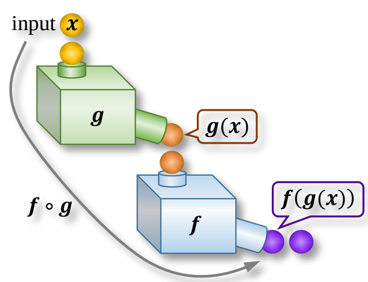
P.4 Composition of Functions and Graphs of Basic Polynomial Functions



In the last three sections, we have created new functions with the use of function operations such as addition, subtraction, multiplication, and division. In this section, we will introduce one more function operation, called **composition** of functions. This operation will allow us to represent situations in which some quantity depends on a variable that, in turn, depends on another variable. For instance, the number of employees hired by a firm may depend on the firm's profit, which may in turn depend on the number of items the firm produces. In this situation, we might be interested in the number of employees hired by a firm as a function of the number of items the firm produces. This illustrates a **composite function**. It is obtained by composing the number of employees with respect to the profit and the profit with respect to the number of items produced.

In the second part of this section, we will examine graphs of basic polynomial functions, such as constant, linear, quadratic, and cubic functions.

Composition of Functions

Consider women's shoe size scales in U.S., Italy, and Great Britain.

U.S.	Italy	Britain
4	32	2
5	34	3
6	36	4
7	38	5
8	40	6

g f h

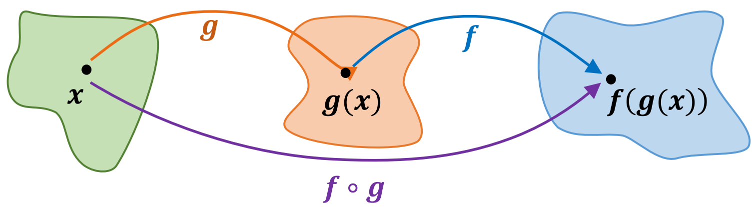
The reader is encouraged to confirm that the function $g(x) = 2x + 24$ gives the women's shoe size in Italy for any given U.S. shoe size x. For example, a U.S. shoe size 7 corresponds to a shoe size of $g(7) = 2 \cdot 7 + 24 = 38$ in Italy. Similarly, the function $f(x) = \frac{1}{2}x - 14$ gives the women's shoe size in Britain for any given Italian shoe size x. For example, an Italian shoe size 38 corresponds to a shoe size of $f(38) = \frac{1}{2} \cdot 38 - 14 = 5$ in Italy. So, by converting the U.S. shoe size first to the corresponding Italian size and then to the Great Britain size, we create a third function, h, that converts the U.S. shoe size directly to the Great Britain size. We say, that function h is the **composition of functions g** and **f**. Can we find a formula for this composite function? By observing the corresponding data for U.S. and Great Britain shoe sizes, we can conclude that $h(x) = x - 2$. This formula can also be derived with the use of algebra.

Since the U.S. shoe size x corresponds to the Italian shoe size $g(x)$, which in turn corresponds to the Britain shoe size $f(g(x))$, the composition function $h(x)$ is given by the formula

$$\boldsymbol{h(x)} = \boldsymbol{f(g(x))} = f(2x + 24) = \frac{1}{2}(2x + 24) - 14 = x + 12 - 14 = \boldsymbol{x - 2},$$

which confirms our earlier observation.

Generally, the composition of functions f and g, denoted $f \circ g$, is the function that acts on the input x by mapping it by the function g to the value $g(x)$, which in turn becomes the input for function f, to be mapped to the value $f(g(x))$. See the diagram below.



Definition 4.1 ► If f and g are functions, then the **composite function $f \circ g$**, or **composition** of f and g, is defined by

$$(f \circ g)(x) = f(g(x)),$$

for all x in the domain of g, such that $g(x)$ is in the domain of f.

Note: We read $f(g(x))$ as "f of g of x".

Example 1 ► **Evaluating a Composite Function**

Suppose $f(x) = x^2$ and $g(x) = 2x + 3$. Find the following:

a. $(f \circ g)(-2)$ **b.** $(g \circ f)(-2)$

Solution ►

a. $(f \circ g)(-2) = f(g(-2)) = f(\underbrace{2(-2) + 3}_{g(-2)}) = f(-1) = \underbrace{(-1)^2}_{f(-1)} = \mathbf{1}$

b. $(g \circ f)(-2) = g(f(-2)) = g(\underbrace{(-2)^2}_{f(-2)}) = g(4) = \underbrace{2(4) + 3}_{g(4)} = \mathbf{11}$

Observation: *In the above example, $(f \circ g)(-2) \neq (g \circ f)(-2)$. This shows that **composition of functions** is **<u>not</u> commutative**.*

Example 2 ► **Finding the Composition of Two Functions**

Given functions f and g, find $(f \circ g)(x)$.

a. $f(x) = 2 - 3x;\ \ g(x) = x^2 - 2x$ **b.** $f(x) = x^2 - x + 3;\ \ g(x) = x + 4$

Solution ►

a. $(f \circ g)(x) = f(g(x)) = f(\underbrace{x^2 - 2x}_{g(x)}) = 2 - 3(\underbrace{x^2 - 2x}_{g(x)}) = \mathbf{-3x^2 + 6x + 2}$

b. $(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{x}) = f\big(g(x)\big) = f(\underbrace{x+4}_{g(x)}) = \underbrace{(x+4)^2 - (x+4) + 3}_{f(x+4)}$

$$= x^2 + 8x + 16 - x - 4 + 3 = \boldsymbol{x^2 + 7x + 15}$$

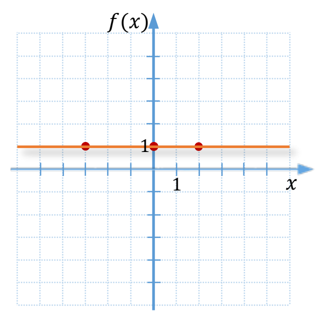
Attention! *Do not confuse the **composition** of functions, $(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}))$, with the **multiplication** of functions, $(\boldsymbol{fg})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{g}(\boldsymbol{x})$.*

Graphs of Basic Polynomial Functions

Since polynomials are functions, they can be evaluated for different x-values and graphed in a system of coordinates. How do polynomial functions look like? Below, we graph several basic polynomial functions up to the third degree, and observe their shape, domain, and range.

CONSTANT

Let us start with a **constant function**, which is defined by a zero degree polynomial, such as $f(x) = 1$. In this example, for any real x-value, the corresponding y-value is constantly equal to 1. So, the graph of this function is a **horizontal line** with the y-intercept at 1.

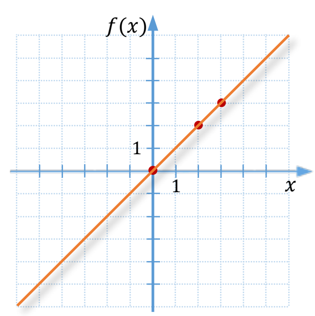


Domain: $\mathbb{R}$
Range: $\{1\}$

Generally, the graph of a **constant function**, $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{c}$, is a horizontal line with the y-intercept at c. The domain of this function is $\mathbb{R}$ and the range is $\{c\}$.

LINEAR

The basic first degree polynomial function is the **identity function** given by the formula $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$. Since both coordinates of any point satisfying this equation are the same, the graph of the identity function is the diagonal line, as shown below.



Domain: $\mathbb{R}$
Range: $\mathbb{R}$

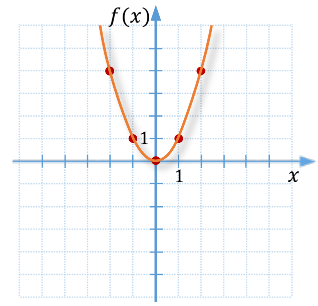
Generally, the graph of any first degree polynomial function, $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{mx} + \boldsymbol{b}$ with $m \neq 0$, is a slanted line. So, the domain and range of such function is $\mathbb{R}$.

QUADRATIC

The basic second degree polynomial function is the **squaring function** given by the formula $\boldsymbol{f(x) = x^2}$. The shape of the graph of this function is refered to as the **basic parabola**. The reader is encouraged to observe the relations between the five points calculated in the table of values below.

x	$f(x) = x^2$	
-2	4	
-1	1	
0	0	*vertex*
1	1	
2	4	

symmetry about the y-axis



Domain: $\mathbb{R}$
Range: $[0, \infty)$

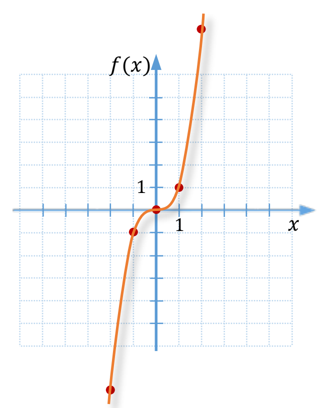
Generally, the graph of any second degree polynomial function, $\boldsymbol{f(x) = ax^2 + bx + c}$ with $a \neq 0$, is a **parabola**. The domain of such function is $\mathbb{R}$ and the range depends on how the parabola is directed, with arms up or down.

CUBIC

The basic third degree polynomial function is the **cubic function**, given by the formula $\boldsymbol{f(x) = x^3}$. The graph of this function has a shape of a 'snake'. The reader is encouraged to observe the relations between the five points calculated in the table of values below.

x	$f(x) = x^3$	
-2	-8	
-1	-1	
0	0	*center*
1	1	
2	8	

symmetry about the origin



Domain: $\mathbb{R}$
Range: $\mathbb{R}$

Generally, the graph of a third degree polynomial function, $\boldsymbol{f(x) = ax^3 + bx^2 + cx + d}$ with $a \neq 0$, has a shape of a 'snake' with different size waves in the middle. The domain and range of such function is $\mathbb{R}$.

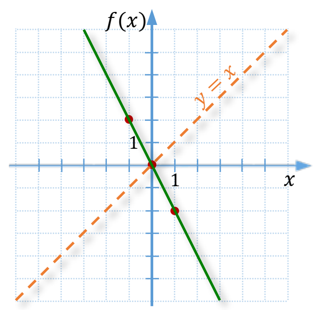
Example 3 ► Graphing Polynomial Functions

Graph each function using a table of values. Give the domain and range of each function by observing its graph. Then, on the same grid, graph the corresponding basic polynomial function. Observe and name the transformation(s) that can be applied to the basic shape in order to obtain the desired function.

a. $f(x) = -2x$ **b.** $f(x) = (x+2)^2$ **c.** $f(x) = x^3 - 2$

Solution ► **a.** The graph of $f(x) = -2x$ is a line passing through the origin and falling from left to right, as shown below in green.

x	$f(x) = -2x$
−1	2
0	0
1	−2



Domain of f: $\mathbb{R}$
Range of f: $\mathbb{R}$

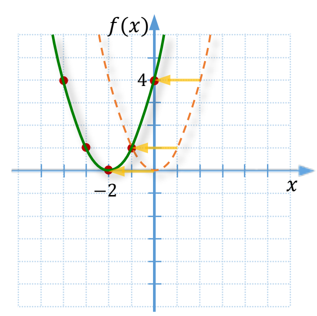
Observe that to obtain the green line, we multiply y-coordinates of the orange line by a factor of -2. Such a transformation is called a **dilation** in the $\boldsymbol{y}$**-axis** by a factor of $\boldsymbol{-2}$. This dilation can also be achieved by applying a **symmetry in the** $\boldsymbol{x}$**-axis** first, and then **stretching** the resulting graph **in the** $\boldsymbol{y}$**-axis** by a factor of $\mathbf{2}$.

b. The graph of $f(x) = (x+2)^2$ is a parabola with a vertex at $(-2, 0)$, and its arms are directed upwards as shown below in green.

x	$f(x) = (x+2)^2$
−4	4
−3	1
−2	0
−1	1
0	4

vertex

symmetry about the $x = -2$



Domain: $\mathbb{R}$
Range: $[0, \infty)$

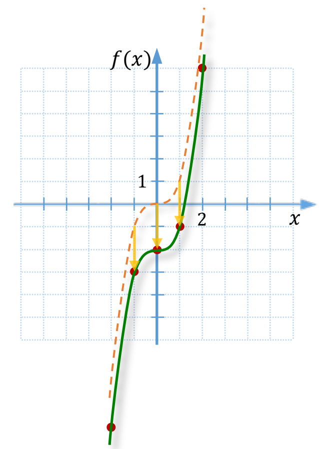
Observe that to obtain the green shape, it is enough to move the graph of the **basic parabola** by two units to the left. This transformation is called a **horizontal translation** by two units to the left. The translation to the left reflects the fact that the vertex of the parabola $f(x) = (x+2)^2$ is located at $x + 2 = 0$, which is equivalent to $x = -2$.

c. The graph of $f(x) = x^3 - 2$ has the shape of a basic cubic function with a center at $(0, -2)$.

x	$f(x) = x^3 - 2$
-2	-10
-1	-3
0	-2
1	-1
2	6

center

symmetry about $(0, -2)$



Domain: $\mathbb{R}$
Range: $\mathbb{R}$

Observe that the green graph can be obtained by shifting the graph of the **basic cubic function** by two units down. This transformation is called a **vertical translation** by two units down.

P.4 Exercises

Vocabulary Check *Complete each blank with the most appropriate term from the given list:* ***composition, product, identity, parabola, translating, up, left, down, right, basic, cubic, graph.***

1. For two functions f and g, the function $f \circ g$ is called the _______________ of f and g.

2. For two functions f and g, the function $f \cdot g$ is called the _______________ of f and g.

3. The function $f(x) = x$ is called an ______________ function.

4. The graph of $f(x) = x^2$ is referred to as the basic ______________.

5. The graph of $f(x) = x^2 + 3$ can be obtained by _______________ the graph of a basic parabola by 3 units _______.

6. The graph of $f(x) = (x + 3)^2$ can be obtained by translating the graph of a _________ parabola by 3 units to the _______.

7. The graph of $f(x) = (x - 3)^3$ can be obtained by translating the graph of a basic ________ function by 3 units to the ________.

8. The ________ of $f(x) = x^3 - 3$ can be obtained by translating the graph of a basic cubic function by 3 units ________.

Concept Check *Suppose $f(x) = x^2 + 2$, $g(x) = 5 - x$, and $h(x) = 2x - 3$. Find the following values or expressions.*

9. $(f \circ g)(1)$ **10.** $(g \circ f)(1)$ **11.** $(f \circ g)(x)$ **12.** $(g \circ f)(x)$

13. $(f \circ h)(-1)$ **14.** $(h \circ f)(-1)$ **15.** $(f \circ h)(x)$ **16.** $(h \circ f)(x)$

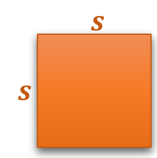
17. $(h \circ g)(-2)$ **18.** $(g \circ h)(-2)$ **19.** $(h \circ g)(x)$ **20.** $(g \circ h)(x)$

21. $(f \circ f)(2)$ **22.** $(f \circ h)\left(\frac{1}{2}\right)$ 23. $(h \circ h)(x)$ **24.** $(g \circ g)(x)$

Analytic Skills *Solve each problem.*

25. The function defined by $f(x) = 12x$ computes the number of inches in x feet, and the function defined by $g(x) = 2.54x$ computes the number of centimeters in x inches. What is $(g \circ f)(x)$ and what does it compute?

26. If hamburgers are \$3.50 each, then $C(x) = 3.5x$ gives the pre-tax cost in dollars for x hamburgers. If sales tax is 12%, then $T(x) = 1.12x$ gives the total cost when the pre-tax cost is x dollars. Write the total cost as a function of the number of hamburgers.



27. The perimeter x of a square with sides of length s is given by the formula $x = 4s$.

a. Solve the above formula for s in terms of x.

b. If y represents the area of this square, write y as a function of the perimeter x.

c. Use the composite function found in part **b.** to find the area of a square with perimeter 6.

28. When a thermal inversion layer occurs over a city, pollutants cannot rise vertically because they are trapped below the layer meaning they must disperse horizontally. Assume a factory smokestack begins emitting a pollutant at 8 a.m., and the pollutant disperses horizontally over a circular area. Suppose that t represents the time, in hours, since the factory began emitting pollutants ($t = 0$ represents 8 a.m.), and assume that the radius of the circle of pollution is $r(t) = 2t$ miles. If $A(t) = \pi r^2$ represents the area of a circle of radius r, find and interpret $(A \circ r)(t)$.

Discussion Point

29. Sears has Super Saturday sales events during which everything is 10% off, even items already on sale. If a coat was already on sale for 25% off, then does this mean that it is 35% off on Super Saturday? What function operation is at work here? Justify.

Concept Check ***Graph*** *each function and state its* ***domain*** *and* ***range****.*

30. $f(x) = -2x + 3$ **31.** $f(x) = 3x - 4$ **32.** $f(x) = -x^2 + 4$

33. $f(x) = x^2 - 2$ **34.** $f(x) = \frac{1}{2}x^2$ **35.** $f(x) = -2x^2 + 1$

36. $f(x) = (x + 1)^2 - 2$ **37.** $f(x) = -x^3 + 1$ **38.** $f(x) = (x - 3)^3$

Analytic Skills *Guess the* ***transformations*** *needed to apply to the graph of a basic parabola* $f(x) = x^2$ *to obtain the graph of the given function* $g(x)$*. Then* ***graph*** *both* $f(x)$ *and* $g(x)$ *on the same grid and confirm the the original guess.*

39. $f(x) = -x^2$

40. $f(x) = x^2 - 3$

41. $f(x) = x^2 + 2$

42. $f(x) = (x + 2)^2$

43. $f(x) = (x - 3)^2$

44. $f(x) = (x + 2)^2 - 1$