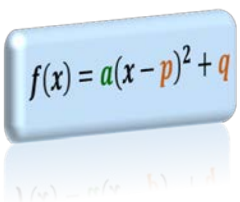
Q.3 Properties and Graphs of Quadratic Functions



In this section, we explore an alternative way of graphing quadratic functions. It turns out that if a quadratic function is given in vertex form, $\boldsymbol{f(x) = a(x-p)^2 + q}$, its graph can be obtained by transforming the shape of the basic **parabola**, $f(x) = x^2$, by applying a **vertical dilation** by the factor of a, as well as a **horizontal translation** by p units and **vertical translation** by q units. This approach makes the graphing process easier than when using a table of values.

In addition, the vertex form allows us to identify the main characteristics of the corresponding graph such as **shape**, **opening**, **vertex**, and **axis of symmetry**. Then, the additional properties of a quadratic function, such as **domain** and **range**, or where the function increases or decreases can be determined by observing the obtained graph.

Properties and Graph of the Basic Parabola $f(x) = x^2$

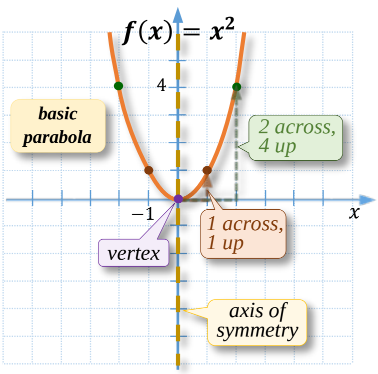


Figure 3.1

Recall the shape of the **basic parabola**, $f(x) = x^2$, as discussed in *section P4*.

x	x^2
-2	4
-1	1
0	0
1	1
2	4

vertex

symmetry about the y-axis

Observe the relations between the points listed in the table above. If we start with plotting the **vertex** $(\mathbf{0}, \mathbf{0})$, then the next pair of points, $(\mathbf{1}, \mathbf{1})$ and $(\mathbf{-1}, \mathbf{1})$, is plotted **1 unit across** from the vertex (both ways) and **1 unit up**. The following pair, $(\mathbf{2}, \mathbf{4})$ and $(\mathbf{-2}, \mathbf{4})$, is plotted **2 units across** from the vertex and **4 units up**. The graph of the parabola is obtained by connecting these 5 main points by a curve, as illustrated in *Figure 3.1*.

The graph of this parabola is symmetric in the y-axis, so the equation of the **axis of symmetry** is $\boldsymbol{x = 0}$.

The **domain** of the basic parabola is the set of all real numbers, $\mathbb{R}$, as $f(x) = x^2$ is a polynomial, and polynomials can be evaluated for any real x-value.

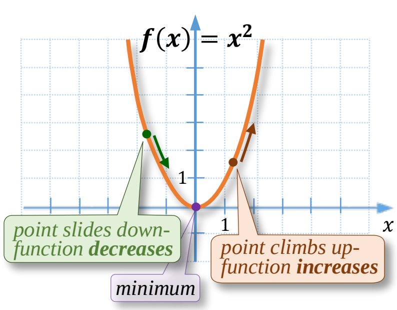


Figure 3.2

The **arms** of the parabola are directed **upwards**, which means that the vertex is the lowest point of the graph. Hence, the **range** of the basic parabola function, $f(x) = x^2$, is the interval $[\mathbf{0}, \infty)$, and the **minimum value** of the function is $\mathbf{0}$.

Suppose a point 'lives' on the graph and travels from left to right. Observe that in the case of the basic parabola, if x-coordinates of the 'travelling' point are smaller than 0, the point slides down along the graph. Similarly, if x-coordinates are larger than 0, the point climbs up the graph. (See *Figure 3.2*) To describe this property in mathematical language, we say that the function $f(x) = x^2$ **decreases** in the interval $(-\infty, 0]$ and **increases** in the interval $[0, \infty)$.

Properties and Graphs of a Dilated Parabola $f(x) = ax^2$

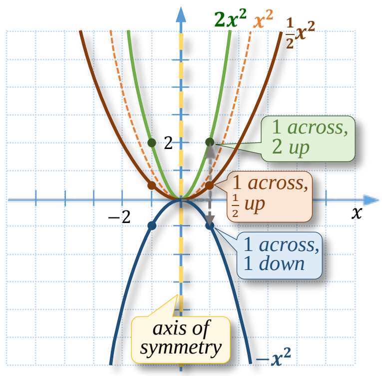


Figure 3.3

Figure 3.3 shows graphs of several functions of the form $f(x) = ax^2$. Observe how the shapes of these parabolas change for various values of a in comparison to the shape of the basic parabola $y = x^2$.

The common point for all of these parabolas is the vertex $(0,0)$. Additional points, essential for graphing such parabolas, are shown in the table below.

x	ax^2
-2	$4a$
-1	a
0	0
1	a
2	$4a$

vertex

1 *unit apart from zero,* ***a*** *units up*

2 *units apart from zero,* ***4a*** *units up*

For example, to graph $f(x) = 3x^2$, it is convenient to plot the **vertex** first, which is at the point $(\mathbf{0}, \mathbf{0})$. Then, we may move the pen **1** unit **across** from the vertex (either way) and **3** units **up** to plot the points $(-1,3)$ and $(1,3)$. If the grid allows, we might want to plot the next two points, $(-2,12)$ and $(2,12)$, by moving the pen **2** units **across** from the vertex and $4 \cdot 3 = \mathbf{12}$ units **up**, as in *Figure 3.4*.

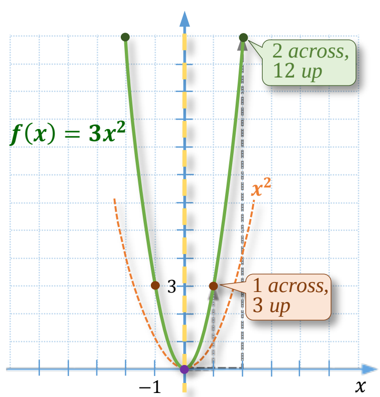


Figure 3.4

Notice that the obtained shape (in green) is **narrower** than the shape of the basic parabola (in orange). However, similarly as in the case of the basic parabola, the shape of the dilated function is still **symmetrical about the y-axis**, $\boldsymbol{x = 0}$.

Now, suppose we want to graph the function $f(x) = -\frac{1}{2}x^2$. As before, we may start by plotting the vertex at $(0,0)$. Then, we move the pen **1** unit **across** from the vertex (either way) and **half** a unit **down** to plot the points $\left(-1, -\frac{1}{2}\right)$ and $\left(1, -\frac{1}{2}\right)$, as in *Figure 3.5*. The next pair of points can be plotted by moving the pen **2** units **across** from the vertex and **2** units **down**, as the ordered pairs $(-2,-2)$ and $(2,-2)$ satisfy the equation $f(x) = -\frac{1}{2}x^2$.

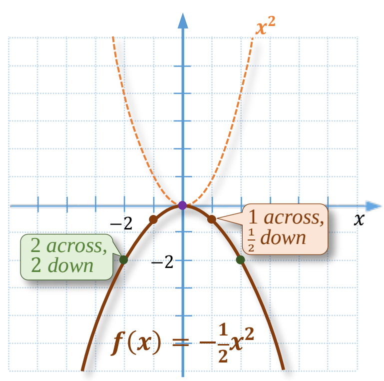


Figure 3.5

Notice that this time the obtained shape (in green) is **wider** than the shape of the basic parabola (in orange). Also, as a result of the **negative a-value**, the parabola opens **down**, and the **range** of this function is $(-\infty, \mathbf{0}]$.

Generally, the **shape** of a quadratic function of the form $f(x) = ax^2$ is
- **narrower** than the shape of the basic parabola, if $|a| > 1$;
- **wider** than the shape of the basic parabola, if $0 < |a| < 1$; and
- **the same** as the shape of the **basic parabola**, $y = x^2$, if $|a| = 1$.

The parabola opens **up**, for $a > 0$, and **down**, for $a < 0$.
Thus the **vertex** becomes the **lowest point** of the graph, if $a > 0$, and the **highest point** of the graph, if $a < 0$.

The **range** of $f(x) = ax^2$ is $[0, \infty)$, if $a > 0$, and $(-\infty, 0]$, if $a < 0$.

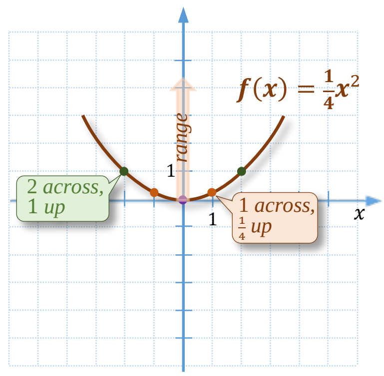
The **axis of symmetry** of the dilated parabola $f(x) = ax^2$ remains the same as that of the basic parabola, which is $x = 0$.

Example 1 ► **Graphing a Dilated Parabola and Describing Its Shape, Opening, and Range**

For each quadratic function, describe its shape and opening. Then graph it and determine its range.

a. $f(x) = \frac{1}{4}x^2$ **b.** $g(x) = -2x^2$

Solution ►



a. Since the leading coefficient of the function $f(x) = \frac{1}{4}x^2$ is positive, the parabola **opens up**. Also, since $0 < \frac{1}{4} < 1$, we expect the shape of the parabola to be **wider** than that of the basic parabola.

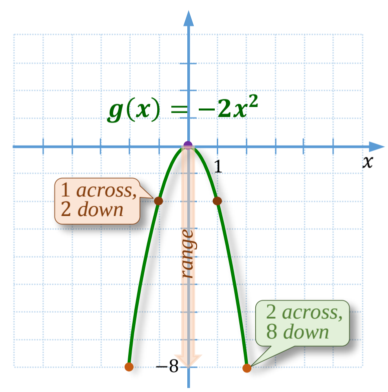
To graph $f(x) = \frac{1}{4}x^2$, first we plot the vertex at $(0,0)$ and then points $\left(\pm 1, \frac{1}{4}\right)$ and $\left(\pm 2, \frac{1}{4} \cdot 4\right) = (\pm 2, 1)$. After connecting these points with a curve, we obtain the graph of the parabola.

By projecting the graph onto the y-axis, we observe that the range of the function is $[0, \infty)$.

b. Since the leading coefficient of the function $g(x) = -2x^2$ is negative, the parabola **opens down**. Also, since $|-2| > 1$, we expect the shape of the parabola to be **narrower** than that of the basic parabola.

To graph $g(x) = -2x^2$, first we plot the vertex at $(0,0)$ and then points $(\pm 1, -2)$ and $(\pm 2, -2 \cdot 4) = (\pm 2, -8)$. After connecting these points with a curve, we obtain the graph of the parabola.

By projecting the graph onto the y-axis, we observe that the range of the function is $(-\infty, 0]$.



Properties and Graphs of the Basic Parabola with Shifts

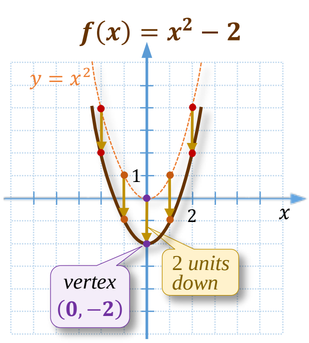


Figure 3.6

Suppose we would like to graph the function $f(x) = x^2 - 2$. We could do this via a table of values, but there is an easier way if we already know the shape of the basic parabola $y = x^2$.

Observe that for every x-value, the value of $x^2 - 2$ is obtained by subtracting 2 from the value of x^2. So, to graph $f(x) = x^2 - 2$, it is enough to **move each point** (x, x^2) of the basic parabola by **two units down**, as indicated in *Figure 3.6*.

The shift of y-values by 2 units down causes the **range** of the new function, $f(x) = x^2 - 2$, to become $[-2, \infty)$. Observe that this vertical shift also changes the minimum value of this function, from 0 to -2.

The **axis of symmetry** remains unchanged, and it is $x = 0$.

Generally, the graph of a quadratic function of the form $\boldsymbol{f(x) = x^2 + q}$ can be obtained by

- **shifting** the graph of the basic parabola $\boldsymbol{q}$ **steps up**, if $\boldsymbol{q > 0}$;
- **shifting** the graph of the basic parabola $|\boldsymbol{q}|$ **steps down**, if $\boldsymbol{q < 0}$.

The **vertex** of such parabola is at $(\boldsymbol{0, q})$. The **range** of it is $[\boldsymbol{q}, \infty)$.
The **minimum** (lowest) **value** of the function is $\boldsymbol{q}$.
The **axis of symmetry** is $\boldsymbol{x = 0}$.

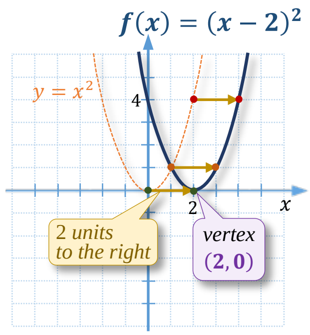


Figure 3.7

Now, suppose we wish to graph the function $f(x) = (x-2)^2$. We can graph it by joining the points calculated in the table below.

$\boldsymbol{x}$	$\boldsymbol{(x-2)^2}$
0	**4**
1	**1**
2	**0**
3	**1**
4	**4**

Observe that the parabola $f(x) = (x-2)^2$ assumes its lowest value at the vertex. The lowest value of the perfect square $(x-2)^2$ is zero, and it is attained at the x-value of 2. Thus, the vertex of this parabola is $(2,0)$.

Notice that the **vertex** $(\boldsymbol{2, 0})$ of $f(x) = (x-2)^2$ is positioned 2 units to the right from the vertex $(0,0)$ of the basic parabola.

This suggests that the graph of the function $f(x) = (x-2)^2$ can be obtained without the aid of a table of values. It is enough to shift the graph of the basic parabola **2 units** to the **right**, as shown in *Figure 3.7.*

Observe that the horizontal shift does not influence the **range** of the new parabola $f(x) = (x-2)^2$. It is still $[\boldsymbol{0}, \infty)$, the same as for the basic parabola. However, the **axis of symmetry** has changed to $\boldsymbol{x = 2}$.

Generally, the graph of a quadratic function of the form $\boldsymbol{f(x) = (x-p)^2}$ can be obtained by

- **shifting** the graph of the basic parabola $\boldsymbol{p}$ **steps** to the **right**, if $\boldsymbol{p > 0}$;
- **shifting** the graph of the basic parabola $|\boldsymbol{p}|$ **steps** to the **left**, if $\boldsymbol{p < 0}$.

The **vertex** of such a parabola is at $(\boldsymbol{p, 0})$. The **range** of it is $[\boldsymbol{0}, \infty)$.

The **minimum value** of the function is **0**.

The **axis of symmetry** is $\boldsymbol{x = p}$.

Example 2 ► **Graphing Parabolas and Observing Transformations of the Basic Parabola**

Graph each parabola by plotting its vertex and following the appropriate opening and shape. Then describe transformations of the basic parabola that would lead to the obtained graph. Finally, state the range and the equation of the axis of symmetry.

a. $f(x) = (x+3)^2$ **b.** $g(x) = -x^2 + 1$

Solution

a. The perfect square $(x+3)^2$ attains its lowest value at $x=-3$. So, the **vertex** of the parabola $f(x)=(x+3)^2$ is $(\mathbf{-3,0})$. Since the leading coefficient is 1, the parabola takes the shape of $y=x^2$, and its **arms open up**.

The graph of the function f can be obtained by **shifting** the graph of the basic parabola **3 units to the left**, as shown in *Figure 3.8.*

The **range** of function f is $[\mathbf{0},\infty)$, and the equation of the **axis of symmetry** is $\boldsymbol{x=-3}$.

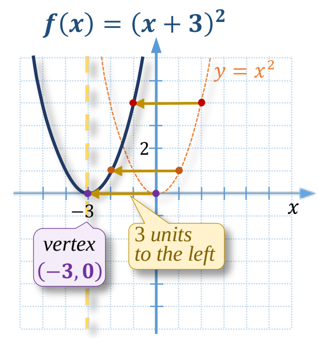


Figure 3.8

b. The expression $-x^2+1$ attains its highest value at $x=0$. So, the **vertex** of the parabola $g(x)=-x^2+1$ is $(\mathbf{0,1})$. Since the leading coefficient is -1, the parabola takes the shape of $y=x^2$, but its **arms open down**.

The graph of the function g can be obtained by:

- first, **flipping the graph** of the basic parabola **over the $\boldsymbol{x}$-axis**, and then
- **shifting** the graph of $y=-x^2$ **1 unit up**, as shown in *Figure 3.9.*

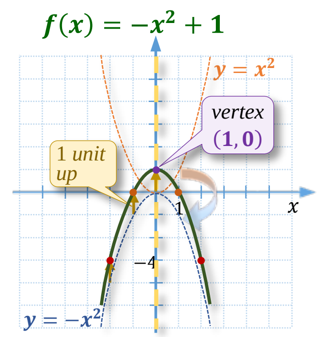


Figure 3.9

The **range** of the function g is $(-\infty,\mathbf{1}]$, and the equation of the **axis of symmetry** is $\boldsymbol{x=0}$.

> ***Note:*** The order of transformations in the above example is essential. The reader is encouraged to check that **shifting** the graph of $y=x^2$ by 1 unit up first and then **flipping** it over the x-axis results in a different graph than the one in *Figure 3.9.*

Properties and Graphs of Quadratic Functions Given in the Vertex Form $f(x)=a(x-p)^2+q$

So far, we have discussed properties and graphs of quadratic functions that can be obtained from the graph of the basic parabola by applying mainly a single transformation. These transformations were: dilations (including flips over the x-axis), and horizontal and vertical shifts. Sometimes, however, we need to apply more than one transformation. We have already encountered such a situation in *Example 2b*, where a flip and a horizontal shift was applied. Now, we will look at properties and graphs of any function of the form $\boldsymbol{f(x)=a(x-p)^2+q}$, referred to as the **vertex form** of a quadratic function.

Suppose we wish to graph $f(x) = 2(x+1)^2 - 3$. This can be accomplished by connecting the points calculated in a table of values, such as the one below, or by observing the coordinates of the vertex and following the shape of the graph of $y = 2x^2$. Notice that the vertex of our parabola is at $(-1, -3)$. This information can be taken directly from the equation $f(x) = 2(x+1)^2 - 3 = 2\left(x - (-\mathbf{1})\right)^2 - \mathbf{3}$,

opposite to the number in the bracket *the same last number*

without the aid of a table of values.

x	$2(x+1)^2 - 3$
-3	5
-2	-1
-1	-3
0	-1
1	5

1 unit apart from zero, 2 units up

vertex

The rest of the points follow the pattern of the shape for the $y = 2x^2$ parabola: 1 across, 2 up; 2 across, $4 \cdot 2 = 8$ up. So, we connect the points as in *Figure 3.10*.

Notice that the graph of function f could also be obtained as a result of translating the graph of $y = 2x^2$ by 1 unit left and 3 units down, as indicated in *Figure 3.10* by the blue vectors.

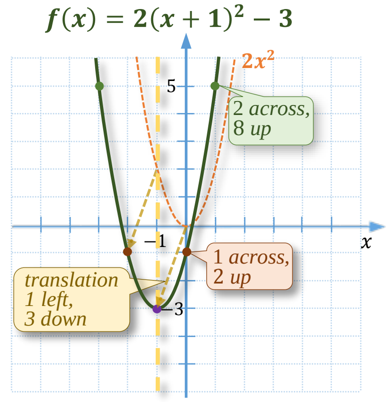


Figure 3.10

Here are the main properties of the graph of function f:
- It has a **shape** of $y = \mathbf{2x^2}$;
- It is a parabola that **opens up**;
- It has a **vertex** at $(\mathbf{-1}, \mathbf{-3})$;
- It is **symmetrical** about the line $\boldsymbol{x = -1}$;
- Its **minimum value** is $\mathbf{-3}$, and this minimum is attained at $x = -1$;
- Its domain is the set of all real numbers, and its **range** is the interval $[\mathbf{-3}, \infty)$;
- It **decreases** for $x \in (-\infty, \mathbf{-1}]$ and **increases** for $x \in [\mathbf{-1}, \infty)$.

The above discussion of properties and graphs of a quadratic function given in vertex form leads us to the following general observations:

Characteristics of Quadratic Functions Given in Vertex Form $f(x) = a(x-p)^2 + q$

1. The graph of a quadratic function given in **vertex form**

$$f(x) = a(x-p)^2 + q, \text{ where } a \neq 0,$$

is a **parabola** with **vertex** (p, q) and **axis of symmetry** $x = p$.

2. The graph **opens up** if a is **positive** and **down** if a is **negative**.

3. If $a > 0$, q is the **minimum value**. If $a < 0$, q is the **maximum value**.

3. The graph is **narrower** than that of $y = x^2$ if $|a| > 1$.
The graph is **wider** than that of $y = x^2$ if $0 < |a| < 1$.

4. The **domain** of function f is the set of real numbers, $\mathbb{R}$.
The **range** of function f is $[q, \infty)$ if a is **positive** and $(-\infty, q]$ if a is **negative**.

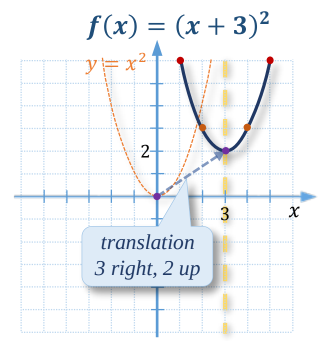
Example 3 ► **Identifying Properties and Graphing Quadratic Functions Given in Vertex Form $f(x) = a(x-p)^2 + q$**

For each function, identify its **vertex**, **opening**, **axis of symmetry**, and **shape**. Then graph the function and state its **domain** and **range**. Finally, describe **transformations** of the basic parabola that would lead to the obtained graph.

a. $f(x) = (x-3)^2 + 2$ **b.** $g(x) = -\frac{1}{2}(x+1)^2 + 3$

Solution ►

a. The vertex of the parabola $f(x) = (x-3)^2 + 2$ is $(\mathbf{3}, \mathbf{2})$; the graph **opens up**, and the equation of the axis of symmetry is $\boldsymbol{x = 3}$. To graph this function, we can plot the vertex first and then follow the shape of the basic parabola $y = \boldsymbol{x^2}$.



The domain of function f is $\mathbb{R}$, and the range is $[\mathbf{2}, \infty)$.

The graph of f can be obtained by shifting the graph of the basic parabola **3 units to the right** and **2 units up**.

b. The vertex of the parabola $g(x) = -\frac{1}{2}(x+1)^2 + 3$ is $(-\mathbf{1}, \mathbf{3})$; the graph **opens down**, and the equation of the axis of symmetry is $\boldsymbol{x = -1}$. To graph this function, we can plot the vertex first and then follow the shape of the parabola $y = -\frac{1}{2}\boldsymbol{x^2}$. This means that starting from the vertex, we move the pen one unit across (both ways) and drop half a unit to plot the next two points, $\left(0, \frac{5}{2}\right)$ and symmetrically $\left(-2, \frac{5}{2}\right)$. To plot the following two points, again, we start from the vertex and move our pen two units across and 2 units down (as $-\frac{1}{2} \cdot 4 = -2$). So, the next two points are $(1,1)$ and symmetrically $(-4,1)$, as indicated in *Figure 3.11*.

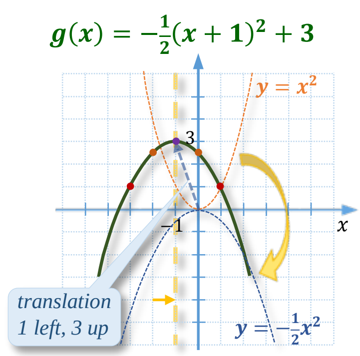


Figure 3.11

The domain of function g is $\mathbb{R}$, and the range is $(-\infty, \mathbf{3}]$.

The graph of g can be obtained from the graph of the basic parabola in two steps:

1. **Dilate** the basic parabola by multiplying its y-values by the factor of $-\frac{1}{2}$.
2. Shift the graph of the dilated parabola $y = -\frac{1}{2}\boldsymbol{x^2}$, **1 unit to the left** and **3 units up**, as indicated in *Figure 3.11*.

Q.3 Exercises

Vocabulary Check *Complete each blank with the most appropriate term or phrase from the given list:* ***axis of symmetry, horizontal, maximum, minimum, opens, shape, up, vertex, vertical.***

1. A quadratic function can be stated in a form $f(x) = a(x-p)^2 + q$. This form is referred to as the ____________ form because it allows for quick identification of the vertex of the corresponding parabola.

2. The line $x = p$ is the ____________________________ of the parabola $f(x) = a(x-p)^2 + q$.

3. The _________ of the function $f(x) = a(x-p)^2 + q$ is the same as the shape of the parabola $y = ax^2$.

4. If $a > 0$, the parabola $f(x) = a(x-p)^2 + q$ opens ____. If $a < 0$, the parabola $f(x) = a(x-p)^2 + q$ ___________ down.

5. The value p in the formula $f(x) = a(x-p)^2$, represents the ________________ shift of the parabola $y = ax^2$ needed to obtain the graph of function f.

6. The value q in the formula $f(x) = ax^2 + q$, represents the ________________ shift of the parabola $y = ax^2$ needed to obtain the graph of function f.

7. If $a > 0$, then the vertex (p, q) of the parabola $f(x) = a(x-p)^2 + q$ is its _______________ . If $a < 0$, then the vertex (p, q) of the parabola $f(x) = a(x-p)^2 + q$ is its _______________ .

Concept Check

8. Match each quadratic function **a.-d.** with its graph **I-IV**.

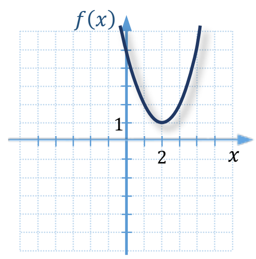
a. $f(x) = (x-2)^2 - 1$

b. $f(x) = (x-2)^2 + 1$

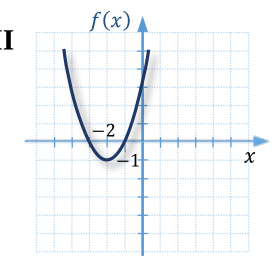
c. $f(x) = (x+2)^2 + 1$

d. $f(x) = (x+2)^2 - 1$

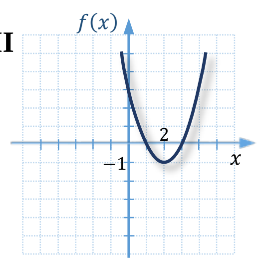
I



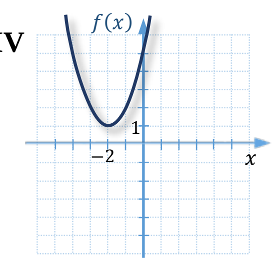
II



III



IV



9. Match each quadratic function **a.-d.** with its graph **I-IV**.

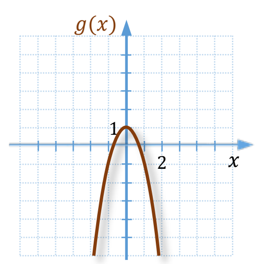
a. $g(x) = -(x-2)^2 + 1$

b. $g(x) = x^2 - 1$

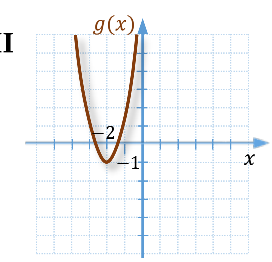
c. $g(x) = -2x^2 + 1$

d. $g(x) = 2(x+2)^2 - 1$

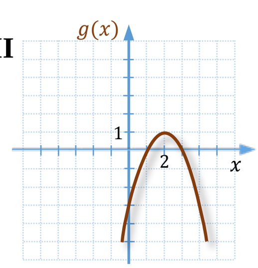
I



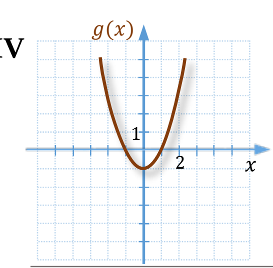
II



III



IV



Concept Check

10. Match each quadratic function with the characteristics of its parabolic graph.

a.	$f(x) = 5(x-3)^2 + 2$	**I**	vertex $(3,2)$, opens down
b.	$f(x) = -4(x+2)^2 - 3$	**II**	vertex $(3,2)$, opens up
c.	$f(x) = -\frac{1}{2}(x-3)^2 + 2$	**III**	vertex $(-2,-3)$, opens down
d.	$f(x) = \frac{1}{4}(x+2)^2 - 3$	**IV**	vertex $(-2,-3)$, opens up

For each quadratic function, describe the ***shape*** *(as* ***wider***, ***narrower***, *or the* ***same*** *as the shape of $y = x^2$) and* ***opening*** *(**up** or **down**) of its graph. Then* ***graph it*** *and determine its* ***range***.

11. $f(x) = 3x^2$

12. $f(x) = -\frac{1}{2}x^2$

13. $f(x) = -\frac{3}{2}x^2$

14. $f(x) = \frac{5}{2}x^2$

15. $f(x) = -x^2$

16. $f(x) = \frac{1}{3}x^2$

Graph *each parabola by plotting its vertex, and following its shape and opening. Then,* ***describe transformations*** *of the basic parabola that would lead to the obtained graph. Finally, state the* ***domain*** *and* ***range***, *and the equation of the* ***axis of symmetry***.

17. $f(x) = (x-3)^2$

18. $f(x) = -x^2 + 2$

19. $f(x) = x^2 - 5$

20. $f(x) = -(x+2)^2$

21. $f(x) = -2x^2 - 1$

22. $f(x) = \frac{1}{2}(x+2)^2$

For each parabola, state its ***vertex***, ***shape***, ***opening***, *and the equation of its* ***axis of symmetry***. *Then,* ***graph*** *the function and describe* ***transformations*** *of the basic parabola that would lead to the obtained graph.*

23. $f(x) = 3x^2 - 1$

24. $f(x) = -\frac{3}{4}x^2 + 3$

25. $f(x) = -\frac{1}{2}(x+4)^2 + 2$

26. $f(x) = \frac{5}{2}(x-2)^2 - 4$

27. $f(x) = 2(x-3)^2 + \frac{3}{2}$

28. $f(x) = -3(x+1)^2 + 5$

29. $f(x) = -\frac{2}{3}(x+2)^2 + 4$

30. $f(x) = \frac{4}{3}(x-3)^2 - 2$

Discussion Point

31. Four students, **A**, **B**, **C**, and **D**, tried to graph the function $f(x) = -2(x+1)^2 - 3$ by transforming the graph of the basic parabola, $y = x^2$. Here are the transformations that each student applied

*Student **A**:*
- shift 1 unit left and 3 units down
- dilation of y-values by the factor of -2

*Student **B**:*
- dilation of y-values by the factor of -2
- shift 1 unit left
- shift 3 units down

*Student **C**:*
- flip over the x-axis
- shift 1 unit left and 3 units down
- dilation of y-values by the factor of 2

*Student **D**:*
- shift 1 unit left
- dilation of y-values by the factor of 2
- shift 3 units down
- flip over the x-axis

With the assumption that all transformations were properly applied, discuss whose graph was correct and what went wrong with the rest of the graphs. Is there any other sequence of transformations that would result in a correct graph?

*For each parabola, state the coordinates of its **vertex** and then **graph** it. Finally, state the **extreme value** (**maximum** or **minimum**, whichever applies) and the **range** of the function.*

32. $f(x) = 3(x-1)^2$

33. $f(x) = -\frac{5}{2}(x+3)^2$

34. $f(x) = (x+2)^2 - 3$

35. $f(x) = -3(x+4)^2 + 5$

36. $f(x) = -2(x-5)^2 - 2$

37. $f(x) = 2(x-4)^2 + 1$

38. $f(x) = \frac{1}{2}(x+1)^2 + \frac{3}{2}$

39. $f(x) = -\frac{1}{2}(x-1)^2 - 3$

40. $f(x) = -\frac{1}{4}(x-3)^2 + 4$

41. $f(x) = \frac{3}{4}\left(x+\frac{5}{2}\right)^2 - \frac{3}{2}$

Analytic Skills

*Given the graph of a parabola, state the most probable **equation** of the corresponding function. Hint: Use the vertex form of a quadratic function.*

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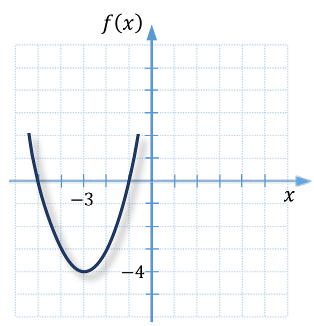

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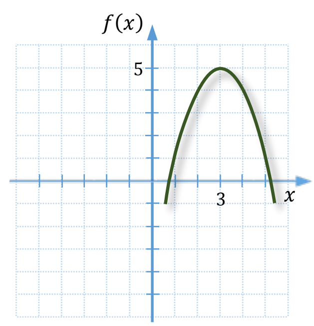

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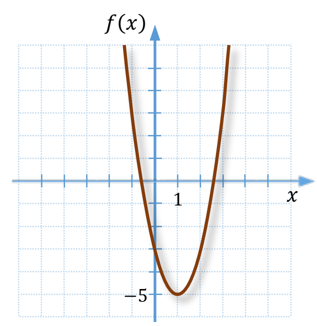

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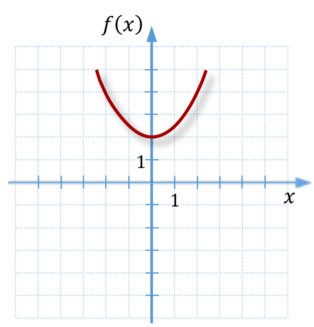

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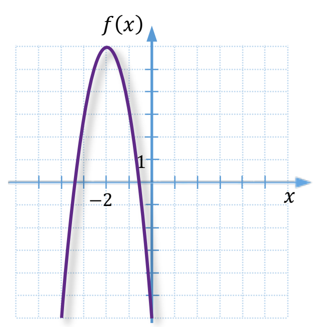

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