Problems for Math Match 2019

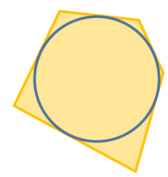
1. Let $n > 3$ be a positive integer. Equilateral triangle ABC is divided into n^2 smaller congruent equilateral triangles (with sides parallel to its sides). Let m be the number of rhombuses that contain two small equilateral triangles and d the number of rhombuses that contain eight small equilateral triangles. Find the difference $m - d$ in terms of n.

2. Let ABC be a right triangle with right angle at B. Let $ACDE$ be a square drawn exterior to triangle ABC. If M is the center of this square, find the measure of $\angle MBC$.

3. Two runners start running laps at the same time, from the same starting position. Sue runs a lap in 50 seconds while George runs a lap in 30 seconds. When will the runners next be side by side? Generalize your answer for the problem when Sue runs a lap in m seconds while George runs a lap in n seconds.

4. Two friends go to a pizza place regularly where a rectangular pizza is served. This pizza is called the hole in one pizza because the chef always cuts out randomly a circular piece of any radius from the rectangular pizza. How can the two friends equally divide the pizza with one cut?

5. You have a cake in a shape of a circumscribed n-polygon (a convex polygon that contains an inscribed circle). How would you divide the cake equally among five people so that each person receives the same amount of icing?



6. 101 wise men stand in a circle. Each of them either thinks that the Earth orbits Jupiter or that Jupiter orbits the Earth. Once a minute, all the wise men express their opinion at the same time. Right after that, every wise man who stands between two people with a different opinion from him changes his opinion himself. The rest do not change. Prove that at one point they will all stop changing opinions. After what time we can guarantee that the wise man will no longer change their opinion.

7. Show that if 101 different integers are chosen from 1 to 200, inclusive, thare must be 2 with the property that one is divisible by the other.

8. Show that, for any integer n, the number $n^3 - 9n + 27$ is not divisible by 81.