Quadratic Equations and Functions

In this chapter, we discuss various ways of solving quadratic equations, $ax^2 + bx + c = 0$, including equations quadratic in form, such as $x^{-2} + x^{-1} - 20 = 0$, and solving formulas for a variable that appears in the first and second power, such as k in $k^2 - 3k = 2N$. Frequently used strategies of solving quadratic equations include the **completing the square** procedure and its generalization in the form of the **quadratic formula**. Completing the square allows for rewriting quadratic functions in vertex form, $f(x) = a(x-h)^2 + k$, which is very useful for graphing as it provides information about the location, shape, and direction of the parabola.

In the second part of this chapter, we examine properties and graphs of quadratic functions, including basic transformations of these graphs.

Finally, these properties are used in solving application problems, particularly problems involving **optimization**.

In the last section of this chapter, we study how to solve polynomial and rational inequalities using **sign analysis**.

Q.1 Methods of Solving Quadratic Equations

As defined in *section F4*, a quadratic equation is a second-degree polynomial equation in one variable that can be written in standard form as

$$\boldsymbol{ax^2 + bx + c = 0},$$

where a, b, and c are real numbers and $\boldsymbol{a \neq 0}$. Such equations can be solved in many different ways, as presented below.

Solving by Graphing

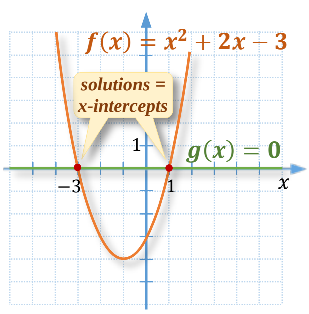

Figure 1.1

To solve a quadratic equation, for example $x^2 + 2x - 3 = 0$, we can consider its left side as a function $f(x) = x^2 + 2x - 3$ and the right side as a function $g(x) = 0$. To satisfy the original equation, both function values must be equal. After graphing both functions on the same grid, one can observe that this happens at points of intersection of the two graphs.

So the **solutions** to the original equation are the x-coordinates of the intersection points of the two graphs. In our example, these are the $\boldsymbol{x}$**-intercepts** or the **roots** of the function $f(x) = x^2 + 2x - 3$, as indicated in *Figure 1.1*.

Thus, the solutions to $x^2 + 2x - 3 = 0$ are $\boldsymbol{x = -3}$ and $\boldsymbol{x = 1}$.

Note: Notice that the graphing method, although visually appealing, is not always reliable. For example, the solutions to the equation $49x^2 - 4 = 0$ are $x = \frac{2}{7}$ and $x = -\frac{2}{7}$. Such numbers would be very hard to read from the graph.

Thus, the graphing method is advisable to use when searching for integral solutions or estimations of solutions.

To find exact solutions, we can use one of the algebraic methods presented below.

Solving by Factoring

Many quadratic equations can be solved by factoring and employing the zero-product property, as in *section F4*.

For example, the equation $x^2 + 2x - 3 = 0$ can be solved as follows:

$$(x+3)(x-1) = 0$$

so, by zero-product property,

$$x + 3 = 0 \text{ or } x - 1 = 0,$$

which gives us the solutions

$$\boldsymbol{x = -3} \text{ or } \boldsymbol{x = 1}.$$

Solving by Using the Square Root Property

Quadratic equations of the form $\boldsymbol{ax^2 + c = 0}$ can be solved by applying the **square root property**.

Square Root Property:

For any positive real number a, if $\boldsymbol{x^2 = a}$, then $\boldsymbol{x = \pm\sqrt{a}}$.

This is because $\sqrt{x^2} = |x|$. So, after applying the square root operator to both sides of the equation $x^2 = a$, we have

$$\sqrt{x^2} = \sqrt{a}$$
$$|x| = \sqrt{a}$$
$$x = \pm\sqrt{a}$$

The $\pm\sqrt{a}$ is a shorter recording of two solutions: $\sqrt{a}$ and $-\sqrt{a}$.

For example, the equation $49x^2 - 4 = 0$ can be solved as follows:

$$49x^2 - 4 = 0 \quad / +4$$
$$49x^2 = 4 \quad / \div 49$$
$$x^2 = \frac{4}{49}$$

apply square root to both sides of the equation

$$\sqrt{x^2} = \sqrt{\frac{4}{49}}$$

Here we use the **square root property**. Remember the $\pm$ sign!

$$x = \pm\sqrt{\frac{4}{49}}$$
$$\boldsymbol{x = \pm\frac{2}{7}}$$

Note: Using the square root property is a common solving strategy for quadratic equations where **one side is a perfect square** of an unknown quantity and the **other side is a constant** number.

Example 1 ► **Solve by the Square Root Property**

Solve each equation using the square root property.

a. $(x-3)^2 = 49$ **b.** $2(3x-6)^2 - 54 = 0$

Solution ► **a.** Applying the square root property, we have

$$\sqrt{(x-3)^2} = \sqrt{49}$$

$$x - 3 = \pm 7 \qquad / +3$$

$$x = 3 \pm 7$$

so

$$\boldsymbol{x = 10} \text{ or } \boldsymbol{x = -4}$$

b. To solve $2(3x-6)^2 - 54 = 0$, we isolate the perfect square first and then apply the square root property. So,

$$2(3x-6)^2 - 54 = 0 \qquad / +54, \div 2$$

$$(3x-6)^2 = \frac{54}{2}$$

$$\sqrt{(3x-6)^2} = \sqrt{27}$$

$$3x - 6 = \pm 3\sqrt{3} \qquad / +6$$

$$3x = 6 \pm 3\sqrt{3} \qquad / \div 3$$

$$x = \frac{6 \pm 3\sqrt{3}}{3}$$

$$x = \frac{\cancel{3}(2 \pm \sqrt{3})}{\cancel{3}}$$

$$x = 2 \pm \sqrt{3}$$

Thus, the solution set is $\{\mathbf{2 - \sqrt{3}},\ \mathbf{2 + \sqrt{3}}\}$.

Caution: To simplify expressions such as $\frac{6+3\sqrt{3}}{3}$, we **factor the numerator** first. The common errors to avoid are

incorrect order of operations $\frac{6+3\sqrt{3}}{3} = \frac{9\sqrt{3}}{3} = 3\sqrt{3}$

or

incorrect canceling $\frac{6+\cancel{3}\sqrt{3}}{\cancel{3}} = 6 + \sqrt{3}$

or

incorrect canceling $\frac{\overset{2}{\cancel{6}}+3\sqrt{3}}{\cancel{3}} = 2 + 3\sqrt{3}$

Solving by Completing the Square

So far, we have seen how to solve quadratic equations, $ax^2 + bx + c = 0$, if the expression $ax^2 + bx + c$ is factorable or if the coefficient b is equal to zero. To solve other quadratic equations, we may try to rewrite the variable terms in the form of a perfect square, so that the resulting equation can already be solved by the square root property.

For example, to solve $x^2 + 6x - 3 = 0$, we observe that the variable terms $x^2 + 6x$ could be written in **perfect square** form if we add 9, as illustrated in *Figure 1.2*. This is because

$$x^2 + 6x + 9 = (x+3)^2$$

observe that 3 comes from taking half of 6

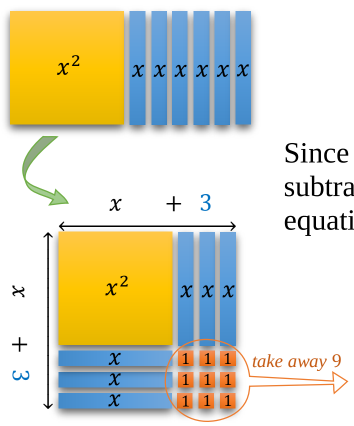

Figure 1.2

Since the original equation can only be changed to an equivalent form, if we add 9, we must subtract 9 as well. *(Alternatively, we could add 9 to both sides of the equation.)* So, the equation can be transformed as follows:

Completing the Square Procedure

$$x^2 + 6x - 3 = 0 \qquad / +12$$

$$\underbrace{x^2 + 6x + 9}_{perfect\ square} - 9 - 3 = 0$$

$$(x+3)^2 = 12$$

square root property

$$\sqrt{(x+3)^2} = \sqrt{12}$$

$$x + 3 = \pm 2\sqrt{3}$$

$$x = \mathbf{-3 \pm 2\sqrt{3}}$$

Generally, to **complete the square** for the first two terms of the equation

$$\boldsymbol{x^2 + bx + c = 0},$$

we take half of the x-coefficient, which is $\frac{b}{2}$, and square it. Then, we add and subtract that number, $\left(\frac{b}{2}\right)^2$. *(Alternatively, we could add* $\left(\frac{b}{2}\right)^2$ *to both sides of the equation.)*
This way, we produce an equivalent equation

$$\boldsymbol{x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c = 0},$$

and consequently,

$$\boldsymbol{\left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c = 0}.$$

We can write this equation directly, by following the rule:
Write the sum of $\boldsymbol{x}$ and **half of the middle coefficient**, **square the binomial**, and **subtract the perfect square of the constant** appearing in the bracket.

To **complete the square** for the first two terms of a quadratic equation with a leading coefficient of $a \neq 1$,

$$\boldsymbol{ax^2 + bx + c = 0},$$

we

- divide the equation by $\boldsymbol{a}$ *(alternatively, we could factor a out of the first two terms)* so that the leading coefficient is 1, and then
- complete the square as in the previous case, where $\boldsymbol{a = 1}$.

So, after division by $\boldsymbol{a}$, we obtain

$$\boldsymbol{x^2 + \frac{b}{a}x + \frac{c}{a} = 0}.$$

Since half of $\frac{b}{a}$ is $\frac{b}{2a}$, then we complete the square as follows:

$$\boldsymbol{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0}.$$

Remember to **subtract the perfect square of the constant** appearing in the bracket!

Example 2 ► **Solve by Completing the Square**

Solve each equation using the completing the square method.

a. $x^2 + 5x - 1 = 0$ **b.** $3x^2 - 12x - 5 = 0$

Solution ► **a.** First, we complete the square for $x^2 + 5x$ by adding and subtracting $\left(\frac{5}{2}\right)^2$ and then we apply the square root property. So, we have

$$\underbrace{x^2 + 5x + \left(\frac{5}{2}\right)^2}_{perfect\ square} - \left(\frac{5}{2}\right)^2 - 1 = 0$$

$$\left(x + \frac{5}{2}\right)^2 - \frac{25}{4} - 1 \cdot \frac{4}{4} = 0$$

$$\left(x + \frac{5}{2}\right)^2 - \frac{29}{4} = 0 \qquad / +\frac{29}{4}$$

apply square root to both sides of the equation

$$\left(x + \frac{5}{2}\right)^2 = \frac{29}{4}$$

remember to use the $\pm$ sign!

$$x + \frac{5}{2} = \pm\sqrt{\frac{29}{4}}$$

$$x+\frac{5}{2}=\pm\frac{\sqrt{29}}{2} \qquad / -\frac{5}{2}$$

$$x=\frac{-5\pm\sqrt{29}}{2}$$

Thus, the solution set is $\left\{\frac{-5-\sqrt{29}}{2}, \frac{-5+\sqrt{29}}{2}\right\}$.

Note: Unless specified otherwise, we are expected to state the **exact solutions** rather than their calculator approximations. Sometimes, however, especially when solving application problems, we may need to use a calculator to approximate the solutions. The reader is encouraged to check that the two decimal **approximations** of the above solutions are

$$\frac{-5-\sqrt{29}}{2}\approx \mathbf{-5.19} \quad \text{and} \quad \frac{-5+\sqrt{29}}{2}\approx \mathbf{0.19}$$

b. In order to apply the strategy as in the previous example, we divide the equation by the leading coefficient, 3. So, we obtain

$$3x^2-12x-5=0 \qquad / \div 3$$

$$x^2-4x-\frac{5}{3}=0$$

Then, to complete the square for x^2-4x, we may add and subtract 4. This allows us to rewrite the equation equivalently, with the variable part in perfect square form.

$$(x-2)^2-4-\frac{5}{3}=0$$

$$(x-2)^2=4\cdot\frac{3}{3}+\frac{5}{3}$$

$$(x-2)^2=\frac{17}{3}$$

$$x-2=\pm\sqrt{\frac{17}{3}}$$

$$x=\mathbf{2}\pm\frac{\sqrt{17}}{\sqrt{3}}$$

Note: The final answer could be written as a single fraction as shown below:

$$x=\frac{2\sqrt{3}\pm\sqrt{17}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{6\pm\sqrt{51}}{3}$$

Solving with Quadratic Formula

Applying the completing the square procedure to the quadratic equation

$$ax^2+bx+c=0,$$

with real coefficients $a\neq 0$, b, and c, allows us to develop a general formula for finding the solution(s) to any such equation.

Quadratic Formula ► The solution(s) to the equation $ax^2 + bx + c = 0$, where $a \neq 0$, b, c are real coefficients, are given by the formula

$$\boxed{\boldsymbol{x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}}$$

Here $\boldsymbol{x_{1,2}}$ denotes the two solutions, $\boldsymbol{x_1} = \frac{-b+\sqrt{b^2-4ac}}{2a}$, and $\boldsymbol{x_2} = \frac{-b-\sqrt{b^2-4ac}}{2a}$.

Proof: ► First, since $a \neq 0$, we can divide the equation $ax^2 + bx + c = 0$ by a. So, the equation to solve is

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Then, we complete the square for $x^2 + \frac{b}{a}x$ by adding and subtracting the perfect square of half of the middle coefficient, $\left(\frac{b}{2a}\right)^2$. So, we obtain

$$\underbrace{x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2}_{perfect\ square} - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 \qquad / +\left(\frac{b}{2a}\right)^2, -\frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \cdot \frac{4a}{4a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \qquad / -\frac{b}{2a}$$

and finaly,

QUADRATIC FORMULA

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}},$$

which concludes the proof.

Example 3 ► **Solving Quadratic Equations with the Use of the Quadratic Formula**

Using the Quadratic Formula, solve each equation, if possible. Then visualize the solutions graphically.

a. $2x^2 + 3x - 20 = 0$ **b.** $3x^2 - 4 = 2x$ **c.** $x^2 - \sqrt{2}x + 3 = 0$

Solution

a. To apply the quadratic formula, first, we identify the values of a, b, and c. Since the equation is in standard form, $a = 2$, $b = 3$, and $c = -20$. The solutions are equal to

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2(-20)}}{2 \cdot 2} = \frac{-3 \pm \sqrt{9 + 160}}{4}$$

$$= \frac{-3 \pm 13}{4} = \begin{cases} \dfrac{-3 + 13}{4} = \dfrac{10}{4} = \mathbf{\dfrac{5}{2}} \\ \dfrac{-3 - 13}{4} = \dfrac{-16}{4} = \mathbf{-4} \end{cases}$$

Thus, the solution set is $\left\{\mathbf{-4}, \mathbf{\frac{5}{2}}\right\}$.

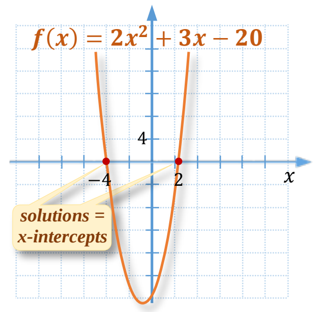

Figure 1.3

These solutions can be seen as x-intercepts of the function $f(x) = 2x^2 + 3x - 20$, as shown in *Figure 1.3.*

b. Before we identify the values of a, b, and c, we need to write the given equation $3x^2 - 4 = 2x$ in standard form. After subtracting $4x$ from both sides of the given equation, we obtain

$$3x^2 - 2x - 4 = 0$$

Since $a = 3$, $b = -2$, and $c = -5$, we evaluate the quadratic formula,

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 3(-4)}}{2 \cdot 3} = \frac{2 \pm \sqrt{4 + 48}}{6} = \frac{2 \pm \sqrt{52}}{6}$$

$$= \frac{2 \pm \sqrt{4 \cdot 13}}{6} = \frac{2 \pm 2\sqrt{13}}{6} = \frac{\cancel{2}(1 \pm \sqrt{13})}{\cancel{6}} = \mathbf{\frac{1 \pm \sqrt{13}}{3}}$$

simplify by factoring

So, the solution set is $\left\{\mathbf{\frac{1-\sqrt{13}}{3}}, \mathbf{\frac{1+\sqrt{13}}{3}}\right\}$.

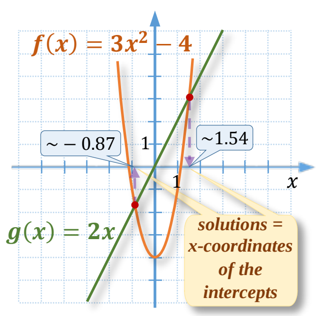

Figure 1.4

We may visualize solutions to the original equation, $3x^2 - 4 = 2x$, by graphing functions $f(x) = 3x^2 - 4$ and $g(x) = 2x$. The x-coordinates of the intersection points are the solutions to the equation $f(x) = g(x)$, and consequently to the original equation. As indicated in *Figure 1.4*, the approximations of these solutions are $\frac{1-\sqrt{13}}{3} \approx -0.87$ and $\frac{1+\sqrt{13}}{3} \approx 1.54$.

c. Substituting $a = 1$, $b = -\sqrt{2}$, and $c = 3$ into the Quadratic Formula, we obtain

$$x_{1,2} = \frac{\sqrt{2} \pm \sqrt{\left(-\sqrt{2}\right)^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} = \frac{\sqrt{2} \pm \sqrt{4 - 12}}{2} = \frac{\sqrt{2} \pm \sqrt{-8}}{2}$$

not a real number!

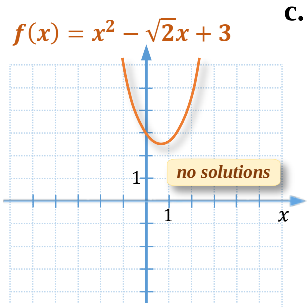

Figure 1.5

Since a square root of a negative number is not a real value, we have **no real solutions**. Thus the solution set to this equation is $\emptyset$. In a graphical representation, this means that the graph of the function $f(x) = x^2 - \sqrt{2}x + 3$ does not cross the x-axis. See *Figure 1.5*.

Observation: Notice that we could find the solution set in *Example 3c* just by evaluating the radicand $b^2 - 4ac$. Since this radicand was negative, we concluded that there is no solution to the given equation as a root of a negative number is not a real number. There was no need to evaluate the whole expression of Quadratic Formula.

So, the radicand in the Quadratic Formula carries important information about the number and nature of roots. Because of it, this radicand earned a special name, the discriminant.

Definition 1.1 ► The radicand $\boldsymbol{b^2 - 4ac}$ in the Quadratic Formula is called the **discriminant** and it is denoted by Δ.

Notice that in terms of Δ, the Quadratic Formula takes the form

$$\boxed{x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}}$$

Observing the behaviour of the expression $\sqrt{\Delta}$ allows us to classify the number and type of solutions (roots) of a quadratic equation with rational coefficients.

Characteristics of Roots (Solutions) Depending on the Discriminant

Suppose $ax^2 + bx + c = 0$ has **rational** coefficients $a \neq 0$, b, c, and $\Delta = b^2 - 4ac$.

- If $\Delta < \mathbf{0}$, then the equation has **no real solutions**, as $\sqrt{negative}$ is not a real number.
- If $\Delta = \mathbf{0}$, then the equation has **one rational solution**, $\frac{-b}{2a}$.
- If $\Delta > \mathbf{0}$, then the equation has **two solutions**, $\frac{-b-\sqrt{\Delta}}{2a}$ and $\frac{-b+\sqrt{\Delta}}{2a}$.

  These solutions are
  - **irrational**, if Δ is **not** a **perfect square number**
  - **rational**, if Δ is a **perfect square number** (as $\sqrt{perfect\ square} = integer$)

  In addition, if $\Delta \geq \mathbf{0}$ is a **perfect square number**, then the equation could be solved by **factoring**.

Example 4 ► **Determining the Number and Type of Solutions of a Quadratic Equation**

Using the discriminant, determine the number and type of solutions of each equation without solving the equation. If the equation can be solved by factoring, show the factored form of the trinomial.

a. $2x^2 + 7x - 15 = 0$

b. $4x^2 - 12x + 9 = 0$

c. $3x^2 - x + 1 = 0$

d. $2x^2 - 7x + 2 = 0$

Solution ► **a.** $\Delta = 7^2 - 4 \cdot 2 \cdot (-15) = 49 + 120 = 169$

Since 169 is a perfect square number, the equation has **two rational solutions** and it can be solved by factoring. Indeed, $2x^2 + 7x - 15 = (2x - 3)(x + 5)$.

b. $\Delta = (-12)^2 - 4 \cdot 4 \cdot 9 = 144 - 144 = 0$

$\Delta = 0$ indicates that the equation has **one rational solution** and it can be solved by factoring. Indeed, the expression $4x^2 - 12x + 9$ is a perfect square, $(2x - 3)^2$.

c. $\Delta = (-1)^2 - 4 \cdot 3 \cdot 1 = 1 - 12 = -11$

Since $\Delta < 0$, the equation has **no real solutions** and therefore it can not be solved by factoring.

d. $\Delta = (-7)^2 - 4 \cdot 2 \cdot 2 = 49 - 16 = 33$

Since $\Delta > 0$ but it is not a perfect square number, the equation has **two real solutions** but it cannot be solved by factoring.

Example 5 ► **Solving Equations Equivalent to Quadratic**

Solve each equation.

a. $2 + \frac{7}{x} = \frac{5}{x^2}$ **b.** $2x^2 = (x + 2)(x - 1) + 1$

Solution ► **a.** This is a rational equation, with the set of $\mathbb{R} \setminus \{0\}$ as its domain. To solve it, we multiply the equation by the $LCD = x^2$. This brings us to a quadratic equation

$$2x^2 + 7x = 5$$

or equivalently

$$2x^2 + 7x - 5 = 0,$$

which can be solved by following the Quadratic Formula for $a = 2$, $b = 7$, and $c = -5$. So, we have

$$x_{1,2} = \frac{-7 \pm \sqrt{(-7)^2 - 4 \cdot 2(-5)}}{2 \cdot 2} = \frac{-7 \pm \sqrt{49 + 40}}{4} = \frac{\mathbf{-7 \pm \sqrt{89}}}{\mathbf{4}}$$

Since both solutions are in the domain, the solution set is $\left\{\frac{\mathbf{-7-\sqrt{89}}}{\mathbf{4}}, \frac{\mathbf{-7+\sqrt{89}}}{\mathbf{4}}\right\}$

b. To solve $1 - 2x^2 = (x + 2)(x - 1)$, we simplify the equation first and rewrite it in standard form. So, we have

$$1 - 2x^2 = x^2 + x - 2 \qquad / \ -x, +2$$

$$-3x^2 - x + 3 = 0 \qquad / \cdot (-1)$$

$$3x^2 + x - 3 = 0$$

Since the left side of this equation is not factorable, we may use the Quadratic Formula. So, the solutions are

$$x_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 3(-3)}}{2 \cdot 3} = \frac{1 \pm \sqrt{1 + 36}}{6} = \boldsymbol{\frac{1 \pm \sqrt{37}}{6}}.$$

Q.1 Exercises

Concept Check *True or False.*

1. A quadratic equation is an equation that can be written in the form $ax^2 + bx + c = 0$, where a, b, and c are any real numbers.

2. If the graph of $f(x) = ax^2 + bx + c$ intersects the x-axis twice, the equation $ax^2 + bx + c = 0$ has two solutions.

3. If the equation $ax^2 + bx + c = 0$ has no solution, the graph of $f(x) = ax^2 + bx + c$ does not intersect the x-axis.

4. The Quadratic Formula cannot be used to solve the equation $x^2 - 5 = 0$ because the equation does not contain a linear term.

5. The solution set for the equation $x^2 = 16$ is $\{4\}$.

6. To complete the square for $x^2 + bx$, we add $\left(\frac{b}{2}\right)^2$.

7. If the discriminant is positive, the equation can be solved by factoring.

Concept Check

For each function f,
a) *graph $f(x)$ using a table of values;*
b) *find the x-intercepts of the graph;*
c) *solve the equation $f(x) = 0$ by factoring and compare these solutions to the x-intercepts of the graph.*

8. $f(x) = -x^2 - 3x + 2$ **9.** $f(x) = x^2 + 2x - 3$ **10.** $f(x) = 3x + x(x - 2)$

11. $f(x) = 2x - x(x - 3)$ **12.** $f(x) = 4x^2 - 4x - 3$ **13.** $f(x) = -\frac{1}{2}(2x^2 + 5x - 12)$

Solve each equation using the ***square root property****.*

14. $x^2 = 49$ **15.** $x^2 = 32$ **16.** $a^2 - 50 = 0$

17. $n^2 - 24 = 0$ **18.** $3x^2 - 72 = 0$ **19.** $5y^2 - 200 = 0$

20. $(x-4)^2 = 64$

21. $(x+3)^2 = 16$

22. $(3n-1)^2 = 7$

23. $(5t+2)^2 = 12$

24. $x^2 - 10x + 25 = 45$

25. $y^2 + 8y + 16 = 44$

26. $4a^2 + 12a + 9 = 32$

27. $25(y-10)^2 = 36$

28. $16(x+4)^2 = 81$

29. $(4x+3)^2 = -25$

30. $(3n-2)(3n+2) = -5$

31. $2x - 1 = \frac{18}{2x-1}$

Solve each equation using the ***completing the square*** *procedure.*

32. $x^2 + 12x = 0$

33. $y^2 - 3y = 0$

34. $x^2 - 8x + 2 = 0$

35. $n^2 + 7n = 3n - 4$

36. $p^2 - 4p = 4p - 16$

37. $y^2 + 7y - 1 = 0$

38. $2x^2 - 8x = -4$

39. $3a^2 + 6a = -9$

40. $3y^2 - 9y + 15 = 0$

41. $5x^2 - 60x + 80 = 0$

42. $2t^2 + 6t - 10 = 0$

43. $3x^2 + 2x - 2 = 0$

44. $2x^2 - 16x + 25 = 0$

45. $9x^2 - 24x = -13$

46. $25n^2 - 20n = 1$

47. $x^2 - \frac{4}{3}x = -\frac{1}{9}$

48. $x^2 + \frac{5}{2}x = -1$

49. $x^2 - \frac{2}{5}x - 3 = 0$

Given $f(x)$ and $g(x)$, find all values of x for which $f(x) = g(x)$.

50. $f(x) = x^2 - 9$ and $g(x) = 4x - 6$

51. $f(x) = 2x^2 - 5x$ and $g(x) = -x + 14$

Discussion Point

52. Explain the errors in the following solutions of the equation $5x^2 - 8x + 2 = 0$:

a. $x = \frac{8 \pm \sqrt{-8^2 - 4\cdot5\cdot2}}{2\cdot8} = \frac{8 \pm \sqrt{64-40}}{16} = \frac{8 \pm \sqrt{24}}{16} = \frac{8 \pm 2\sqrt{6}}{16} = \mathbf{\frac{1}{2} \pm 2\sqrt{6}}$

b. $x = \frac{8 \pm \sqrt{(-8)^2 - 4\cdot5\cdot2}}{2\cdot8} = \frac{8 \pm \sqrt{64-40}}{16} = \frac{8 \pm \sqrt{24}}{16} = \frac{8 \pm 2\sqrt{6}}{16} = \begin{cases} \frac{10\sqrt{6}}{16} = \mathbf{\frac{5\sqrt{6}}{8}} \\ \frac{6\sqrt{6}}{16} = \mathbf{\frac{3\sqrt{6}}{8}} \end{cases}$

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Solve each equation with the aid of the ***Quadratic Formula****, if possible. Illustrate your solutions graphically, using a table of values or a graphing utility.*

53. $x^2 + 3x + 2 = 0$

54. $y^2 - 2 = y$

55. $x^2 + x = -3$

56. $2y^2 + 3y = -2$

57. $x^2 - 8x + 16 = 0$

58. $4n^2 + 1 = 4n$

Solve each equation with the aid of the ***Quadratic Formula****. Give the* ***exact*** *and* ***approximate*** *solutions up to two decimal places.*

59. $a^2 - 4 = 2a$

60. $2 - 2x = 3x^2$

61. $0.2x^2 + x + 0.7 = 0$

62. $2t^2 - 4t + 2 = 3$

63. $y^2 + \frac{y}{3} = \frac{1}{6}$

64. $\frac{x^2}{4} - \frac{x}{2} = 1$

65. $5x^2 = 17x - 2$

66. $15y = 2y^2 + 16$

67. $6x^2 - 8x = 2x - 3$

Concept Check *Use the discriminant to determine the* ***number and type of solutions*** *for each equation. Also, decide whether the equation can be solved by* ***factoring*** *or whether the quadratic formula should be used. Do not actually solve.*

68. $3x^2 - 5x - 2 = 0$ **69.** $4x^2 = 4x + 3$ **70.** $x^2 + 3 = -2\sqrt{3}x$

71. $4y^2 - 28y + 49 = 0$ **72.** $3y^2 - 10y + 15 = 0$ **73.** $9x^2 + 6x = -1$

Find the value(s) of the constant k, so that each equation will have exactly one rational solution.

74. $x^2 + ky + 49 = 0$ **75.** $9y^2 - 30y + k = 0$ **76.** $kx^2 + 8x + 1 = 0$

Discussion Point

77. Is it possible for the solution of a quadratic equation with integral coefficients to include only one irrational number? Why or why not?

Solve each equation using any algebraic method. State the solutions in their exact form .

78. $-2x(x + 2) = -3$ **79.** $(x + 2)(x - 4) = 1$ **80.** $(x + 2)(x + 6) = 8$

81. $(2x - 3)^2 = 8(x + 1)$ **82.** $(3x + 1)^2 = 2(1 - 3x)$ **83.** $2x^2 - (x + 2)(x - 3) = 12$

84. $(x - 2)^2 + (x + 1)^2 = 0$ **85.** $1 + \frac{2}{x} + \frac{5}{x^2} = 0$ **86.** $x = \frac{2(x+3)}{x+5}$

87. $2 + \frac{1}{x} = \frac{3}{x^2}$ **88.** $\frac{3}{x} + \frac{x}{3} = \frac{5}{2}$ **89.** $\frac{1}{x} + \frac{1}{x+4} = \frac{1}{7}$