P.3 Division of Polynomials

In this section we will discuss dividing polynomials. The result of division of polynomials is not always a polynomial. For example, $x + 1$ divided by x becomes

$$\frac{x+1}{x} = \frac{x}{x} + \frac{1}{x} = 1 + \frac{1}{x},$$

which is not a polynomial. Thus, the set of polynomials is not closed under the operation of division. However, we can perform division with remainders, mirroring the algorithm of division of natural numbers. We begin with dividing a polynomial by a monomial and then by another polynomial.

Division of Polynomials by Monomials

To divide a polynomial by a monomial, we divide each term of the polynomial by the monomial, and then simplify each quotient. In other words, we use the reverse process of addition of fractions, as illustrated below.

$$\boldsymbol{\frac{a+b}{d} = \frac{a}{d} + \frac{b}{d}}$$

Example 1 ► **Dividing Polynomials by Monomials**

Divide and simplify.

a. $(6x^3 + 15x^2 - 2x) \div (3x)$ **b.** $\frac{xy^2 - 8x^2y + 6x^3y^2}{-2xy^2}$

Solution ►

a. $(6x^3 + 15x^2 - 2x) \div (3x) = \frac{6x^3 + 15x^2 - 2x}{3x} = \frac{6x^3}{3x} + \frac{15x^2}{3x} - \frac{2x}{3x} = \boldsymbol{2x^2 + 5x - \frac{2}{3}}$

b. $\frac{xy^2 - 8x^2y + 6x^3y^2}{-2xy^2} = -\frac{xy^2}{2xy^2} + \frac{8x^2y}{2xy^2} - \frac{6x^3y^2}{2xy^2} = \boldsymbol{-\frac{1}{2} + \frac{4x}{y} - 3x^2}$

Division of Polynomials by Polynomials

To divide a polynomial by another polynomial, we follow an algorithm similar to the long division algorithm used in arithmetic. For example, observe the steps taken in the long division algorithm when dividing 158 by 13 and the corresponding steps when dividing $x^2 + 5x + 8$ by $x + 3$.

Step 1: Place the dividend under the long division symbol and the divisor in front of this symbol.

$13\,\overline{)\,158}$ $\underbrace{x+3}_{divisor}\,\overline{)\,\underbrace{x^2 + 5x + 8}_{dividend}}$

Remember: Both polynomials should be written in **decreasing order of powers**. Also, any **missing terms** after the leading term should be displayed with a **zero coefficient**. This will ensure that the terms in each column are of the same degree.

Step 2: Divide the first term of the dividend by the first term of the divisor and record the quotient above the division symbol.

$$\begin{array}{r|l} & \ 1 \\ \hline \underbrace{13} & \underbrace{15}8 \end{array} \qquad\qquad \begin{array}{r|l} & \overbrace{\ x \qquad\qquad}^{quatient} \\ \hline \underbrace{x}+3 & \underbrace{x^2}+5x+8 \end{array}$$

Step 3: Multiply the quotient from *Step 2* by the divisor and write the product under the dividend, lining up the columns with the same degree terms.

$$\begin{array}{r|l} & \ \ 1 \\ \hline 13 & 158 \\ & 13 \end{array} \qquad\qquad \begin{array}{r|l} & x \\ \hline x+3 & x^2+5x+8 \\ & x^2+3x \end{array}$$

Step 4: Underline and subtract by adding opposite terms in each column. We suggest to record the new sign in a circle, so that it is clear what is being added.

$$\begin{array}{r|l} & \ \ 1 \\ \hline 13 & 158 \\ & \underline{-13} \\ & \ \ \ 2 \end{array} \qquad\qquad \begin{array}{r|l} & x \\ \hline x+3 & x^2+5x+8 \\ & \underline{-(x^2 \overset{\ominus}{+} 3x)} \\ & \qquad 2x \end{array}$$

Step 5: Drop the next term (or digit) and repeat the algorithm until the degree of the remainder is lower than the degree of the divisor.

$$\begin{array}{r|l} & \ \ 12 \\ \hline 13 & 158 \\ & \underline{-13} \\ & \ \ \ 28 \\ & \underline{-26} \\ & \ \ \ \ 2 \end{array} \qquad\qquad \begin{array}{r|l} & x \ + \ 2 \\ \hline x+3 & x^2+5x+8 \\ & \underline{-(x^2 \overset{\ominus}{+} 3x)} \\ & \qquad 2x+8 \\ & \qquad \underline{-(2x \overset{\ominus}{+} 6)} \\ & \qquad\qquad 2 \leftarrow remainder \end{array}$$

In the example of long division of numbers, we have $158 = 13 \cdot 12 + 2$.

So, the quotient can be written as $\frac{158}{13} = \mathbf{12} + \frac{\mathbf{2}}{\mathbf{13}}$.

In the example of long division of polynomials, we have

$$x^2 + 5x + 8 = (x+3) \cdot (x \ + \ 2) + 2.$$

So, the quotient can be written as $\frac{x^2+5x+8}{x+3} = \boldsymbol{x+2} + \frac{\mathbf{2}}{\boldsymbol{x+3}}$.

Generally, if P, D, Q, and R are polynomials, such that $P(x) = D(x) \cdot Q(x) + R(x)$, then the ratio of polynomials P and D can be written as

$$\boxed{\frac{\boldsymbol{P(x)}}{\boldsymbol{D(x)}} = \boldsymbol{Q(x)} + \frac{\boldsymbol{R(x)}}{\boldsymbol{D(x)}},}$$

where $Q(x)$ is the quotient polynomial, and $R(x)$ is the remainder from the division of $P(x)$ by the divisor $D(x)$.

Observe: The degree of the remainder must be lower than the degree of the divisor, as otherwise, we could apply the division algorithm one more time.

Example 2 ► **Dividing Polynomials by Polynomials**

Divide.

a. $(3x^3 - 2x^2 + 5) \div (x^2 - 3)$ **b.** $\frac{2p^3+2p+3p^2}{5+2p}$

Solution ► **a.** When writing the polynomials in the long division format, we use a zero placeholder term in place of the missing linear terms in both, the dividend and the divisor. So, we have

$$
\begin{array}{rl}
 & \mathbf{3x - 2} \\
x^2 + 0x - 3 & \overline{)\,3x^3 - 2x^2 + 0x + 5} \\
 & -(3x^3 + 0x^2 + 9x) \\
 & \qquad -2x^2 - 9x + 5 \\
 & \qquad -(-2x^2 - 0x + 6) \\
 & \qquad\qquad -\mathbf{9x - 1}
\end{array}
$$

Thus, $(3x^3 - 2x^2 + 5) \div (x^2 - 3) = 3x - 2 + \frac{-9x-1}{x^2-3} = \mathbf{3x - 2 - \frac{9x+1}{x^2-3}}$.

b. To perform this division, we arrange both polynomials in decreasing order of powers, and replace the constant term in the dividend with a zero. So, we have

$$
\begin{array}{rl}
 & \mathbf{p^2 - p + \frac{7}{2}} \\
2p + 5 & \overline{)\,2p^3 + 3p^2 + 2p + 0} \\
 & -(2p^3 + 5p^2) \\
 & \qquad -2p^2 + 2p \\
 & \qquad -(-2p^2 - 5p) \\
 & \qquad\qquad 7p + 0 \\
 & \qquad\qquad -\left(7p + \frac{35}{2}\right) \\
 & \qquad\qquad\qquad -\mathbf{\frac{35}{2}}
\end{array}
$$

Thus, $\frac{2p^3+2p+3p^2}{5+2p} = p^2 - p + \frac{7}{2} + \frac{-\frac{35}{2}}{2p+5} = \mathbf{p^2 - p + \frac{7}{2} - \frac{35}{4p+10}}$.

Observe in the above answer that $\frac{-\frac{35}{2}}{2p+5}$ is written in a simpler form, $-\frac{35}{4p+10}$. This is because $\frac{-\frac{35}{2}}{2p+5} = -\frac{35}{2} \cdot \frac{1}{2p+5} = -\frac{35}{4p+10}$.

Quotient Functions

Similarly as in the case of polynomials, we can define quotients of functions.

Definition 3.1 ► Suppose f and g are functions of x with the corresponding domains D_f and D_g.

Then the **quotient function**, denoted $\frac{f}{g}$, is defined as

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

The **domain** of the quotient function is the intersection of the domains of the two functions, D_f and D_g, excluding the x-values for which $g(x) = 0$. So,

$$D_{\frac{f}{g}} = D_f \cap D_g \backslash \{x | g(x) = 0\}$$

Example 3 ► **Dividing Polynomial Functions**

Suppose $P(x) = 2x^2 - x - 6$ and $Q(x) = x - 2$. Find the following:

a. $\left(\frac{P}{Q}\right)(x)$ and $\left(\frac{P}{Q}\right)(2a)$,

b. $\left(\frac{P}{Q}\right)(-3)$ and $\left(\frac{P}{Q}\right)(2)$,

c. domain of $\frac{P}{Q}$.

Notice that this equation holds only for $x \neq 2$.

Solution ► **a.** By *Definition 3.1*, $\left(\frac{P}{Q}\right)(x) = \frac{P(x)}{Q(x)} = \frac{2x^2-x-6}{x-2} = \frac{(2x+3)(x-2)}{x-2} = \mathbf{2x + 3}$

So, $\left(\frac{P}{Q}\right)(-3) = 2(-3) + 3 = \mathbf{-3}$. One can verify that the same value is found by evaluating $\frac{P(-3)}{Q(-3)}$.

b. Since the equation $\frac{(2x+3)(x-2)}{x-2} = 2x + 3$ is true only for $x \neq 2$, the simplified formula $\left(\frac{P}{Q}\right)(x) = 2x + 3$ cannot be used to evaluate $\left(\frac{P}{Q}\right)(2)$. However, by *Definition 3.1*, we have

$\left(\frac{P}{Q}\right)(2)$ is undefined, so 2 is not in the domain of $\frac{P}{Q}$

$$\left(\frac{P}{Q}\right)(2) = \frac{P(2)}{Q(2)} = \frac{2(2)^2 - (2) - 6}{(2) - 2} = \frac{8-2-6}{0} = \frac{0}{0} = \boldsymbol{undefined}$$

To evaluate $\left(\frac{P}{Q}\right)(2a)$, we first notice that if $a \neq 1$, then $2a \neq 2$. So, we can use the simplified formula $\left(\frac{P}{Q}\right)(x) = 2x + 3$ and evaluate $\left(\frac{P}{Q}\right)(2a) = 2(2a) + 3 = \mathbf{4a + 3}$.

c. The domain of any polynomial is the set of all real numbers. So, the domain of $\frac{P}{Q}$ is the set of all real numbers except for the x-values for which the denominator $Q(x) =$

$x - 2$ is equal to zero. Since the solution to the equation $x - 2 = 0$ is $x = 2$, then the value 2 must be excluded from the set of all real numbers. Therefore, $\boldsymbol{D_{\frac{P}{Q}}} = \mathbb{R} \setminus \{\mathbf{2}\}$.

P.3 Exercises

Vocabulary Check *Complete each blank with the most appropriate term from the given list:* ***divisor, remainder, decreasing, zero, monomial, lower, domain, factor.***

1. The algorithm of long division of polynomials requires that both polynomials are written in _____________ order of powers and any missing term is replaced with a _________ term which plays the role of a placeholder.

2. To check a division problem, multiply the ____________ by the quotient and then add the _____________ .

3. When dividing a polynomial by a ______________, there is no need to use the long division algorithm.

4. In polynomial division, when the degree of the remainder is _________ than the degree of the divisor, the division process ends.

5. The _____________ of a quotient function $\frac{f}{g}$ does not contain the zeros of the devisor function g.

6. If the remainder in the division of $P(x)$ by $Q(x)$ is zero, then $Q(x)$ is a ____________ of $P(x)$.

Concept Check

7. *True or False?* When a tenth-degree polynomial is divided by a second-degree polynomial, the quotient is a fifth-degree polynomial.

8. *True or False?* When a polynomial is divided by a second-degree polynomial, the remainder is a first-degree polynomial.

Divide.

9. $\frac{20x^3-15x^2+5x}{5x}$

10. $\frac{27y^4+18y^2-9y}{9y}$

11. $\frac{8x^2y^2-24xy}{4xy}$

12. $\frac{5c^3d+10c^2d^2-15cd^3}{5cd}$

13. $\frac{9a^5-15a^4+12a^3}{-3a^2}$

14. $\frac{20x^3y^2+44x^2y^3-24x^2y}{-4x^2y}$

15. $\frac{64x^3-72x^2+12x}{8x^3}$

16. $\frac{4m^2n^2-21mn^3+18mn^2}{14m^2n^3}$

17. $\frac{12ab^2c+10a^2bc+18abc^2}{6a^2bc}$

Divide.

18. $(x^2 + 3x - 18) \div (x + 6)$

19. $(3y^2 + 17y + 10) \div (3y + 2)$

20. $(x^2 - 11x + 16) \div (x + 8)$

21. $(t^2 - 7t - 9) \div (t - 3)$

22. $\frac{6y^3-y^2-10}{3y+4}$

23. $\frac{4a^3+6a^2+14}{2a+4}$

24. $\frac{4x^3+8x^2-11x+3}{4x+1}$

25. $\frac{10z^3-26z^2+17z-13}{5z-3}$

26. $\frac{2x^3+4x^2-x+2}{x^2+2x-1}$

27. $\frac{3x^3-2x^2+5x-4}{x^2-x+3}$

28. $\frac{4k^4+6k^3+3k-1}{2k^2+1}$

29. $\frac{9k^4+12k^3-4k-1}{3k^2-1}$

30. $\frac{2p^3+7p^2+9p+3}{2p+2}$

31. $\frac{5t^2+19t+7}{4t+12}$

32. $\frac{x^4-4x^3+5x^2-3x+2}{x^2+3}$

33. $\frac{p^3-1}{p-1}$

34. $\frac{x^3+1}{x+1}$

35. $\frac{y^4+16}{y+2}$

36. $\frac{x^5-32}{x-2}$

Concept Check *For each pair of polynomials, $P(x)$ and $D(x)$, find such polynomials $Q(x)$ and $R(x)$ that $P(x) = Q(x) \cdot D(x) + R(x)$.*

37. $P(x) = 4x^3 - 4x^2 + 13x - 2$ and $D(x) = 2x - 1$

38. $P(x) = 3x^3 - 2x^2 + 3x - 5$ and $D(x) = 3x - 2$

Concept Check *For each pair of functions, $\boldsymbol{f}$ and $\boldsymbol{g}$, find the quotient function $\left(\frac{f}{g}\right)(\boldsymbol{x})$ and state its **domain**.*

39. $f(x) = 6x^2 - 4x, \;\; g(x) = 2x$

40. $f(x) = 6x^2 + 9x, \;\; g(x) = -3x$

41. $f(x) = x^2 - 36, \;\; g(x) = x + 6$

42. $f(x) = x^2 - 25, \;\; g(x) = x - 5$

43. $f(x) = 2x^2 - x - 3, \;\; g(x) = 2x - 3$

44. $f(x) = 3x^2 + x - 4, \;\; g(x) = 3x + 4$

45. $f(x) = 8x^3 + 125, \;\; g(x) = 2x + 5$

46. $f(x) = 64x^3 - 27, \;\; g(x) = 4x - 3$

Let $\boldsymbol{P(x) = x^2 - 4}$, $\boldsymbol{Q(x) = 2x}$, and $\boldsymbol{R(x) = x - 2}$. Find each of the following. If the value can't be evaluated, say DNE (does not exist).

47. $\left(\frac{R}{Q}\right)(x)$

48. $\left(\frac{P}{R}\right)(x)$

49. $\left(\frac{R}{P}\right)(x)$

50. $\left(\frac{R}{Q}\right)(2)$

51. $\left(\frac{R}{Q}\right)(0)$

52. $\left(\frac{P}{R}\right)(3)$

53. $\left(\frac{R}{P}\right)(-2)$

54. $\left(\frac{R}{P}\right)(2)$

55. $\left(\frac{P}{R}\right)(a)$, for $a \neq 2$

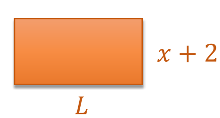
56. $\left(\frac{R}{Q}\right)\left(\frac{3}{2}\right)$

57. $\frac{1}{2}\left(\frac{Q}{R}\right)(x)$

58. $\left(\frac{Q}{R}\right)(a-1)$

Analytic Skills *Solve each problem.*

59. The area A of a rectangle is $5x^2 + 12x + 4$ and its width W is $x + 2$.
a. Find the length L of the rectangle.
b. Find the length if the width is 8 meters.



60. The area A of a triangle is $6x^2 - x - 15$. Find its height h, if the base of the triangle is $3x - 5$. Then, find the height h, if the base is 7 centimeters.