Polynomials and Polynomial Functions

One of the simplest types of algebraic expressions are polynomials. They are formed only by addition and multiplication of variables and constants. Since both addition and multiplication produce unique values for any given inputs, polynomials are in fact functions. One of the simplest polynomial functions are linear functions, such as $P(x) = 2x + 1$, or quadratic functions, such as $Q(x) = x^2 + x - 6$. Due to the comparably simple form, polynomial functions appear in a wide variety of areas of mathematics and science, economics, business, and many other areas of life. Polynomial functions are often used to model various natural phenomena, such as the shape of a mountain, the distribution of temperature, the trajectory of projectiles, etc. The shape and properties of polynomial functions are helpful when constructing such structures as roller coasters or bridges, solving optimization problems, or even analysing stock market prices.

In this chapter, we will introduce polynomial terminology, perform operations on polynomials, and evaluate and compose polynomial functions.

P.1 Addition and Subtraction of Polynomials

Terminology of Polynomials

Recall that products of constants, variables, or expressions are called **terms** (see *section R3*, *Definition 3.1*). **Terms** that are **products** of only **numbers** and **variables** are called **monomials**. Examples of monomials are $-2x$, xy^2, $\frac{2}{3}x^3$, etc.

Definition 1.1 ► A **polynomial** is a sum of monomials.

A **polynomial** in a single variable is the sum of terms of the form ax^n, where a is a **numerical coefficient**, x is the variable, and n is a whole number.

An ***n*-th degree polynomial** in x-variable has the form

$$\boldsymbol{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0},$$

where $a_n, a_{n-1}, \dots, a_2, a_1, a_0 \in \mathbb{R}$, $a_n \neq 0$.

Note: A polynomial can always be considered as a sum of monomial terms even though there are negative signs when writing it.
For example, polynomial $x^2 - 3x - 1$ can be seen as the sum of signed terms

$$x^2 + -3x + -1$$

Definition 1.2 ► The **degree of a monomial** is the sum of exponents of all its variables.

For example, the degree of $5x^3y$ is 4, as the sum of the exponent of x^3, which is 3 and the exponent of y, which is 1. To record this fact, we write $\deg(5x^3y) = 4$.

The **degree of a polynomial** is the highest degree out of all its terms.

For example, the degree of $2x^2y^3 + 3x^4 - 5x^3y + 7$ is 5 because $\deg(2x^2y^3) = 5$ and the degrees of the remaining terms are not greater than 5.

Polynomials that are sums of two terms, such as $x^2 - 1$, are called **binomials**.
Polynomials that are sums of three terms, such as $x^2 + 5x - 6$ are called **trinomials**.

The **leading term** of a polynomial is the highest degree term.
The **leading coefficient** is the numerical coefficient of the leading term.
So, the leading term of the polynomial $1 - x - x^2$ is $-x^2$, even though it is not the first term. The leading coefficient of the above polynomial is -1, as $-x^2$ can be seen as $(-1)x^2$.

A first degree term is often referred to as a **linear term**. A second degree term can be referred to as a **quadratic term**. A zero degree term is often called a **constant** or a **free term**.

Below are the parts of an n-th degree polynomial in a single variable x:

$$\underbrace{a_n x^n}_{\text{leading term}} + a_{n-1}x^{n-1} + \cdots + \underbrace{a_2x^2}_{\text{quadratic term}} + \underbrace{a_1x}_{\text{linear term}} + \underbrace{a_0}_{\text{constant (free) term}}$$

(leading coefficient → a_n)

Note: Single variable polynomials are usually arranged in descending powers of the variable. Polynomials in more than one variable are arranged in decreasing degrees of terms. If two terms are of the same degree, they are arranged with respect to the descending powers of the variable that appers first in alphabetical order.

For example, polynomial $x^2 + x - 3x^4 - 1$ is customarily arranged as follows

$$-3x^4 + x^2 + x - 1,$$

while polynomial $3x^3y^2 + 2y^6 - y^2 + 4 - x^2y^3 + 2xy$ is usually arranged as below.

$$\underbrace{2y^6}_{\text{6th degree term}} \underbrace{+3x^3y^2 - x^2y^3}_{\text{5th degree terms arranged with respect to } x} \underbrace{+2xy - y^2}_{\text{2nd degree terms arranged with respect to } x} \underbrace{+4}_{\text{zero degree term}}$$

Example 1 ► **Writing Polynomials in Descending Order and Identifying Parts of a Polynomial**

Suppose $P = x - 6x^3 - x^6 + 4x^4 + 2$ and $Q = 2y - 3xyz - 5x^2 + xy^2$. For each polynomial:

a. Write the polynomial in descending order.
b. State the degree of the polynomial and the number of its terms.
c. Identify the leading term, the leading coefficient, the coefficient of the linear term, the coefficient of the quadratic term, and the free term of the polynomial.

Solution ► **a.** After arranging the terms in descending powers of x, polynomial P becomes

$$\boldsymbol{-x^6 + 4x^4 - 6x^3 + x + 2},$$

while polynomial Q becomes

$$\boldsymbol{xy^2 - 3xyz - 5x^2 + 2y}.$$

Notice that the first two terms, xy^2 and $-3xyz$, are both of the same degree. So, to decide which one should be written first, we look at powers of x. Since these powers are again the same, we look at powers of y. This time, the power of y in xy^2 is higher than the power of y in $-3xyz$. So, the term xy^2 should be written first.

b. The polynomial P has **5 terms**. The highest power of x in P is 6, so the **degree** of the polynomial P is **6**.
The polynomial Q has **4 terms**. The highest degree terms in Q are xy^2 and $-3xyz$, both third degree. So the **degree** of the polynomial Q is **3**.

$x = \mathbf{1} \cdot x$
$-x = (\mathbf{-1})x$

c. The leading term of the polynomial $P = -x^6 + 4x^4 - 6x^3 + x - 2$ is $-x^6$, so the **leading coefficient** equals $-\mathbf{1}$.
The linear term of P is x, so the **coefficient of the linear term** equals **1**.
P doesn't have any quadratic term so the coefficient of the quadratic term equals **0**.
The **free term** of P equals $-\mathbf{2}$.

The leading term of the polynomial $Q = xy^2 - 3xyz - 5x^2 + 2y$ is xy^2, so the **leading coefficient** is equal to **1**.
The linear term of Q is $2y$, so the **coefficient of the linear term** equals **2**.
The quadratic term of Q is $-5x^2$, so the **coefficient of the quadratic term** equals $-\mathbf{5}$.
The polynomial Q does not have a free term, so the **free term** equals **0**.

Example 2 ► **Classifying Polynomials**

Describe each polynomial as a *constant*, *linear*, *quadratic*, or *n-th degree* polynomial. Decide whether it is a *monomial*, *binomial*, or *trinomial*, if applicable.

a. $x^2 - 9$ **b.** $-3x^7y$
c. $x^2 + 2x - 15$ **d.** π
e. $4x^5 - x^3 + x - 7$ **f.** $x^4 + 1$

Solution ►

a. $x^2 - 9$ is a second degree polynomial with two terms, so it is a **quadratic binomial**.

b. $-3x^7y$ is an **8-th degree monomial**.

c. $x^2 + 2x - 15$ is a second degree polynomial with three terms, so it is a **quadratic trinomial**.

d. π is a 0-degree term, so it is a **constant monomial**.

e. $4x^5 - x^3 + x - 7$ is a **5-th degree polynomial**.

f. $x^4 + 1$ is a **4-th degree binomial**.

Polynomials as Functions and Evaluation of Polynomials

Each term of a polynomial in one variable is a product of a number and a power of the variable. The polynomial itself is either one term or a sum of several terms. Since taking a power of a given value, multiplying, and adding given values produce unique answers, polynomials are also functions. While f, g, or h are the most commonly used letters to represent functions, other letters can also be used. To represent polynomial functions, we customarily use capital letters, such as P, Q, R, etc.

Any polynomial function P of degree n, has the form

$$\boldsymbol{P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0}$$

where $a_n, a_{n-1}, \ldots, a_2, a_1, a_0 \in \mathbb{R}$, $a_n \neq 0$, and $n \in \mathbb{W}$.

Since polynomials are functions, they can be evaluated for different x-values.

Example 3 ► **Evaluating Polynomials**

Given $P(x) = 3x^3 - x^2 + 4$, evaluate the following expressions:

a. $P(0)$ **b.** $P(-1)$
c. $2 \cdot P(1)$ **d.** $P(a)$

Solution ►

a. $P(0) = 3 \cdot 0^3 - 0^2 + 4 = \mathbf{4}$

b. $P(-1) = 3 \cdot (-1)^3 - (-1)^2 + 4 = 3 \cdot (-1) - 1 + 4 = -3 - 1 + 4 = \mathbf{0}$

When evaluating at negative x-values, it is essential to use brackets in place of the variable before substituting the desired value.

c. $2 \cdot P(1) = 2 \cdot \underbrace{(3 \cdot 1^3 - 1^2 + 4)}_{this\ is\ P(1)} = 2 \cdot (3 - 1 + 4) = 2 \cdot 6 = \mathbf{12}$

d. To find the value of $P(a)$, we replace the variable x in $P(x)$ with a. So, this time the final answer,

$$P(a) = \mathbf{3a^3 - a^2 + 4},$$

is an expression in terms of a rather than a specific number.

Since polynomials can be evaluated at any real x-value, then the **domain** (*see Section G3, Definition 5.1*) of any polynomial is the set $\mathbb{R}$ of all real numbers.

Addition and Subtraction of Polynomials

Recall that terms with the same variable part are referred to as **like terms** (see *Section R3, Definition 3.1*). Like terms can be **combined** by adding their coefficients. For example,

$$\underbrace{2x^2y - 5x^2y = (2-5)x^2y}_{\substack{by\ distributive\ property \\ (factoring)}} = -3x^2y$$

In practice, this step is not necessary to write.

Unlike terms, such as $2x^2$ and $3x$, **cannot be combined**.

Example 4 ► **Simplifying Polynomial Expressions**

Simplify each polynomial expression.

a. $5x - 4x^2 + 2x + 7x^2$ **b.** $8p - (2 - 3p) + (3p - 6)$

Solution ►

a. To simplify $5x - 4x^2 + 2x + 7x^2$, we combine like terms, starting from the highest degree terms. It is suggested to underline the groups of like terms, using different type of underlining for each group, so that it is easier to see all the like terms and not to miss any of them. So,

$$\underline{\underline{5x}}\ \underline{-4x^2}\ \underline{\underline{+2x}}\ \underline{+7x^2} = \mathbf{3x^2 + 7x}$$

Remember that the sign in front of a term belongs to this term.

b. To simplify $8p - (2 - 3p) + (3p - 6)$, first we remove the brackets using the distributive property of multiplication and then we combine like terms. So, we have

$-(2 - 3p) = (-1)(2 - 3p)$

$$8p - (2 - 3p) + (3p - 6)$$
$$= \underline{8p} - 2 \underline{+3p} \underline{+3p} - 6$$
$$= \mathbf{11p - 8}$$

Example 5 ► **Adding or Subtracting Polynomials**

Perform the indicated operations.

a. $(6a^5 - 4a^3 + 3a - 1) + (2a^4 + a^2 - 5a + 9)$
b. $(4y^3 - 3y^2 + y + 6) - (y^3 + 3y - 2)$
c. $[9p - (3p - 2)] - [4p - (3 - 7p) + p]$

Solution ►

a. To add polynomials, combine their like terms. So,

remove any bracket preceeded by a "+" sign

$$(6a^5 - 4a^3 + 3a - 1) + (2a^4 + a^2 - 5a + 9)$$
$$= 6a^5 \underline{-4a^3}\ \underline{\underline{+3a}} - 1 + 2a^4 \underline{+a^3}\ \underline{\underline{-5a}} + 9$$
$$= \mathbf{6a^5 + 2a^4 - 3a^3 - 8a + 8}$$

b. To subtract a polynomial, add its opposite. In practice, remove any bracket preceeded by a negative sign by reversing the signs of all the terms of the polynomial inside the bracket. So,

$$(4y^3 - 3y^2 + y + 6) - (y^3 + 3y - 2)$$
$$= \underline{4y^3} - 3y^2 \underline{\underline{+y}} + 6\ \underline{-y^3}\ \underline{\underline{-3y}} + 2$$
$$= \mathbf{3y^3 - 3y^2 - 2y + 8}$$

To remove a bracket preceeded by a "−" sign, reverse each sign inside the bracket.

c. First, perform the operations within the square brackets and then subtract the resulting polynomials. So,

$$[9p-(3p-2)]-[4p-(3-7p)+p]$$
$$=[9p-3p+2]-[4p-3+7p+p]$$
$$=[6p+2]-[12p-3]$$
$$=6p+2-12p+3$$
$$=\mathbf{-6p+5}$$

collect like terms before removing the next set of brackets

Addition and Subtraction of Polynomial Functions

Similarly as for polynomials, addition and subtraction can also be defined for general functions.

Definition 1.3 ► Suppose f and g are functions of x with the corresponding domains D_f and D_g.

Then the **sum function** $\boldsymbol{f+g}$ is defined as

$$(\boldsymbol{f+g})(\boldsymbol{x})=\boldsymbol{f}(\boldsymbol{x})+\boldsymbol{g}(\boldsymbol{x})$$

and the **difference function** $\boldsymbol{f-g}$ is defined as

$$(\boldsymbol{f-g})(\boldsymbol{x})=\boldsymbol{f}(\boldsymbol{x})-\boldsymbol{g}(\boldsymbol{x}).$$

The **domain** of the sum or difference function is the intersection $\boldsymbol{D_f\cap D_g}$ of the domains of the two functions.

A frequently used application of a sum or difference of polynomial functions comes from the business area. The fact that profit P equals revenue R minus cost C can be recorded using function notation as

$$P(x)=(R-C)(x)=R(x)-C(x),$$

where x is the number of items produced and sold. Then, if $R(x)=6.5x$ and $C(x)=3.5x+900$, the profit function becomes

$$\boldsymbol{P}(\boldsymbol{x})=R(x)-C(x)=6.5x-(3.5x+900)=6.5x-3.5x-900=\mathbf{3x-900}.$$

Example 6 ► **Adding or Subtracting Polynomial Functions**

Suppose $P(x)=x^2-6x+4$ and $Q(x)=2x^2-1$. Find the following:

a. $(P+Q)(x)$ and $(P+Q)(2)$
b. $(P-Q)(x)$ and $(P-Q)(-1)$
c. $(P+Q)(k)$
d. $(P-Q)(2a)$

Solution ► **a.** Using the definition of the sum of functions, we have

$$(\boldsymbol{P}+\boldsymbol{Q})(\boldsymbol{x}) = P(x)+Q(x) = \underbrace{x^2-6x+4}_{P(x)} + \underbrace{2x^2-1}_{Q(x)} = \mathbf{3x^2-6x+3}$$

Therefore, $(\boldsymbol{P}+\boldsymbol{Q})(\mathbf{2}) = 3\cdot 2^2 - 6\cdot 2 + 3 = 12 - 12 + 3 = \mathbf{3}$.

Alternatively, $(P+Q)(2)$ can be calculated without refering to the function $(P+Q)(x)$, as shown below.

$$(\boldsymbol{P}+\boldsymbol{Q})(\mathbf{2}) = P(2)+Q(2) = \underbrace{2^2-6\cdot 2+4}_{P(2)} + \underbrace{2\cdot 2^2-1}_{Q(2)}$$

$$= 4 - 12 + 4 + 8 - 1 = \mathbf{3}.$$

b. Using the definition of the difference of functions, we have

$$(\boldsymbol{P}-\boldsymbol{Q})(\boldsymbol{x}) = P(x)-Q(x) = \underbrace{x^2-6x+4}_{P(x)} - \underbrace{(2x^2-1)}_{Q(x)}$$

$$= x^2-6x+4-2x^2+1 = \mathbf{-x^2-6x+5}$$

To evaluate $(P-Q)(-1)$, we will take advantage of the difference function calculated above. So, we have

$$(\boldsymbol{P}-\boldsymbol{Q})(\mathbf{-1}) = -(-1)^2 - 6(-1) + 5 = -1 + 6 + 5 = \mathbf{10}.$$

c. By *Definition 1.3*,

$$(\boldsymbol{P}+\boldsymbol{Q})(\boldsymbol{k}) = P(k)+Q(k) = k^2-6k+4+2k^2-1 = \mathbf{3k^2-6k+3}$$

Alternatively, we could use the sum function already calculated in the solution to *Example 6a*. Then, the result is instant: $(\boldsymbol{P}+\boldsymbol{Q})(\boldsymbol{k}) = \mathbf{3k^2-6k+3}$.

d. To find $(P-Q)(2a)$, we will use the difference function calculated in the solution to *Example 6b*. So, we have

$$(\boldsymbol{P}-\boldsymbol{Q})(\mathbf{2a}) = -(2a)^2 - 6(2a) + 5 = \mathbf{-4a^2-12a+5}.$$

P.1 Exercises

Vocabulary Check *Complete each blank with the most appropriate term or phrase from the given list:* ***monomials, polynomial, binomial, trinomial, leading, constant, like, degree.***

1. Terms that are products of only numbers and variables are called _________________.

2. A _______________ is a sum of monomials.

3. A polynomial with two terms is called a ______________.

4. A polynomial with three terms is called a _____________.

5. The _____________ coefficient is the coefficient of the highest degree term.

6. A free term, also called a ___________ term, is the term of zero degree.

7. Only __________ terms can be combined.

8. The __________ of a monomial is the sum of exponents of all its variables.

Concept Check *Determine whether the expression is a monomial.*

9. $-\pi x^3 y^2$ **10.** $5x^{-4}$ **11.** $5\sqrt{x}$ **12.** $\sqrt{2}x^4$

Concept Check *Identify the degree and coefficient.*

13. xy^3 **14.** $-x^2 y$ **15.** $\sqrt{2}xy$ **16.** $-3\pi x^2 y^5$

Concept Check *Arrange each polynomial in descending order of powers of the variable. Then, identify the degree and the leading coefficient of the polynomial.*

17. $5 - x + 3x^2 - \frac{2}{5}x^3$ **18.** $7x + 4x^4 - \frac{4}{3}x^3$ **19.** $8x^4 + 2x^3 - 3x + x^5$

20. $4y^3 - 8y^5 + y^7$ **21.** $q^2 + 3q^4 - 2q + 1$ **22.** $3m^2 - m^4 + 2m^3$

Concept Check *State the degree of each polynomial and identify it as a monomial, binomial, trinomial, or n-th degree polynomial if $n > 2$.*

23. $7n - 5$ **24.** $4z^2 - 11z + 2$ **25.** 25

26. $-6p^4 q + 3p^3 q^2 - 2pq^3 - p^4$ **27.** $-mn^6$ **28.** $16k^2 - 9p^2$

Concept Check *Let $P(x) = -2x^2 + x - 5$ and $Q(x) = 2x - 3$. Evaluate each expression.*

29. $P(-1)$ **30.** $P(0)$ **31.** $2P(1)$ **32.** $P(a)$

33. $Q(-1)$ **34.** $Q(5)$ **35.** $Q(a)$ **36.** $Q(3a)$

37. $3Q(-2)$ **38.** $3P(a)$ **39.** $3Q(a)$ **40.** $Q(a+1)$

Concept Check *Simplify each polynomial expression.*

41. $5x + 4y - 6x + 9y$ **42.** $4x^2 + 2x - 6x^2 - 6$

43. $6xy + 4x - 2xy - x$ **44.** $3x^2 y + 5xy^2 - 3x^2 y - xy^2$

45. $9p^3 + p^2 - 3p^3 + p - 4p^2 + 2$ **46.** $n^4 - 2n^3 + n^2 - 3n^4 + n^3$

47. $4 - (2 + 3m) + 6m + 9$ **48.** $2a - (5a - 3) - (7a - 2)$

49. $6 + 3x - (2x + 1) - (2x + 9)$ **50.** $4y - 8 - (-3 + y) - (11y + 5)$

Perform the indicated operations.

51. $(x^2 - 5y^2 - 9z^2) + (-6x^2 + 9y^2 - 2z^2)$ **52.** $(7x^2 y - 3xy^2 + 4xy) + (-2x^2 y - xy^2 + xy)$

53. $(-3x^2 + 2x - 9) - (x^2 + 5x - 4)$

54. $(8y^2 - 4y^3 - 3y) - (3y^2 - 9y - 7y^3)$

55. $(3r^6 + 5) + (-7r^2 + 2r^6 - r^5)$

56. $(5x^{2a} - 3x^a + 2) + (-x^{2a} + 2x^a - 6)$

57. $(-5a^4 + 8a^2 - 9) - (6a^3 - a^2 + 2)$

58. $(3x^{3a} - x^a + 7) - (-2x^{3a} + 5x^{2a} - 1)$

59. $(10xy - 4x^2y^2 - 3y^3) - (-9x^2y^2 + 4y^3 - 7xy)$

60. Subtract $(-4x + 2z^2 + 3m)$ from the sum of $(2z^2 - 3x + m)$ and $(z^2 - 2m)$.

61. Subtract the sum of $(2z^2 - 3x + m)$ and $(z^2 - 2m)$ from $(-4x + 2z^2 + 3m)$.

62. $[2p - (3p - 6)] - [(5p - (8 - 9p)) + 4p]$

63. $-[3z^2 + 5z - (2z^2 - 6z)] + [(8z^2 - (5z - z^2)) + 2z^2]$

64. $5k - (5k - [2k - (4k - 8k)]) + 11k - (9k - 12k)$

Discussion Point

65. If $P(x)$ and $Q(x)$ are polynomials of degree 3, is it possible for the sum of the two polynomials to have degree 2? If so, give an example. If not, explain why.

*For each pair of functions, find **a)** $(f + g)(x)$ and **b)** $(f - g)(x)$.*

66. $f(x) = 5x - 6, \quad g(x) = -2 + 3x$

67. $f(x) = x^2 + 7x - 2, \quad g(x) = 6x + 5$

68. $f(x) = 3x^2 - 5x, \quad g(x) = -5x^2 + 2x + 1$

69. $f(x) = 2x^n - 3x - 1, \quad g(x) = 5x^n + x - 6$

70. $f(x) = 2x^{2n} - 3x^n + 3, \quad g(x) = -8x^{2n} + x^n - 4$

Let $\boldsymbol{P(x) = x^2 - 4}$, $\boldsymbol{Q(x) = 2x + 5}$, *and* $\boldsymbol{R(x) = x - 2}$. *Find each of the following.*

71. $(P + R)(-1)$

72. $(P - Q)(-2)$

73. $(Q - R)(3)$

74. $(R - Q)(0)$

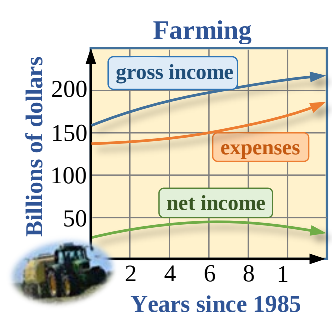
75. $(R - Q)(k)$

76. $(P + Q)(a)$

77. $(Q - R)(a + 1)$

78. $(P + R)(2k)$

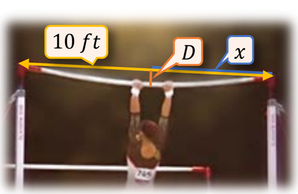
Analytic Skills *Solve each problem.*



79. From 1985 through 1995, the gross farm income G and farm expenses E (in billions of dollars) in the United States can be modeled by

$$G(t) = -0.246t^2 + 7.88t + 159 \quad \text{and} \quad E(t) = 0.174t^2 + 2.54t + 131$$

where t is the number of years since 1985. Write a model for the net farm income $N(t)$ for these years.



80. Suppose the maximum deflection D (in centimeters) of a gimnastic bar is given by the polynomial function $D(x) = 0.015x^4 - 0.3x^3 + 1.5x^2$, where x (in feet) is the distance from one end of the bar. The maximum deflection occurs when x is equal to half of the length of the bar. Determine the maximum deflection for the 10 ft long bar, as shown in the accompanying figure.

81. Write a polynomial that gives the sum of the areas of a square with sides of length x and a circle with radius x. Find the combined area when $x = 10$ centimeters. *Round the answer to the nearest centimeter square.*

82. Suppose the cost in dollars for a band to produce x compact discs is given by $C(x) = 4x + 2000$. If the band sells CDs for $16 each, complete the following.

a. Write a function $R(x)$ that gives the revenue for selling x compact discs.

b. If profit equals revenue minus cost, write a formula $P(x)$ for the profit.

c. Evaluate $P(3000)$ and interpret the answer.