the equation into the corresponding system of equations with no absolute value anymore. Then, we solve the resulting linear equations. So, we have

$$|2 - 3x| = 7$$

$$2 - 3x = 7 \quad \text{or} \quad 2 - 3x = -7$$

$$2 - 7 = 3x \quad \text{or} \quad 2 + 7 = 3x$$

$$x = \frac{-5}{3} \quad \text{or} \quad x = \frac{9}{3} = 3$$

Therefore, the solution set of this equation is $\left\{-\frac{\mathbf{5}}{\mathbf{3}}, \mathbf{3}\right\}$.

b. To solve $5|x| - 3 = 12$, first, we **isolate the absolute value**, and then replace the equation by the corresponding system of two linear equations.

$$5|x| - 3 = 12$$

$$5|x| = 15$$

$$|x| = 3$$

$$x = 3 \quad \text{or} \quad x = -3$$

So, the solution set of the given equation is $\{\mathbf{-3}, \mathbf{3}\}$.

c. By properties of absolute value, $\left|\frac{1-x}{4}\right| = 0$ if and only if $\frac{1-x}{4} = 0$, which happens when the numerator $1 - x = 0$. So, the only solution to the given equation is $x = \mathbf{1}$.

d. Since an absolute value is never negative, the equation $|6x + 5| = -4$ does not have any solution.

e. To solve $|2x-3| = |x+5|$, we remove the absolute value symbols by changing the equation into the corresponding system of linear equations with no absolute value. Then, we solve the resulting equations. So, we have

$$|2x-3| = |x+5|$$

$$\begin{array}{rcl} 2x-3 = x+5 & or & 2x-3 = -(x+5) \\ 2x-x = 5+3 & or & 2x-3 = -x-5 \\ x = 8 & or & 3x = -2 \\ & & x = \frac{-2}{3} \end{array}$$

Therefore, the solution set of this equation is $\left\{-\frac{2}{3}, 8\right\}$.

f. We solve $|x-3| = |3-x|$ as in *Example 2e*.

$$|x-3| = |3-x|$$

$$\begin{array}{rcl} x-3 = 3-x & or & x-3 = -(3-x) \\ 2x = 6 & or & x-3 = -3+x \\ x = 3 & or & 0 = 0 \end{array}$$

Since the equation $0 = 0$ is always true, any real x-value satisfies the original equation $|x-3| = |3-x|$. So, the solution set to the original equation is $\mathbb{R}$.

Remark: *Without solving the equation in Example 2f, one could observe that the expressions $x-3$ and $3-x$ are opposite to each other and as such, they have the same absolute value. Therefore, the equation is always true.*

Summary of Solving Absolute Value Equations

Step 1 **Isolate the absolute value** expression on one side of the equation.

Step 2 **Check for special cases**, such as

$$|A| = 0 \leftrightarrow A = 0$$
$$|A| = -r \rightarrow \textbf{No solution}$$

Step 2 **Remove the absolute value symbol** by replacing the equation with the corresponding system of equations with the joining word *or*,

$$|A| = r \quad (r > 0) \qquad\qquad |A| = |B|$$

$$A = r \ \ or \ \ A = -r \qquad\qquad A = B \ \ or \ \ A = -B$$

Step 3 **Solve** the resulting equations.

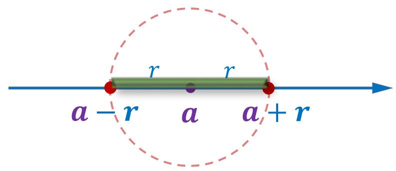
Step 4 **State the solution set** as a union of the solutions of each equation in the system.

Absolute Value Inequalities with One Absolute Value Symbol

GEOMETRIC VISUALIZATION

Suppose we wish to solve inequalities of the form $|x-a|<r$ or $|x-a|>r$, where r is a positive real number. Similarly as in the case of absolute value equations, we can either use a geometric interpretation with the aid of a number line, or we can rely on an algebraic procedure.

Using the geometrical visualization of $|x-a|$ as the distance between x and a on a number line, the inequality $|x-a|<r$ tells us that the number x is less than r units from number a. One could think of drawing a circle centered at a, with radius r. Then, the solutions of the inequality $|x-a|<r$ are all the points on a number line that lie inside such a circle (*see the green segment below*).



Therefore, the solution set is the interval $(a-r, a+r)$.

This result can be achieved algebraically by rewriting the absolute value inequality

$$|x-a|<r$$

in an equivalent three-part inequality form

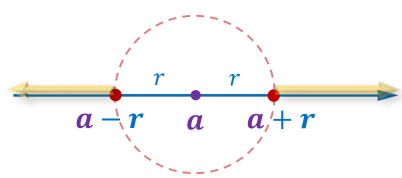
$$-r<x-a<r,$$

and then solving it for x

$$a-r<x<a+r,$$

which confirms that the solution set is indeed $(a-r, a+r)$.

Similarly, the inequality $|x-a|>r$ tells us that the number x is more than r units from number a. As illustrated in the diagram below, the solutions of this inequality are all points on a number line that lie outside of the circle centered at a, with radius r.



Therefore, the solution set is the union $(-\infty, a-r)\cup(a+r, \infty)$.

As before, this result can be achieved algebraically by rewriting the absolute value inequality

$$|x-a|>r$$

in an equivalent system of two inequalities joined by the word *or*

$$x-a<-r \quad \text{or} \quad r<x-a,$$

and then solving it for x

$$\boldsymbol{x} < \boldsymbol{a} - \boldsymbol{r} \quad \textit{or} \quad \boldsymbol{a} + \boldsymbol{r} < \boldsymbol{x},$$

which confirms that the solution set is $(-\infty, \boldsymbol{a} - \boldsymbol{r}) \cup (\boldsymbol{a} + \boldsymbol{r}, \infty)$.

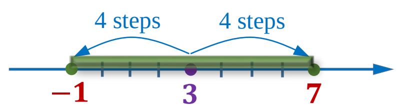
Example 3 ► **Solving Absolute Value Inequalities Geometrically**

For each inequality, state its geometric interpretation, illustrate the situation on a number line, and then find its solution set.

a. $|x - 3| \le 4$ **b.** $|x + 5| > 3$

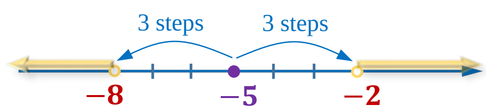
Solution ►

a. Geometrically, $|x - 3|$ represents the distance between x and 3. Thus, in $|x - 3| \le 4$, x is a number whose distance from 3 is at most 4, in either direction.
So, $3 - 4 \le x \le 3 + 4$, which is equivalent to $-1 \le x \le 7$.



Therefore, the solution set is $[\mathbf{-1}, \mathbf{7}]$.

b. By rewriting $|x + 5|$ as $|x - (-5)|$, we can interpret this expression as the distance between x and -5. Thus, in $|x + 5| > 3$, x is a number whose distance from -5 is more than 3, in either direction. Thus, $x < -5 - 3$ or $-5 + 3 < x$, which results in $x < -8$ or $x > -2$.



Therefore, the solution set equals $(-\infty, \mathbf{-8}) \cup (\mathbf{-2}, \infty)$.

The algebraic strategy can be applied to any inequality of the form

$$|\boldsymbol{expr.}| < (\le)\boldsymbol{r}, \quad \text{or} \quad |\boldsymbol{expr.}| > (\ge)\boldsymbol{r}, \;\; \text{as long as } \boldsymbol{r} > \mathbf{0}.$$

Depending on the type of inequality, we follow these rules:

$$|\boldsymbol{expr.}| < \boldsymbol{r}$$

$$-\boldsymbol{r} < \boldsymbol{expr.} < \boldsymbol{r}$$

or

$$|\boldsymbol{expr.}| > \boldsymbol{r}$$

$$\boldsymbol{expr.} < -\boldsymbol{r} \quad \text{or} \quad \boldsymbol{r} < \boldsymbol{expr.}$$

These rules also apply to weak inequalities, such as $\le$ or $\ge$.

In the above rules, we assume that $\boldsymbol{r} > \mathbf{0}$. **What if** $\boldsymbol{r} = \mathbf{0}$ **?**

Observe that, the inequality $|\boldsymbol{expr.}| < \mathbf{0}$ is never true, so this inequality doesn't have any solution. Since $|\boldsymbol{expr.}| < \mathbf{0}$ is never true, the inequality $|\boldsymbol{expr.}| \le \mathbf{0}$ is equivalent to the equation $|\boldsymbol{expr.}| = \mathbf{0}$.

On the other hand, $|\boldsymbol{expr.}| \geq \mathbf{0}$ is always true, so the solution set equals to $\mathbb{R}$. However, since $|\boldsymbol{expr.}|$ is either positive or zero, the solution to $|\boldsymbol{expr.}| > \mathbf{0}$ consists of all real numbers except for the solutions of the equation $\boldsymbol{expr.} = \mathbf{0}$.

What if $r < 0$?

Observe that both inequalities $|\boldsymbol{expr.}| > \boldsymbol{negative}$ and $|\boldsymbol{expr.}| \geq \boldsymbol{negative}$ are always true, so the solution set of such inequalities is equal to $\mathbb{R}$.

On the other hand, both inequalities $|\boldsymbol{expr.}| < \boldsymbol{negative}$ and $|\boldsymbol{expr.}| \leq \boldsymbol{negative}$ are never true, so such inequalities result in **NO SOLUTION**.

Example 4 ► **Solving Absolute Value Inequalities with One Absolute Value Symbol**

Solve each inequality. Give the solution set in both interval and graph form.

a. $|5x + 9| \leq 4$ **b.** $|-2x - 5| > 1$

e. $16 \leq |2x - 3| + 9$ **f.** $1 - 2|4x - 7| > -5$

Solution ► **a.** To solve $|5x + 9| \leq 4$, first, we remove the absolute value symbol by rewriting the inequality in the three-part inequality, as below.

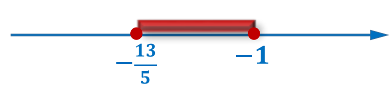
$$|5x + 9| \leq 4$$

$$-4 \leq 5x + 9 \leq 4 \qquad / -9$$

$$-13 \leq 5x \leq -5 \qquad / \div 5$$

$$-\frac{13}{5} \leq x \leq -1$$

The solution is shown in the graph below.



The inequality is satisfied by all $x \in \left[-\frac{13}{5}, -\mathbf{1}\right]$.

b. As in the previous example, first, we remove the absolute value symbol by replacing the inequality with the corresponding system of inequalities, joined by the word *or*. So, we have

$$|-2x - 5| > 1$$

$$-2x - 5 < -1 \quad or \quad 1 < -2x - 5 \qquad / +5$$

$$-2x < 4 \quad or \quad 6 < -2x \qquad / \div (-2)$$

$$x > -2 \quad or \quad -3 > x$$

The joining word *or* indicates that we look for the union of the obtained solutions. This union is shown in the graph below.

The inequality is satisfied by all $x \in (-\infty, -\mathbf{3}) \cup (-\mathbf{2}, \infty)$.

c. To solve $16 \le |2x-3| + 9$, first, we <u>**isolate the absolute value**</u>, and then replace the inequality with the corresponding system of two linear equations. So, we have

$$16 \le |2x-3| + 9 \qquad / -9$$

$$7 \le |2x-3|$$

$$\begin{array}{ccc} 2x-3 \le -7 & \textit{or} & 7 \le 2x-3 \\ 2x \le -4 & \textit{or} & 2x \ge 10 \\ x \le -2 & \textit{or} & x \ge 5 \end{array} \qquad \begin{array}{l} / +3 \\ / \div 2 \\ \end{array}$$

The joining word *or* indicates that we look for the union of the obtained solutions. This union is shown in the graph below.

So, the inequality is satisfied by all $x \in (-\infty, \mathbf{-2}) \cup (\mathbf{5}, \infty)$.

d. As in the previous example, first, we <u>**isolate the absolute value**</u>, and then replace the inequality with the corresponding system of two inequalities.

$$1 - 2|4x-7| > -5 \qquad / -1$$

$$-2|4x-7| > -6 \qquad / \div(-2)$$

reverse the signs

$$|4x-7| < 3$$

$$-3 < 4x-7 < 3 \qquad / +7$$

$$4 < 4x < 10 \qquad / \div 4$$

$$1 < x < \frac{10}{4} = \frac{5}{2}$$

So the solution set is the interval $\left(\mathbf{1}, \frac{\mathbf{5}}{\mathbf{2}}\right)$, visualized in the graph below.

Example 5 ► **Solving Absolute Value Inequalities in Special Cases**

Solve each inequality.

a. $\left|\frac{1}{2}x + \frac{5}{3}\right| \ge -3$ **b.** $|4x-7| \le 0$

c. $|3-4x| > 0$ **d.** $1 - 2\left|\frac{3}{2}x - 5\right| > 3$

Solution ► **a.** Since an absolute value is always bigger or equal to zero, the inequality $\left|\frac{1}{2}x + \frac{5}{3}\right| \ge -3$ is always true. Thus, it is satisfied by **any real number**. So the solution set is $\mathbb{R}$.

b. Since $|4x-7|$ is never negative, the inequality $|4x-7| \leq 0$ is satisfied only by solutions to the equation $|4x-7| = 0$. So, we solve

$$|4x-7| = 0$$

$$4x - 7 = 0 \qquad / +7$$

$$4x = 7 \qquad / \div 4$$

$$x = \frac{7}{4}$$

Therefore, the inequality is satisfied only by $\boldsymbol{x = \frac{7}{4}}$.

c. Inequality $|3-4x| > 0$ is satisfied by all real x-values except for the solution to the equation $3 - 4x = 0$. Since

$$3 - 4x = 0 \qquad / +4x$$

$$3 = 4x \qquad / \div 4$$

$$\frac{3}{4} = x,$$

then the solution to the original inequality is $\boldsymbol{\left(-\infty, \frac{3}{4}\right) \cup \left(\frac{3}{4}, \infty\right)}$.

d. To solve $1 - 2\left|\frac{3}{2}x - 5\right| > 3$, first, we **isolate the absolute value**. So, we have

$$1 - 2\left|\frac{3}{2}x - 5\right| > 3 \qquad / +2\left|\frac{3}{2}x - 5\right|, -3$$

$$-2 > 2\left|\frac{3}{2}x - 5\right| \qquad / \div 2$$

$$-1 > \left|\frac{3}{2}x - 5\right|$$

Since $\left|\frac{3}{2}x - 5\right|$ is never negative, it can't be less than -1. So, there is **no solution** to the original inequality.

Summary of Solving Absolute Value Inequalities with One Absolute Value Symbol

Let $\boldsymbol{r}$ be a positive real number. To solve absolute value inequalities with one absolute value symbol, follow the steps:

Step 1 **Isolate the absolute value** expression on one side of the inequality.

Step 2 **Check for special cases**, such as

$\boldsymbol{|A| < 0} \rightarrow$ **No solution**
$\boldsymbol{|A| \leq 0} \leftrightarrow \boldsymbol{A = 0}$
$\boldsymbol{|A| \geq 0} \rightarrow$ **All real numbers**
$\boldsymbol{|A| > 0} \rightarrow$ **All real numbers except for solutions of $\boldsymbol{A = 0}$**
$\boldsymbol{|A| > (\geq) -r} \rightarrow$ **All real numbers**
$\boldsymbol{|A| < (\leq) -r} \rightarrow$ **No solution**

Step 3 **Remove the absolute value symbol** by replacing the equation with the corresponding system of equations as below:

$$\boldsymbol{|A| < r} \qquad\qquad \boldsymbol{|A| > r}$$

$$\boldsymbol{-r < A < r} \qquad\qquad \boldsymbol{A < -r} \text{ or } \boldsymbol{r < A}$$

This also applies to weak inequalities, such as $\leq$ or $\geq$.

Step 3 **Solve** the resulting equations.

Step 4 **State the solution set** as a union of the solutions of each equation in the system.

Applications of Absolute Value Inequalities

One of the typical applications of absolute value inequalities is in error calculations. When discussing errors in measurements, we refer to the *absolute error* or the *relative error*. For example, if M is the actual measurement of an object and x is the approximated measurement, then the ***absolute error*** is given by the formula $\boldsymbol{|x - M|}$ and the ***relative error*** is calculated according to the rule $\frac{\boldsymbol{|x-M|}}{\boldsymbol{M}}$.

In quality control situations, the relative error often must be less than some predetermined amount. For example, suppose a machine that fills two-litre milk cartons is set for a relative error no greater than 1%. We might be interested in how much milk a two-litre carton can actually contain? What is the absolute error that this machine is allowed to make?

Since $M = 2$ litres and the relative error = $1\% = 0.01$, we are looking for all x-values that would satisfy the inequality

$$\frac{|x-2|}{2} < 0.01.$$

This is the **relative error**.

This is equivalent to

$$|x-2| < 0.02$$

This is the **absolute error**.

$$-0.02 < x - 2 < 0.02$$

$$1.98 < x < 2.02,$$

so, a two-litre carton of milk can contain any amount of milk between 1.98 and 2.02 litres. The absolute error in this situation is $0.02\ l = 20\ ml$.

Example 6 ► **Solving Absolute Value Application Problems**

A technician is testing a scale with a 50 kg block of steel. The scale passes this test if the relative error when weighing this block is less than 0.1%. If x is the reading on the scale, then for what values of x does the scale pass this test?

Solution ► If the relative error must be less than 0.1%=0.001, then x must satisfy the inequality

$$\frac{|x-50|}{50} < 0.001$$

After solving it for x,

$$|x - 50| < 0.05$$

$$-0.05 < x - 50 < 0.05$$

$$49.95 < x < 50.05$$

We conclude that the scale passes the test if it shows a weight between **49.95** and **50.05** kg. This also tells us that the scale may err up to $0.05\ kg = 5\ dkg$ when weighing this block.

L.6 Exercises

Vocabulary Check *Complete each blank with one of the suggested words or the most appropriate term from the given list:* ***absolute value, addition, distance, division, multiplication, opposite, solution set, subtraction, three-part****.*

1. The ______________ ______________ of a real number represents the distance of this number from zero, on a number line.

2. The expression $|a - b|$ represents the ______________ between a and b, on a number line.

3. The absolute value symbol 'splits' over ______________ and ______________ but not over _______________ and _______________.

4. The absolute value of any expression is also equal to the absolute value of the ______________ expression.

5. If X represents an expression and r represents a positive real number, the ______________ __________ of the inequality $|X| > r$ consists of $\underset{one\ /two}{\underline{\hspace{2cm}}}$ interval(s) of numbers.

6. To solve $|X| < r$, remove the absolute value symbol by writing the corresponding __________________ inequality instead.

Concept Check *Simplify, if possible, leaving as little as possible inside the absolute value symbol.*

7. $|-2x^2|$ **8.** $|3x|$ **9.** $\left|\frac{-5}{y}\right|$ **10.** $\left|\frac{3}{-y}\right|$

11. $|7x^4y^3|$ **12.** $|-3x^5y^4|$ **13.** $\left|\frac{x^2}{y}\right|$ **14.** $\left|\frac{-4x}{y^2}\right|$

15. $\left|\frac{-3x^3}{6x}\right|$ **16.** $\left|\frac{5x^2}{-25x}\right|$ **17.** $|(x-1)^2|$ **18.** $|x^2 - 1|$

19. ***Concept Check*** *How many solutions will* $|ax + b| = k$ *have for each situation?*

a. $k < 0$ **b.** $k = 0$ **c.** $k > 0$

20. *Concept Check* *Match each absolute value equation or inequality in Column I with the graph of its solution set in Column II.*

	Column I		Column II
a.	$\lvert x\rvert = 3$	**A.**	
b.	$\lvert x\rvert > 3$	**B.**	
c.	$\lvert x\rvert < 3$	**C.**	
d.	$\lvert x\rvert \geq 3$	**D.**	
e.	$\lvert x\rvert \leq 3$	**E.**	

Concept Check *Solve each equation.*

21. $|-x| = 4$

22. $|5x| = 20$

23. $|y - 3| = 8$

24. $|2y + 5| = 9$

25. $7|3x - 5| = 35$

26. $-3|2x - 7| = -12$

27. $\left|\frac{1}{2}x + 3\right| = 11$

28. $\left|\frac{2}{3}x - 1\right| = 5$

29. $|2x - 5| = -1$

30. $|7x + 11| = 0$

31. $2 + 3|a| = 8$

32. $10 - |2a - 1| = 4$

33. $\left|\frac{2x-1}{3}\right| = 5$

34. $\left|\frac{3-5x}{6}\right| = 3$

35. $|2p + 4| = |3p - 1|$

36. $|5 - q| = |q + 7|$

37. $\left|\frac{1}{2}x + 3\right| = \left|\frac{1}{5}x - 1\right|$

38. $\left|\frac{2}{3}x - 8\right| = \left|\frac{1}{6}x + 3\right|$

39. $\left|\frac{3x-6}{2}\right| = \left|\frac{5+x}{5}\right|$

40. $\left|\frac{6-5x}{4}\right| = \left|\frac{7+3x}{3}\right|$

Concept Check *Solve each inequality. Give the solution set in both interval and graph form.*

41. $|x + 4| < 3$

42. $|x - 5| > 7$

43. $|x - 12| \geq 5$

44. $|x + 14| \leq 5$

45. $|5x + 3| \leq 8$

46. $|3x - 2| \geq 10$

47. $|7 - 2x| > 5$

48. $|-5x + 4| < 3$

49. $\left|\frac{1}{4}y - 6\right| \leq 24$

50. $\left|\frac{2}{5}x + 3\right| > 5$

51. $\left|\frac{3x-2}{4}\right| \geq 10$

52. $\left|\frac{2x+3}{3}\right| < 10$

53. $|-2x + 4| - 8 \geq -5$

54. $|6x - 2| + 3 < 9$

55. $7 - 2|x + 4| \geq 5$

56. $9 - 3|x - 2| < 3$

Concept Check *Solve each inequality.*

57. $\left|\frac{2}{3}x + 4\right| \leq 0$

58. $\left|-2x + \frac{4}{5}\right| > 0$

59. $\left|\frac{6x-2}{5}\right| < -3$

60. $|-3x + 5| > -3$

61. $|-x + 4| + 5 \geq 4$

62. $|4x + 1| - 2 < -5$

Discussion Point

63. Assume that you have solved an inequality of the form $|x - a| < b$, where a is a real number and b is a positive real number. Explain how you can write the solutions of the inequality $|x - a| \geq b$ without solving this inequality. Justify your answer.

Analytic Skills *Solve each problem.*

64. The recommended daily intake (RDI) of calcium for females aged 19–50 is 1000 mg. Actual needs vary from person to person, with a tolerance of up to 100 mg. Write this statement as an absolute value inequality, with x representing the actual needs for calcium intake, and then solve this inequality.

65. A police radar gun is calibrated to have an allowable maximum error of 2 km/h. If a car is detected at 61 km/h, what are the minimum and maximum possible speeds that the car was traveling?

66. A patient placed on a restricted diet is allowed 1300 calories per day with a tolerance of no more than 50 calories.

a. Write an inequality to represent this situation. Use c to represent the number of allowable calories per day.

b. Use the inequality in part (**a**) to find the range of allowable calories per day in the patient's diet.

67. The average annual income of residents in an apartment house is \$39,000. The income of a particular resident is not within \$5000 of the average.

a. Write an absolute value inequality that describes the income I of the resident.

b. Solve the inequality in part (**a**) to find the possible income of this resident.

68. On a highway, the speed limit is 110 km/h, and slower cars are required to travel at least 90 km/h. What absolute value inequality must any legal speed s satisfy?

69. An adult's body temperature T is considered to be normal if it is at least 36.4°C and at most 37.6°C. Express this statement as an absolute value inequality.