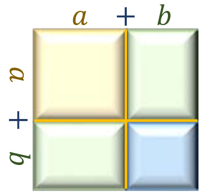
P.2 Multiplication of Polynomials

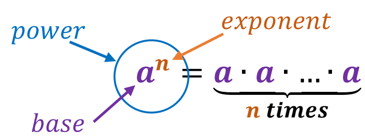


As shown in the previous section, addition and subtraction of polynomials results in another polynomial. This means that the **set of polynomials** is **closed under** the operation of **addition** and **subtraction**. In this section, we will show that the set of polynomials is also closed under the operation of **multiplication**, meaning that a product of polynomials is also a polynomial.

Properties of Exponents

Since multiplication of polynomials involves multiplication of powers, let us review properties of exponents first.

Recall:



For example, $x^4 = x \cdot x \cdot x \cdot x$ and we read it "x to the fourth power" or shorter "x to the fourth". If $n = 2$, the power x^2 is customarily read "x squared". If $n = 3$, the power x^3 is often read "x cubed".

Let $a \in \mathbb{R}$, and $m, n \in \mathbb{W}$. The table below shows basic exponential rules with some examples justifying each rule.

Power Rules for Exponents

General Rule	Description	Example
$a^m \cdot a^n = a^{m+n}$	To **multiply powers** of the same bases, keep the base and **add** the **exponents**.	$x^2 \cdot x^3 = (x \cdot x) \cdot (x \cdot x \cdot x) = x^{2+3} = x^5$
$\frac{a^m}{a^n} = a^{m-n}$	To **divide powers** of the same bases, keep the base and **subtract** the **exponents**.	$\frac{x^5}{x^2} = \frac{(x \cdot x \cdot x \cdot \cancel{x} \cdot \cancel{x})}{(\cancel{x} \cdot \cancel{x})} = x^{5-2} = x^3$
$(a^m)^n = a^{mn}$	To raise a **power to a power, multiply the exponents.**	$(x^2)^3 = (x \cdot x)(x \cdot x)(x \cdot x) = x^{2 \cdot 3} = x^6$
$(ab)^n = a^n b^n$	To raise a **product to a power**, raise **each factor to that power**.	$(2x)^3 = 2^3 x^3$
$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	To raise a **quotient to a power**, raise the **numerator and** the **denominator to that power**.	$\left(\frac{x}{3}\right)^2 = \frac{x^2}{3^2}$
$a^0 = 1$ for $a \neq 0$ 0^0 is undefined	A **nonzero number** raised **to the power of zero** equals **one**.	$x^0 = x^{n-n} = \frac{x^n}{x^n} = 1$

Example 1 ► **Simplifying Exponential Expressions**

Simplify.

a. $(-3xy^2)^4$ **b.** $(5p^3q)(-2pq^2)$

c. $\left(\frac{-2x^5}{x^2y}\right)^3$ **d.** $x^{2a}x^a$

Solution ►

a. To simplify $(-3xy^2)^4$, we apply the fourth power to each factor in the bracket. So,

$$(-3xy^2)^4 = \underbrace{(-3)^4}_{\substack{\text{even power}\\ \text{of a negative}\\ \text{is a positive}}} \cdot x^4 \cdot \underbrace{(y^2)^4}_{\substack{\text{multiply}\\ \text{exponents}}} = \mathbf{3^4x^4y^8}$$

$p = p^1$

b. To simplify $(5p^3q)(-2pq^2)$, we multiply numbers, powers of p, and powers of q. So,

$$(5p^3q)(-2pq^2) = (-2)\cdot 5 \cdot \underbrace{p^3 \cdot p}_{\substack{\text{add}\\ \text{exponents}}} \cdot \underbrace{q \cdot q^2}_{\substack{\text{add}\\ \text{exponents}}} = \mathbf{-10p^4q^3}$$

c. To simplify $\left(\frac{-2x^5}{x^2y}\right)^3$, first we reduce the common factors and then we raise every factor of the numerator and denominator to the third power. So, we obtain

$$\left(\frac{-2x^{\not{5}\,3}}{\not{x^2}y}\right)^3 = \left(\frac{-2x^3}{y}\right)^3 = \frac{(-2)^3(x^3)^3}{y^3} = \mathbf{\frac{-8x^9}{y^3}}$$

d. When multiplying powers with the same bases, we add exponents, so $x^{2a}x^a = \mathbf{x^{3a}}$

Multiplication of Polynomials

Multiplication of polynomials involves finding products of monomials. To multiply monomials, we use the commutative property of multiplication and the product rule of powers.

Example 2 ► **Multiplying Monomials**

Find each product.

a. $(3x^4)(5x^3)$ **b.** $(5b)(-2a^2b^3)$ **c.** $-4x^2(3xy)(-x^2y)$

Solution ►

a. $$(3x^4)(5x^3) = 3 \cdot \underbrace{x^4 \cdot 5}_{\substack{\text{commutative}\\ \text{property}}} \cdot x^3 = 3 \cdot 5 \cdot \underbrace{x^4 \cdot x^3}_{\substack{\text{product}\\ \text{rule of powers}}} = \mathbf{15x^7}$$

The intermediate steps are not necessary to write. The final answer is immediate if we follow the order: **sign**, **number**, **power** of each variable.

To find the product of monomials, find the following:
- the final **sign**,
- the **number**,
- the **power**.

b. $$(5b)(-2a^2b^3) = 5(-2)a^2bb^3 = \mathbf{-10a^2b^4}$$

c. $$-4x^2(3xy)(-x^2y) = \underbrace{(-4)\cdot 3 \cdot (-1)}_{\substack{\text{multiply}\\ \text{coefficients}}} \underbrace{x^2xx^2}_{\substack{\text{apply product}\\ \text{rule of powers}}} \underbrace{yy} = \mathbf{12x^5y^2}$$

To multiply polynomials by a monomial, we use the distributive property of multiplication.

Example 3 ► **Multiplying Polynomials by a Monomial**

Find each product.

a. $-2x(3x^2 - x + 7)$ **b.** $(5b - ab^3)(3ab^2)$

Solution ► **a.** To find the product $-2x(3x^2 - x + 7)$, we distribute the monomial $-2x$ to each term inside the bracket. So, we have

$$-2x(3x^2 - x + 7) = \underbrace{-2x(3x^2) - 2x(-x) - 2x(7)}_{\textit{this step can be done mentally}} = \mathbf{-6x^3 + 2x^2 - 14x}$$

b. $$(5b - ab^3)(3ab^2) = \underbrace{5b(3ab^2) - ab^3(3ab^2)}_{\textit{this step can be done mentally}}$$

$$= 15ab^3 - 3a^2b^5 = \mathbf{-3a^2b^5 + 15ab^3}$$

arranged in decreasing order of powers

When multiplying polynomials by polynomials we **multiply each term of the first polynomial by each term of the second polynomial**. This process can be illustrated with finding areas of a rectangle whose sides represent each polynomial. For example, we multiply $(2x + 3)(x^2 - 3x + 1)$ as shown below

	x^2	$-3x$	$+1$
$2x$	$2x^3$	$-6x^2$	$2x$
$+3$	$3x^2$	$-9x$	3

So, $(2x + 3)(x^2 - 3x + 1) = 2x^3 - 6x^2 + 2x$

$$+3x^2 - 9x + 3$$

$$= \mathbf{2x^3 - 3x^2 - 7x + 3}$$

line up like terms to combine them

Example 4 ► **Multiplying Polynomials by Polynomials**

Find each product.

a. $(3y^2 - 4y - 2)(5y - 7)$ **b.** $4a^2(2a - 3)(3a^2 + a - 1)$

Solution ► **a.** To find the product $(3y^2 - 4y - 2)(5y - 7)$, we can distribute the terms of the second bracket over the first bracket and then collect the like terms. So, we have

$$(3y^2 - 4y - 2)(5y - 7) = 15y^3 - 20y^2 - 10y$$

$$- 21y^2 + 28y + 14$$

$$= \mathbf{15y^3 - 41y^2 + 18y + 14}$$

b. To find the product $4a^2(2a - 3)(3a^2 + a - 1)$, we will multiply the two brackets first, and then multiply the resulting product by $4a^2$. So,

$$4a^2(2a-3)(3a^2+a-1) = 4a^2\underbrace{\left(6a^3 + 2a^2 - 2a - 9a^2 - 3a + 3\right)}_{\substack{\textit{collect like terms before}\\ \textit{removing the bracket}}}$$

$$= 4a^2(6a^3 - 7a^2 - 5a + 3) = \mathbf{24a^5 - 28a^4 - 20a^3 + 12a^2}$$

In multiplication of binomials, it might be convenient to keep track of the multiplying terms by following the **FOIL** mnemonic, which stands for multiplying the **F**irst, **O**uter, **I**nner, and **L**ast terms of the binomials. Here is how it works:

First Last

$$(2x-3)(x+5) = 2x^2 \underbrace{+10x - 3x}_{} - 15 = \mathbf{2x^2 + 7x - 15}$$

Inner

Outer

*the **sum** of the **Outer** and **Inner** terms becomes the **middle** term*

Example 5 ► Using the FOIL Method in Binomial Multiplication

Find each product.

a. $(x+3)(x-4)$ **b.** $(5x-6)(2x+3)$

Solution ► **a.** To find the product $(x+3)(x-4)$, we may follow the **FOIL** method

x^2 -12

$$(x+3)(x-4) = \mathbf{x^2 - x - 12}$$

$3x$

$-4x$

To find the **linear (middle)** term try to add the inner and outer products mentally.

b. Observe that the linear term of the product $(5x-6)(2x+3)$ is equal to the sum of $-12x$ and $15x$, which is $3x$. So, we have

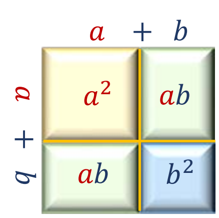
$$(5x-6)(2x+3) = \mathbf{10x^2 + 3x - 18}$$

Special Products

Suppose we want to find the product $(a+b)(a+b)$. This can be done via the FOIL method

$$(\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{a}+\boldsymbol{b}) = a^2 + ab + ab + b^2 = \boldsymbol{a^2 + 2ab + b^2},$$

or via the geometric visualization:



Observe that since the products of the inner and outer terms are both equal to ab, we can use a shortcut and write the middle term of the final product as $2ab$. We encorage the reader to come up with similar observations regarding the product $(a-b)(a-b)$. This regularity in multiplication of a binomial by itself leads us to the **perfect square formula**:

$$\boldsymbol{(a \pm b)^2 = a^2 \pm 2ab + b^2}$$

In the above notation, the " $\pm$ " sign is used to record two formulas at once, the perfect square of a sum and the perfect square of a difference. It tells us to either use a " $+$ " in both places, or a " $-$ " in both places. The $\boldsymbol{a}$ and $\boldsymbol{b}$ can actually represent any expression. For example, to expand $(2x-y^2)^2$, we can apply the perfect square formula by treating $2x$ as a and y^2 as b. Here is the calculation.

$$(2x-y^2)^2 = (2x)^2 - 2(2x)y^2 + (y^2)^2 = \boldsymbol{2x^2 - 4xy^2 + y^4}$$

Conjugate binomials have the **same first terms** and **opposite second terms**.

Another interesting pattern can be observed when multiplying two **conjugate** brackets, such as $(a+b)$ and $(a-b)$. Using the FOIL method,

$$\boldsymbol{(a+b)(a-b)} = a^2 + \cancel{ab} - \cancel{ab} - b^2 = \boldsymbol{a^2 - b^2},$$

we observe, that the products of the inner and outer terms are opposite. So, they add to zero and the product of conjugate brackets becomes the difference of squares of the two terms. This regularity in multiplication of conjugate brackets leads us to the **difference of squares formula**.

$$\boldsymbol{(a+b)(a-b) = a^2 - b^2}$$

Again, $\boldsymbol{a}$ and $\boldsymbol{b}$ can represent any expression. For example, to find the product $(3x+0.1y^2)(3x-0.1y^2)$, we can apply the difference of squares formula by treating $3x$ as a and $0.1y^2$ as b. Here is the calculation.

$$\boldsymbol{(3x+0.1y^3)(3x-0.1y^3)} = (3x)^2 - (0.1y^3)^2 = \boldsymbol{9x^2 - 0.01y^6}$$

We encourage to use the above formulas whenever applicable, as it allows for more efficient calculations and helps to observe patterns useful in future factoring.

Example 6 ► **Useing Special Product Formulas in Polynomial Multiplication**

Find each product. Apply special products formulas, if applicable.

a. $(5x+3y)^2$ **b.** $(x+y-5)(x+y+5)$

Solution ►

a. Applying the perfect square formula, we have

$$(5x+3y)^2 = (5x)^2 + 2(5x)3y + (3y)^2 = \boldsymbol{25x^2 + 30xy + 9y^2}$$

b. The product $(x+y-5)(x+y+5)$ can be found by multiplying each term of the first polynomial by each term of the second polynomial, using the distributive property. However, we can find the product $(x+y-5)(x+y+5)$ in a more efficient way by applying the difference of squares formula. Treating the expression $x+y$ as the first term a and the 5 as the second term b in the formula $(a+b)(a-b)=a^2-b^2$, we obtain

$$(x+y-5)(x+y+5)=(x+y)^2-5^2$$
$$= \underbrace{\boldsymbol{x^2+2xy+y^2}}_{\text{here we apply the perfect square formula}} \boldsymbol{-25}$$

Caution: *The perfect square formula shows that* $(\boldsymbol{a}+\boldsymbol{b})^2 \neq \boldsymbol{a}^2+\boldsymbol{b}^2$.
The difference of squares formula shows that $(\boldsymbol{a}-\boldsymbol{b})^2 \neq \boldsymbol{a}^2-\boldsymbol{b}^2$.
More generally, $(\boldsymbol{a}\pm\boldsymbol{b})^{\boldsymbol{n}} \neq \boldsymbol{a}^{\boldsymbol{n}} \pm \boldsymbol{b}^{\boldsymbol{n}}$ *for any natural* $n \neq 1$.

Product Functions

The operation of multiplication can be defined not only for polynomials but also for general functions.

Definition 2.1 ► Suppose f and g are functions of x with the corresponding domains D_f and D_g.

Then the **product function**, denoted $\boldsymbol{f\cdot g}$ or $\boldsymbol{fg}$, is defined as

$$(\boldsymbol{f\cdot g})(\boldsymbol{x})=\boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{g}(\boldsymbol{x}).$$

The **domain** of the product function is the intersection $\boldsymbol{D_f\cap D_g}$ of the domains of the two functions.

Example 7 ► **Multiplying Polynomial Functions**

Suppose $P(x)=x^2-4x$ and $Q(x)=3x+2$. Find the following:

a. $(PQ)(x)$, $(PQ)(-2)$, and $P(-2)Q(-2)$
b. $(QQ)(x)$ and $(QQ)(1)$
c. $2(PQ)(k)$

Solution ► **a.** Using the definition of the product function, we have

$$(PQ)(x)=P(x)\cdot Q(x)=(x^2-4x)(3x+2)=3x^3+2x^2-12x^2-8x$$
$$=\boldsymbol{3x^3-10x^2-8x}$$

To find $(PQ)(-2)$, we substitute $x=-2$ to the above polynomial function. So,

$$(PQ)(-2)=3(-2)^3-10(-2)^2-8(-2)=3\cdot(-8)-10\cdot 4+16$$
$$=-24-40+16=\boldsymbol{-48}$$

To find $P(-2)Q(-2)$, we calculate

$$P(-2)Q(-2)=\left((-2)^2-4(-2)\right)(3(-2)+2)=(4+8)(-6+2)=12\cdot(-4)$$
$$=\boldsymbol{-48}$$

Observe that both expressions result in the same value. This was to expect, as by the definition, $(PQ)(-2) = P(-2) \cdot Q(-2)$.

b. Using the definition of the product function as well as the perfect square formula, we have

$$(QQ)(x) = Q(x) \cdot Q(x) = [Q(x)]^2 = (3x+2)^2 = \mathbf{9x^2 + 12x + 4}$$

Therefore, $(QQ)(1) = 9 \cdot 1^2 + 12 \cdot 1 + 4 = 9 + 12 + 4 = \mathbf{25}$.

c. Since $(PQ)(x) = 3x^3 - 10x^2 - 8x$, as shown in the solution to *Example 7a*, then $(PQ)(k) = 3k^3 - 10k^2 - 8k$. Therefore,

$$2(PQ)(k) = 2[3k^3 - 10k^2 - 8k] = \mathbf{6k^3 - 20k^2 - 16k}$$

P.2 Exercises

Vocabulary Check *Complete each blank with the most appropriate term from the given list:* ***add, binomials, conjugate, divide, intersection, multiply, perfect.***

1. To multiply powers of the same bases, keep the base and __________ the exponents.

2. To ___________ powers of the same bases, keep the base and subtract the exponents.

3. To calculate a power of a power, _______________ exponents.

4. The domain of a product function fg is the _________________ of the domains of each of the functions, f and g.

5. The FOIL rule applies only to the multiplication of _______________.

6. A difference of squares is a product of two _______________ binomials, such as $(x+y)$ and $(x-y)$.

7. A ______________ square is a product of two identical binomials.

Concept Check

8. Decide whether each expression has been simplified correctly. If not, correct it.

a. $x^2 \cdot x^4 = x^8$ **b.** $-2x^2 = 4x^2$ **c.** $(5x)^3 = 5^3x^3$

d. $-\left(\frac{x}{5}\right)^2 = -\frac{x^2}{25}$ **e.** $(a^2)^3 = a^5$ **f.** $4^5 \cdot 4^2 = 16^7$

g. $\frac{6^5}{3^2} = 2^3$ **h.** $xy^0 = 1$ **i.** $(-x^2y)^3 = -x^6y^3$

Simplify each expression.

9. $3x^2 \cdot 5x^3$ **10.** $-2y^3 \cdot 4y^5$ **11.** $3x^3(-5x^4)$

12. $2x^2y^5(7xy^3)$ **13.** $(6t^4s)(-3t^3s^5)$ **14.** $(-3x^2y)^3$

15. $\frac{12x^3y}{4xy^2}$

16. $\frac{15x^5y^2}{-3x^2y^4}$

17. $(-2x^5y^3)^2$

18. $\left(\frac{4a^2}{b}\right)^3$

19. $\left(\frac{-3m^4}{n^3}\right)^2$

20. $\left(\frac{-5p^2q}{pq^4}\right)^3$

21. $3a^2(-5a^5)(-2a)^0$

22. $-3a^3b(-4a^2b^4)(ab)^0$

23. $\frac{(-2p)^2pq^3}{6p^2q^4}$

24. $\frac{(-8xy)^2y^3}{4x^5y^4}$

25. $\left(\frac{-3x^4y^6}{18x^6y^7}\right)^3$

26. $((-2x^3y)^2)^3$

27. $((-a^2b^4)^3)^5$

28. x^nx^{n-1}

29. $3a^{2n}a^{1-n}$

30. $(5^a)^{2b}$

31. $(-7^{3x})^{4y}$

32. $\frac{-12x^{a+1}}{6x^{a-1}}$

33. $\frac{25x^{a+b}}{-5x^{a-b}}$

34. $\left(x^{a+b}\right)^{a-b}$

35. $(x^2y)^n$

Concept Check *Find each product.*

36. $8x^2y^3(-2x^5y)$

37. $5a^3b^5(-3a^2b^4)$

38. $2x(-3x+5)$

39. $4y(1-6y)$

40. $-3x^4y(4x-3y)$

41. $-6a^3b(2b+5a)$

42. $5k^2(3k^2-2k+4)$

43. $6p^3(2p^2+5p-3)$

44. $(x+6)(x-5)$

45. $(x-7)(x+3)$

46. $(2x+3)(3x-2)$

47. $3p(5p+1)(3p+2)$

48. $2u^2(u-3)(3u+5)$

49. $(2t+3)(t^2-4t-2)$

50. $(2x-3)(3x^2+x-5)$

51. $(a^2-2b^2)(a^2-3b^2)$

52. $(2m^2-n^2)(3m^2-5n^2)$

53. $(x+5)(x-5)$

54. $(a+2b)(a-2b)$

55. $(x+4)(x+4)$

56. $(a-2b)(a-2b)$

57. $(x-4)(x^2+4x+16)$

58. $(y+3)(y^2-3y+9)$

59. $(x^2+x-2)(x^2-2x+3)$

60. $(2x^2+y^2-2xy)(x^2-2y^2-xy)$

Concept Check *True or False? If it is false, show a counterexample by choosing values for a and b that would not satisfy the equation.*

61. $(a+b)^2=a^2+b^2$

62. $a^2-b^2=(a-b)(a+b)$

63. $(a-b)^2=a^2+b^2$

64. $(a+b)^2=a^2+2ab+b^2$

65. $(a-b)^2=a^2+ab+b^2$

66. $(a-b)^3=a^3-b^3$

*Find each product. Use the **difference of squares** or the **perfect square** formula, if applicable.*

67. $(2p+3)(2p-3)$

68. $(5x-4)(5x+4)$

69. $\left(b-\frac{1}{3}\right)\left(b+\frac{1}{3}\right)$

70. $\left(\frac{1}{2}x-3y\right)\left(\frac{1}{2}x+3y\right)$

71. $(2xy+5y^3)(2xy-5y^3)$

72. $(x^2+7y^3)(x^2-7y^3)$

73. $(1.1x+0.5y)(1.1x-0.5y)$

74. $(0.8a+0.2b)(0.8a+0.2b)$

75. $(x+6)^2$

76. $(x-3)^2$

77. $(4x+3y)^2$

78. $(5x-6y)^2$

79. $\left(3a+\frac{1}{2}\right)^2$ **80.** $\left(2n-\frac{1}{3}\right)^2$ **81.** $(a^3b^2-1)^2$

82. $(x^4y^2+3)^2$ **83.** $(3a^2+4b^3)^2$ **84.** $(2x^2-3y^3)^2$

85. $3y(5xy^3+2)(5xy^3-2)$ **86.** $2a(2a^2+5ab)(2a^2+5ab)$ **87.** $3x(x^2y-xy^3)^2$

88. $(-xy+x^2)(xy+x^2)$ **89.** $(4p^2+3pq)(-3pq+4p^2)$ **90.** $(x+1)(x-1)(x^2+1)$

91. $(2x-y)(2x+y)(4x^2+y^2)$ **92.** $(a-b)(a+b)(a^2-b^2)$ **93.** $(a+b+1)(a+b-1)$

94. $(2x+3y-5)(2x+3y+5)$ **95.** $(3m+2n)(3m-2n)(9m^2-4n^2)$

96. $\left((2k-3)+h\right)^2$ **97.** $\left((4x+y)-5\right)^2$

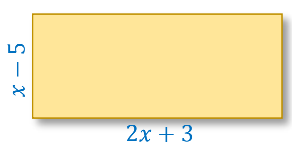
98. $(x^a+y^b)(x^a-y^b)(x^{2a}+y^{2b})$ **99.** $(x^a+y^b)(x^a-y^b)(x^{2a}-y^{2b})$

Concept Check *Use the difference of squares formula, $(a+b)(a-b)=a^2-b^2$, to find each product.*

100. $101\cdot 99$ **101.** $198\cdot 202$ **102.** $505\cdot 495$

Find the area of each figure. Express it as a polynomial in descending powers of the variable x.

103.



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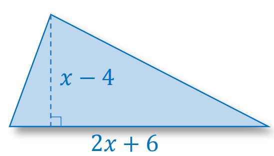

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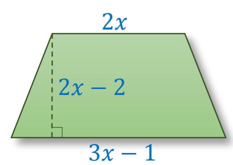

Concept Check *For each pair of functions, $\boldsymbol{f}$ and $\boldsymbol{g}$, find the **product** function $(\boldsymbol{fg})(\boldsymbol{x})$.*

106. $f(x)=5x-6,\quad g(x)=-2+3x$ **107.** $f(x)=x^2+7x-2,\quad g(x)=6x+5$

108. $f(x)=3x^2-5x,\quad g(x)=9+x-x^2$ **109.** $f(x)=x^n-4,\quad g(x)=x^n+1$

Let $\boldsymbol{P(x)=x^2-4}$, $\boldsymbol{Q(x)=2x}$, and $\boldsymbol{R(x)=x-2}$. Find each of the following.

110. $(PR)(x)$ **111.** $(PQ)(x)$ **112.** $(PQ)(a)$

113. $(PR)(-1)$ **114.** $(PQ)(3)$ **115.** $(PR)(0)$

116. $(QR)(x)$ **117.** $(QR)\left(\frac{1}{2}\right)$ **118.** $(QR)(a+1)$

119. $P(a-1)$ **120.** $P(2a+3)$ **121.** $P(1+h)-P(1)$

Analytic Skills *Solve each problem.*

122. The corners are cut from a rectangular piece of cardboard measuring 8 in. by 12 in. The sides are folded up to make a box. Find the volume of the box in terms of the variable x, where x is the length of a side of the square cut from each corner of the rectangle.

123. A rectangular pen has a perimeter of 100 feet. If x represents its width, write a polynomial that gives the area of the pen in terms of x.